GROWTH MODELS WITH INTERNAL COMPETITION*

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Combined statistical physics and computation modelling give new instruments for the study of non-equilibrium systems. We briefly review generalized Eden and Diffusion-Limited Aggregation models as applied to spreading phenomena. We indicate the occurrence of non-universal behaviors.

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1. Introduction

Spreading phenomena are common non-equilibrium processes in nature: surface wetting, viscous fingering, liquid invasion in porous media, grain coalescence in alloys, fracture propagation, but also evolution of territories or population of insect swarms, virus propagation. For spreading processes driven by cooperative or non-linear evolution rules, the systems develop patterns which often reach a high level of complexity [1].

In modern statistical physics, the understanding of a wide variety of natural spreading phenomena is approached by inventing simple models. The complexity generated by these models is often studied in terms of "criticality" [1]. The signatures of strict criticality are known to be *e.g.* the fractality of a pattern or the power law behavior of the size-distribution of spreading events [2]. Such power laws reflect in fact the presence of long-range correlations in the system resulting in the formation of complex patterns.

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In so doing, many models have been imagined. Here, we review a few of our endeavors and those of others in the field of growth models.

2. Eden and multicomponent Eden models

The most simple non-equilibrium spreading model is the Eden model which describes the aggregation of identical particles on a lattice. The model was developed in order to mimic the growth of bacterian cells colonies [3] and was generalized to simulate other one-component spreading phenomena [4-6]. A single step of the Eden growth consists of randomly selecting a particle on the surface of a seed, a cluster thereafter, and randomly filling one of its empty neighbors by a new particle (more generally called an "entitiy"). This irreversible growth rule generates compact and round clusters which tend to fill the entire available space on the lattice. A typical Eden cluster made of $N = 4000$ identical particles is drawn in Fig. 1. The surface of Eden clusters was found to be self-affine [4].

Fig. 1. A typical Eden cluster made of 4000 particles.

However, in most systems, the relevant entities usually present additional degrees of freedom. Examples of multicomponent systems are alloys, fluids, magnets, ceramics, polymers, bacterian cells, viruses.

We extended the simple Eden model in considering that each entity had an internal degree of freedom, called a spin [7–10]. In the multicomponent Eden model [10], the elements of the growth were represented by scalar "spins" σ_i taking q states and coupled by a dimensionless energy taking two values J and 0 like for the Potts model [11]. The parameter J can be related to the affinity of aggregation between different species of particles, or more generally J represents the "intensity" of the internal competition occurring between the q species.

The growth rule is defined as follows. Starting from a single spin of state $\sigma_0 = 1$ as seed, the growth consists in successively selecting at random one site i, then the spin of state $1 \leq \sigma_i \leq q$, on the cluster surface. From this site i, a new spin is glued on an empty nearest neighboring site j chosen at random. The state value of this new spin is $1 \leq \sigma_j \leq g$ chosen with a probability

$$\frac{\exp(J\delta_{\sigma_i \sigma_j})}{\exp(J) + q - 1} \tag{1}$$

among the q states and where $\delta_{x,y}$ is the Kronecker function. After being glued, each spin is frozen forever. This model is quite different from the classical Potts model usually studied in an equilibrium state. As for the Eden model, the so-called "Eden-Potts" model is history-dependent and irreversible. The dynamics of the model is thus expected to be different from the classical Potts dynamics [12].

While the overall cluster growth is strictly equivalent to the classical Eden process leading to round and compact clusters, internal patterns are generated. Domains (a domain is defined as a set of connected spins in the same state) of the q species are nucleating and are competing for $q > 1$. Domains of the same spin species can also coalesce and block other ones during the growth. Such an observation leads to raise the question of a percolation-like mechanism for the seed species domain and whether any critical p_c or q_c exist.

It can be also emphasized that the growth is thus more complex than classical epidemic [5] or forest fire models [6] where only "immune", *i.e.* non-growing sites, constitute obstacles for the spreading of a single domain.

It is of interest to study this problem as a function of the normalized probability p to glue a spin of the same species as the selected one, *i.e.*

$$p = \frac{\exp(J)}{\exp(J) + q - 1}, \tag{2}$$

which reduces obviously to 1 for $q = 1$ since the classical Eden model is recovered.

Fig. 2 shows a typical q-component cluster of $N = 10000$ spins for $q = 4$ on the square lattice and for $p = 0.92$. Each "color" in Fig. 1 represents a spin species. Following the lines of thought of percolation theory [13], we have analyzed the domain competition as a function of q and p in order to know whether the seed domain can grow forever, *i.e.* there is a connected path of spins in the same state which connects the center and the surface of the cluster. Various two-dimensional lattices have been investigated: honeycomb, square and triangular lattices. The results are summarized in Table I.

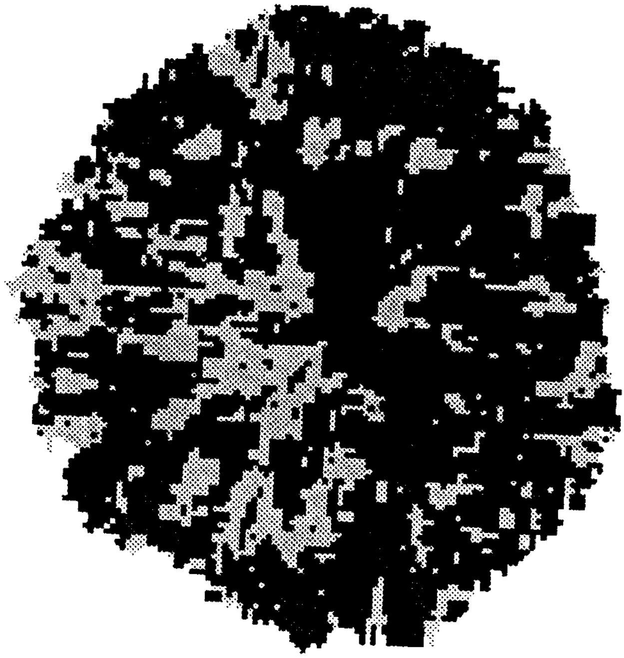

Fig. 2. A typical $q = 4$ multicomponent Eden cluster made of 10000 spins for $p = 0.92$. Each "color" represents a different spin species.

TABLE I

The various dynamical regimes found for the multicomponent Eden model on various two-dimensional lattices.

Lattice	q	Mean domain size
Honeycomb	2	finite
	3	finite
	...	finite
Square	2	finite for $p < p_c$ and infinite for $p > p_c$
	3	finite
	...	finite
Triangular	2	infinite for all p values
	3	finite
	...	finite

For all lattices and all q values, the case $p = 1$ is trivial: since nucleation of new domains is forbidden, the seed domain is the only growing entity which corresponds to an Eden growth.

However, for $p < 1$ the seed domain dominates only the early stages of the growth. After some steps, new domains of the $q - 1$ other spin species nucleate and infect the seed domain surface. The latter new domains grow and the multidomain spreading phenomenon reaches a steady-state for which the fractions c_1 to c_q of the q kinds of species in the cluster are equivalent and equal to $1/q$. This steady-state is reached exponentially and the characteristic time of the transient regimes is proportional to $\exp(J)$, *i.e.* proportional to $p/(1-p)$ [8].

In the steady-state, two different patterns can result from the competition dynamics between domains: *(i)* all domains can have a finite size and the seed domain cannot span through an infinite cluster, the pattern is then cellular and the mean domain size depends on p and q, or *(ii)* if nucleation is small and coalescence large enough, some domains can have an infinite size, the mean domain size diverges then as the cluster grows. We found numerically that the latter scenario can occur only for $q = 2$ and for both square and triangular lattices (see Table I).

On the *square* lattice, a transition between finite and infinite domains takes place at a defined value $p_c = 0.83 \pm 0.03$ for $q = 2$. On this transition, the size-distribution $n(s)$ of domains is a power law $n(s) \sim s^{-\tau}$ indicating that the spreading phenomena are critical. The exponent τ is estimated to be $\tau = 1.63 \pm 0.05$. Moreover, the percolating domains are found to be fractal with a fractal dimension $D_f = 1.5 \pm 0.1$. One should note that the values of the exponents are somewhat different from classical percolation exponents which are $D_f = 91/48$ and $\tau = 187/91$.

However, the dynamics is very different on the *triangular* lattice. Indeed, the seed domain percolates for all values of $0 \leq p < 1$. In fact, the growth process generates a no-scale pattern for all values of p. The size distribution exponent τ is approximately equal to 2. The seed domains and therefore all others are fractal with a dimension around $D_f = 1.8$. We have observed some variation of the exponents with p indicating the presence of finite-size effects. The multicomponent Eden model thus presents a non-universal behavior: the dynamics and the values of the critical exponents seem to depend on non-universal parameters like the lattice structure or the value of q. This numerical work illustrates well the complexity emerging from the introduction of internal degrees of freedom in spreading phenomena. Such a kind of growth should be further analyzed for other lattice types and for other euclidian dimensions. Actually, a general theoretical framework for such multicomponent growth is still lacking.

3. DLA and magnetic DLA models

The Diffusion-Limited Aggregation (DLA) model generates aggregation of particles on a cluster through a Brownian motion [14]. Particles are launched far from the cluster and they diffuse on the lattice. If a particle touches the cluster, the particle sticks immediately. The next particle is then launched. The DLA model generates fractal structures with a fractal dimension close to $D_f = 1.7$ which is known to be universal for various Laplacian growths [15]. A typical structure grown with this model is presented in Fig.3.

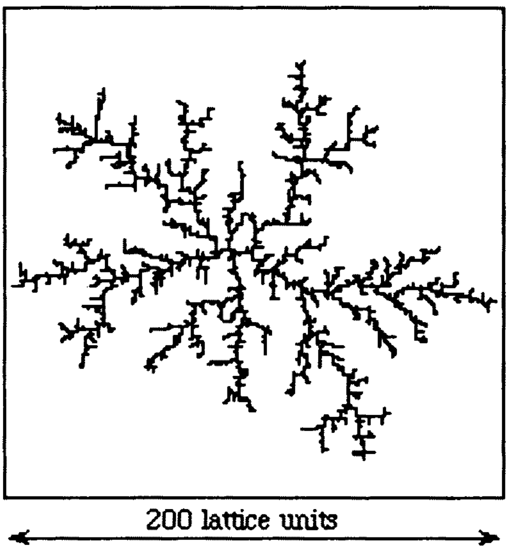

Fig. 3. A typical DLA cluster made of 4000 identical particles.

We introduced an internal degree of freedom, *i.e.* a spin taking two states, in the DLA model extending it to the so-called Magnetic DLA (MDLA) model [9]. The growth begins with an initial spin σ_0 (up or down) on the center of the lattice. A spin is then launched far from the seed, the cluster thereafter. This entity is assumed to diffuse on the lattice. At each step, a choice is then made for both the next site and the next state orientation of the diffusing spin, as the spin is allowed to flip or not flip. The probabilities of jumping to one of the four neighbour sites (on the square lattice) are defined as proportional to $\exp(-\Delta E)$, where ΔE is the local gain of the dimensionless Ising energy between the initial and the final states defined by

$$E = -\frac{J}{s} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - H \sum_i \sigma_i \tag{3}$$

in which the first summation occurs only for the nearest neighbor pairs $\langle i, j \rangle$ while the second sum runs over all spins of the cluster. The probabilities of the 8 possible configurations for each jump on the square lattice (four directions times two different spin species) are renormalized. Then, one specific configuration is chosen through a random number generator like for a Monte-Carlo simulation. If the spin touches the aggregate, the spin sticks immediately and is not allowed to flip later on. In fact, the state of the new spin is only determined by the last step of the walk.

The MDLA model is still a purely kinetic growth model because the diffusion and growth are driven by the probabilities $\exp(J)$ and $\exp(H)$. But these probabilities lead to major differences in the dynamics and in the resulting patterns. In the neighboring of the perimeter, *i.e.* for next nearest neighbors of the cluster sites, the sticking and diffusing probabilities already "feel" the cluster, leading to more constraints than in the DLA process [9].

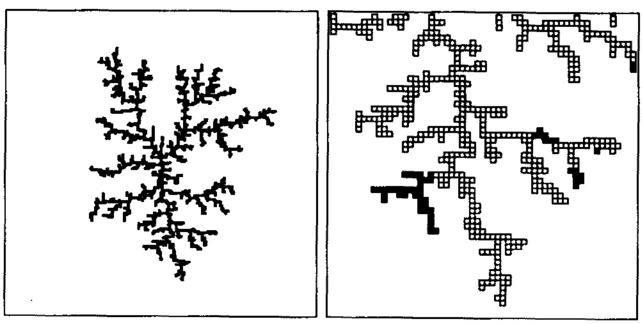

Fig. 4. A typical MDLA cluster of $N = 2000$ spins for $J = 3.0$ and $H = 3.0$. A blow up of a region of the cluster is also presented. Up spins are drawn in white while down spins are drawn in black.

We have examined the square lattice MDLA model only. For finite size systems, the "quenching" of the degree of freedom on the cluster leads to branching or compactness depending on J and H. In the whole ferromagnetic interaction regions ($J > 0$), the structure of the MDLA clusters is DLA-like (see Fig. 4). In the antiferromagnetic regions ($J < 0$), the cluster morphology can however become dendritic with an important thickening of the branches (see Fig. 5). And sometimes, compact structures are generated depending on J and H (see Fig. 6). These more compact clusters are provided with unusual (for DLA) internal lacunes and channels. In other regions, "Eden tree"-like structures are generated (see Fig. 7). In fact, compact morphologies are generated for field values H larger than the interaction coupling J [9].

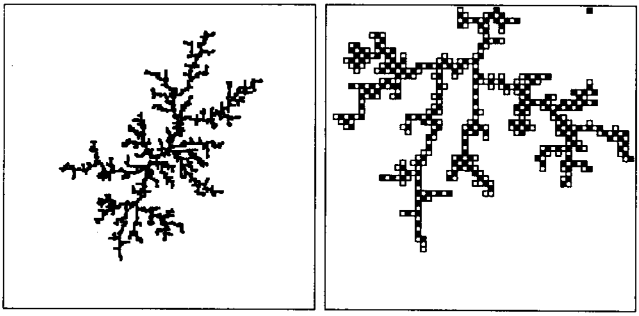

Fig. 5. A typical MDLA cluster of N=2000 spins for $J = -3.0$ and $H = 1.5$. A blow up of a region of the cluster is also presented. Up spins are drawn in white while down spins are drawn in black.

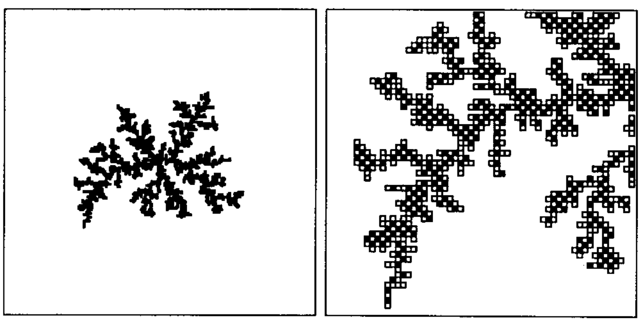

Fig. 6. A typical MDLA cluster of $N = 2000$ spins for $J = -3.0$ and $H = 3.0$. A blow up of a region of the cluster is also presented. Up spins are drawn in white while down spins are drawn in black.

We have found that the fractal dimension is close to $D_f = 1.71$ for decoupled spins, on the ßH axis. However, the fractal dimension D_f takes different values ranging from 1.68 to 1.99 in the other regions. These values are obviously quite different from the ordinary DLA value. In particular, in the ß$J < 0$ region, the coupling and field effects are in conflict. This leads to a wide variety of processes and morphologies. Non-zero values of the field H implies that internal perimeter sites can be more favored than tip sites. This resulted in a thickening of the branches as seen in Fig. 5 and 6. For very large values of H, the spin wandering on the whole "surface" before sticking on it leads to more compact clusters (see Fig. 7).

It is also of interest to study how spin species are distributed in the clusters for $H = 0$. Indeed, the growth starts with a seed species and it is

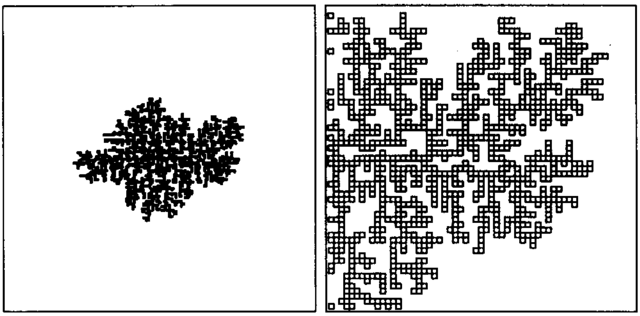

Fig. 7. A typical MDLA cluster of $N = 2000$ spins for $J = -3.0$ and $H = 6.0$. A blow up of a region of the cluster is also presented. Up spins are drawn in white while down spins are drawn in black.

of interest to know whether this species can dominate the growth or not. In fact, this is similar to multicomponent Eden growths. The first stages of the growth are dominated by the seed species. Thereafter, new branches of the other species infect the cluster leading to a steady-state where both species coexist in the same proportion (for $H = 0$). The characteristic duration time for these transient regimes is found to be proportional to $\exp(J)$ [9].

In summary, we found that the introduction of an additional degree of freedom in the DLA model can lead to a wide variety of morphologies. Further simulations are still in order for the MDLA model. Finally, the relevant quantities allowing one to indicate which universality class, or classes, are covered by such a MDLA model and obvious extensions should be determined. A theoretical framework corresponding to that of the usual DLA [16] is also lacking for MDLA.

4. Conclusion

We presented here two examples of multicomponent models of growth. They can show non-universal behaviors or morphologies depending on the growth parameters and on the underlying lattice symmetry. This proves the interest for studying multicomponent growth phenomena. This work suggests also new ways of investigation for generalizing other dynamical models like multicomponent sandpiles [17], magnetic tree growth [18], a.s.o.

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