NOISE–INDUCED TRANSITIONS IN A BISTABLE PROCESS DRIVEN BY NON-MARKOVIAN NOISE: STATIONARY STATES *

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Stationary states of the Verhulst process driven by non-Markovian dichotomic noise containing both Markovian and exponentially damped explicitly non-Markovian components are investigated. Noise-induced stationary states are compared with such states induced by purely Markovian dichotomic noise. It is found that the non-Markovianity of the driving noise may result in the appearance of new noise-induced stationary states, but in some cases, especially for higher values of the deterministic bifurcation parameter, non-Markovianity may result in the damping of Markovian noise-induced states. Besides, non-Markovianity generally diminishes the dispersion of noise-broadened states.

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1. Introduction

Little is known about the properties of non-Markovian stochastic processes, and still less about the properties of such processes driven by other non-Markovian processes (noises). Two years ago the present author began investigations of the general non-Markovian dichotomic noise (DN) and of stochastic flows driven by this noise [1–4]. Recently, properties of an explicitly non-Markovian stochastic process (driven by two Markovian noises) have been discussed by Bartussek *et al.* [5]. Earlier relevant references can be found in Refs. [1–4].

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In these preceding papers the present author proposed a systematic theory of explicitly non-Markovian dichotomic noise (DN) with exponential damping of the memory, and of its "white" limits. In [1, 2] the general properties of such noises have been found, together with preliminary discussion of the properties of processes driven by such non-Markovian noises. It was shown in the subsequent paper [3] that the behavior of the relaxation process driven by this noise exhibits some unexpected features and is distinctly different from that of the process driven by Markovian DN. In last paper of this series we have discussed in more systematic way the exact and approximate master equations governing the behavior of the probability densities describing the stochastic flows driven by non-Markovian DN [4]. One family of these approximations has been found there to lead to exact results for the probability density $P(x,t)$ for the random telegraph process and for the linear stochastic flows (relaxation processes).

It has been found, among others, that in the temporal evolution of $P(x,t)$ of the random telegraph process [2] and of the linear flows [3] there appear transient regions of locally increased probability. Additional peaks in $P(x,t)$ are interpreted [6–8] as the noise-induced transitions between noise-induced locally most probable states having no deterministic counterpart. Assuming this philosophy to be true, the non-Markovianity may lead to a multitude of such transitions: more and more new transient states appear and subsequently vanish during the temporal evolution of the process driven by non-Markovian DN, until the process itself dies out. Addition of deterministic (sinusoidal) driving sustains the appearance and vanishing of such states indefinitely, which, in turn, may lead to stochastic resonance effects [3].

The question arises whether there are also *genuine stationary additional states induced by non-Markovian noise.* For this purpose non-linear deterministically bistable stochastic flows need to be considered. Such processes are known to exhibit noise-induced transitions between additional stationary states of increased probability when driven by *Markovian* dichotomic noise [6-8]. The present paper is devoted to the discussion of the problem how the *non-Markovian* driving changes these additional stationary states: whether it creates new ones, shifts or damps old ones, etc.

The rest of the paper is organized as follows: Section 2 contains general formulation of the relevant master equations, together with the "best" approximations found in [4]. Also the argument for the choice of the specific approximation is presented there. In Section 3 the stochastic flow (Verhulst process) considered in this paper, together with its stationary master equation and numerical results are discussed. Section V collects some final remarks. In Appendix A are listed explicit formulas for the stationary probability densities for the Verhulst process driven by Markovian noises.

2. Non-Markovian master equations

Consider the stochastic flow:

$$\dot{X} = f(X) + g(X)\xi(t)\,, \tag{2.1}$$

driven by the *non-Markovian* asymmetric dichotomic noise (DN) $\xi(t)$ (called also the random telegraph signal), *i.e.* by the random two-state process with zero mean:

$$\xi(t) \in \{\Delta_1, -\Delta_2\}\,, \quad \xi^2(t) = \Delta^2 + \Delta_0\,\xi(t)\,, \quad \langle\xi(t)\rangle = 0\,, \tag{2.2}$$

where $\Delta^2 = \Delta_1\Delta_2$, $\Delta_0 = \Delta_1 - \Delta_2$. Let λ_1 and λ_2 be the probabilities of switching (per unit time) between states $\xi_1 = \Delta_1$ and $\xi_2 = -\Delta_2$. Therefore, $\tau_\alpha = 1/\lambda_\alpha$ will be mean sojourn times in these states.

The master equations for the probability density $P(x,t)$ for the process (1) read [1, 2, 4]:

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left[f(x)P(x,t) + g(x)Q(x,t)\right], \tag{2.3}$$

$$\frac{\partial}{\partial t}Q(x,t) = -\frac{\partial}{\partial x}\left[f(x) + \Delta_0 g(x)\right]Q(x,t) - \Delta^2\frac{\partial}{\partial x}g(x)P(x,t) - \Lambda\int_{t_0}^{t} dt' K(t-t')\,R_1(x,t,t')\,. \tag{2.4}$$

Here $P(x,t)$ and the auxiliary correlation functions $Q(x,t)$, $R_1(x,t,t')$ are defined as follows (the averaging is over all possible realizations of the process $\xi(t)$):

$$P(x,t) = \langle\delta(X(t,[\xi]) - x)\rangle\,, \tag{2.5}$$

$$Q(x,t) = \langle\delta(X(t,[\xi]) - x)\xi(t)\rangle\,, \tag{2.6}$$

$$R_1(x,t,t') = \langle\delta(X(t,[\xi]) - x)\xi(t')\rangle\,, \tag{2.7}$$

$(R_1(x,t,t) = Q(x,t))$, the kernel $K(\tau)$ reads:

$$K(\tau) = \gamma_o\delta(\tau) + \gamma_1 e^{-\nu\tau}\,, \tag{2.8}$$

ν is the inverse memory time, and γ_o, γ_1 describe, respectively, Markovian and non-Markovian contributions to the process $\xi(t)$.

The function R fulfills the master equation containing next higher-order auxiliary probability density, and so on. This means that in the non-Markovian case we have to deal with an infinite hierarchy of equations, and that, to obtain a workable scheme of calculation, some approximation needs to be introduced.

In [4], two similar approximations have been found to lead to correct (*i.e.*, identical with exact) results for simplest stochastic flows: the random telegraph process, $f(x) = 0$, $g(x) = 1$, and for linear relaxation, $f(x) = -ax$, $g(x) = 1$ or $g(x) = x$ in Eq.(2.1), namely the approximation:

$$R_1(x,t,t') \approx \exp\left[-(t-t')\tfrac{\partial}{\partial x}H(x)\right]Q(x,t'), \tag{2.9}$$

with either

$$H(x) = f(x) + \tfrac{1}{2}\Delta_0 g(x), \tag{2.10}$$

or

$$H(x) = f(x), \tag{2.11}$$

which leads to the equation for $Q(x,t)$:

$$\begin{aligned}&\frac{\partial}{\partial t}Q(x,t) + \frac{\partial}{\partial x}[f(x) + \Delta_0 g(x)]Q(x,t) + \Delta^2\frac{\partial}{\partial x}g(x)P(x,t)\\ &= -\Lambda\gamma_1\int_{t_0}^{t} dt' K(t-t')\mathrm{e}^{-(t-t')\frac{\partial}{\partial x}H(x)}Q(x,t'),\end{aligned} \tag{2.12}$$

or, after removing the integral,

$$\begin{aligned}&\left[\nu + \frac{\partial}{\partial t} + \frac{\partial}{\partial x}H(x)\right]\\ &\times\left\{\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big(f(x) + \Delta_0 g(x)\Big) + \gamma_0\Lambda\right]Q(x,t) + \Delta^2\frac{\partial}{\partial x}g(x)P(x,t)\right\} = -\gamma_1\Lambda Q(x,t).\end{aligned} \tag{2.13}$$

Stationary master equation

As we have said, we are principally interested in the noise-induced *stationary* states. Therefore, in the following, only stationary (*i.e.*, $t \to \infty$) values of $P(x,t)$, $Q(x,t)$, denoted by $P(x)$, $Q(x)$, will be discussed. Stationary solution of Eq.(2.3), with natural boundary conditions, reads:

$$Q(x) = -\frac{f(x)}{g(x)}P(x). \tag{2.14}$$

Therefore, stationary master equation (2.13) can be written as:

$$\left\{\nu + \frac{d}{dx}\left[f(x) + \beta\Delta_o g(x)\right]\right\}\left[\frac{d}{dx}\frac{D_{\text{eff}}(x)}{g(x)}P(x) - \gamma_o\Lambda\frac{f(x)}{g(x)}P(x)\right]$$
$$= \gamma_1\Lambda\frac{f(x)}{g(x)}P(x)\,, \tag{2.15}$$

where $\beta = \frac{1}{2}$ for the approximation (2.10), and $\beta = 0$ for the approximation (2.11), and

$$D_{\text{eff}}(x) = \Delta^2 g^2(x) - \Delta_o f(x)g(x) - f^2(x) = \left[\Delta_1 g(x) + f(x)\right]\left[\Delta_2 g(x) - f(x)\right]. \tag{2.16}$$

For further purposes, let us write also the Markovian limit of these equations. At the level of Eq.(2.4) the Markovian version is just the same equation with $\gamma_1 = 0$, $\gamma_0 = 1$. However, proper transition to Markovian limit is given by the scaled transition of the memory time:

$$\gamma_1 = (1-\gamma_0)\nu\,, \quad \nu \to \infty\,, \quad \lim_{\nu\to\infty} K(t-t') = \delta(t-t') \tag{2.17}$$

because when approximations are being used, putting $\gamma_0 = 1$, $\gamma_1 = 0$ may lead to incorrect Markovian limit, whereas the procedure above will lead always to correct results.

In this limit Eq.(2.15) becomes:

$$\frac{d}{dx}\frac{D_{\text{eff}}(x)}{g(x)}P_m(x) = \Lambda\frac{f(x)}{g(x)}P_m(x)\,, \tag{2.18}$$

with the solution given by the well-known formula [9,6]:

$$P_m(x) = \mathcal{N}^{-1}\frac{|g(x)|}{D_{\text{eff}}(x)}\exp\left[\Lambda\int^x dx\frac{f(x)}{D_{\text{eff}}(x)}\right]\Theta\left(D_{\text{eff}}(x)\right), \tag{2.19}$$

where $\mathcal{N}$ is the normalization constant, and $\Theta(x)$ is the Heaviside step function, "expressing that the probability is zero in the 'unstable' region of negative D" [9].

"White" noises

By "white" non-Markovian noises will be called stochastic processes obtained from the non-Markovian DN by the following limiting procedures [1, 4, 9]:

$$\lambda_1 \to \infty,, \quad \Delta_1 \to \infty\,, \quad \Delta_1/\lambda_1 = \Delta_2/\lambda_2 = w_0\,, \tag{2.20}$$

with w_0 kept constant, which defines the so-called *white shot noise* (WSN), being the sequence of separated positive delta-spikes on negative background. The limit:

$$\lambda_1 = \lambda_2 = \lambda \to \infty,\ \Delta_1 = \Delta_2 = \Delta \to \infty,\ \Delta^2/2\lambda = D_0\,, \tag{2.21}$$

defines the *Gaussian white noise* (GWN) as the dense set of positive and negative delta-spikes, which — in the Markovian case — corresponds to the Stratonovich interpretation of the Wiener process. GWN can be obtained also from the WSN as the limit:

$$\lambda_2 \to \infty\,,\ \Delta_2 \to \infty\,,\ w_0 = \Delta_2/\lambda_2 \to 0,\ \lambda_2 w_0^2 = 2D_0\,. \tag{2.22}$$

The limiting procedures (2.8), (2.9) enable us to obtain corresponding equations and formulas for stochastic flows driven by (asymmetric) white shot noise (WSN) with exponentially distributed weights, and for Gaussian white noise (GWN) — *cf.* Eqs. (2.27) and (2.29) below.

For further purposes, let us quote known results for Markovian case. From Eqs.(2.3)–(2.4) we get in the WSN limit the known [9] equation:

$$\frac{\partial}{\partial t}P_m(x,t) = -\frac{\partial}{\partial x}\left\{f(x) - w_0\Delta_2 g(x)\frac{\partial}{\partial x}g(x)\left[1 + w_0\frac{\partial}{\partial x}g(x)\right]^{-1}\right\}P_m(x,t)\,, \tag{2.23}$$

with stationary solution [9]:

$$P_m(x) = \frac{\mathcal{N}^{-1}}{|\Delta_2 g(x) - f(x)|}\exp\left\{\int^x \frac{f(x)dx}{w_0 g(x)\left[\Delta_2 g(x) - f(x)\right]}\right\}\Theta\left(g(x)[\Delta_2 g(x) - f(x)]\right). \tag{2.24}$$

In the GWN limit we get simply the appropriate Fokker-Planck equation (in Stratonovich interpretation):

$$\frac{\partial}{\partial t}P_m(x,t) = \frac{\partial}{\partial x}\left[-f(x) + D_0 g(x)\frac{\partial}{\partial x}g(x)\right]P_m(x,t)\,, \tag{2.25}$$

with well-known stationary solution:

$$P_m(x) = \frac{\mathcal{N}^{-1}}{|g(x)|}\exp\left\{\int^x \frac{f(x)dx}{D_0 g(x)^2}\right\}. \tag{2.26}$$

Now, the GWN limit (2.21) of non-Markovian stationary master equation (2.15) is:

$$D_0\left[\nu + \frac{d}{dx}f(x)\right]\frac{d}{dx}g(x)P(x) = \gamma_0\frac{d}{dx}\frac{f^2(x)}{g(x)}P(x) + \tilde{\gamma}_1\frac{f(x)}{g(x)}P(x)\,, \tag{2.27}$$

where $\tilde{\gamma}_1 = \nu\gamma_0 + \gamma_1$. Markovian limit of this equation is identical with the stationary form of the Fokker–Planck equation above.

The WSN limit depends on whether $\beta \neq 0$ or $\beta = 0$, *i.e.*, whether we accept approximation (2.10) or (2.11). In the former case the WSN limit of Eq. (2.15) reads:

$$w_0 \frac{d}{dx} g(x) \frac{d}{dx} [\Delta_2 g(x) - f(x)] P(x) = \gamma_0 \frac{d}{dx} f(x) P(x) , \tag{2.28}$$

which does not depend on non-Markovian noise parameters ν and γ_1, and is identical with the stationary form of the Markovian WSN master equation, Eq. (2.23) above.

On the other hand, assuming $\beta = 0$, *i.e.*, assuming approximation (2.11), we get:

$$w_0 \left[\nu + \frac{d}{dx} f(x)\right] \frac{d}{dx} \left[\Delta_2 g(x) - f(x)\right] P(x) = \gamma_0 \frac{d}{dx} f(x) P(x) + \tilde{\gamma}_1 \frac{f(x)}{g(x)} P(x) . \tag{2.29}$$

This equation does depend on non-Markovian noise parameters, and, on the other hand, leads to proper Markovian limit. Therefore, it seems that the approximation (2.11) is better (more sensible) than the approximation (2.10). Hence in the following we shall use only the approximation $\beta = 0$.

3. Verhulst process

Consider the stochastic flow (stochastic Verhulst process):

$$\dot{X} = aX - bX^3 + \xi(t) , \quad b > 0 , \tag{3.1}$$

driven by additive non-Markovian DN.

The choice of the third-order Verhulst process is due to the fact that it is the simplest bistable model exhibiting — in the deterministic version — the pitchfork bifurcation of the stationary solution (from $X = 0$ for $a < 0$ to $X = \pm\sqrt{a/b}$ for $a > 0$ at the critical value $a_c = 0$ of the bifurcation parameter a). Therefore, this model is well-suited for the investigations of the noise-induced transitions, and in fact its different versions were frequently used for this purpose (*cf.* *e.g.* ref. [6] — the complete existing literature is much too numerous to cite).

Whereas the stationary solutions $P_m(x)$ for the process (3.1) driven by Markovian noises (DN, WSN, and DWN) can be found explicitly (*cf.* Appendix A), the equations (2.15), (2.27), (2.29) cannot be solved by quadratures.

Comparison of Eqs.(2.15) (non-Markovian) and (2.18) (Markovian) suggests to look for non-Markovian solutions in the form:

$$P(x) = P_m(x)S(x)\,, \tag{3.2}$$

which, among others, implies that $P(x) \neq 0$ only in the domain in which $D_{\text{eff}}(x) \geq 0$. Substitution of Eq.(3.2) to Eq.(2.15) and subtraction of Eq.(2.18) leads to:

$$f(x)D_{\text{eff}}(x)\frac{d^2}{dx^2}S(x)+\left\{\left[\nu+f'(x)\right]D_{\text{eff}}(x)+\gamma_0 \Lambda f^2(x)\right\}\frac{d}{dx}S(x)-\gamma_1 \Lambda f(x)S(x) = 0\,. \tag{3.3}$$

The same procedure gives for WSN and for GWN the same equation as for DN, with formal substitutions:

$$D_{\text{eff}}/\Lambda \to \left[\Delta_2 g(x) - f(x)\right]g(x) \text{ for WSN}\,, \quad D_{\text{eff}}/\Lambda \to D_0 g^2(x) \text{ for GWN}\,. \tag{3.4}$$

Parameter b in the flow (3.1) plays only the role of scaling of x, therefore, in all calculations we have put $b = 1$. For simplicity, we have considered symmetric DN ($\Delta_0 = 0$) only.

Figs. 1–4 show the comparison of the dependence of $P_m(x)$ and $P(x)$ on the bifurcation parameter a for the Verhulst process driven by symmetric DN, by WSN and by GWN. A few different shapes of $P(x)$ (for symmetric DN) are presented in Fig. 5. It is seen that the non-Markovian shapes may differ the Markovian ones (drawn as dashed lines in Fig. 5). Especially, new maxima of $P(x)$ do appear in the non-Markovian case.

This is better documented in Figs. 6–8, where the loci of maxima of $P(x)$ and of $P_m(x)$ are shown as functions of x and of various parameters. Fig. 6 presents comparison of the effect of purely non-Markovian DN, mixed (Markovian and non-Markovian) DN, WSN and GWN. Dotted lines mark loci of maxima of Markovian probability density, dashed lines — these of non-Markovian probability density, and dot-dashed lines — the situation when both coincide (at least within the graph accuracy). Figs. 7 and 8 show the dependence of these loci (for the Verhulst process driven by symmetric DN) on various parameters.

It is to be noted that both the shapes of $P(x)$ and the loci of maxima depend weakly on the parameters Λ (*i.e.* on the noise correlation time), on ν (*i.e.* on the memory characteristic time), and on the parameters γ_1 and γ_0, in contrary to the time-dependent (transient) noise-induced states [3].

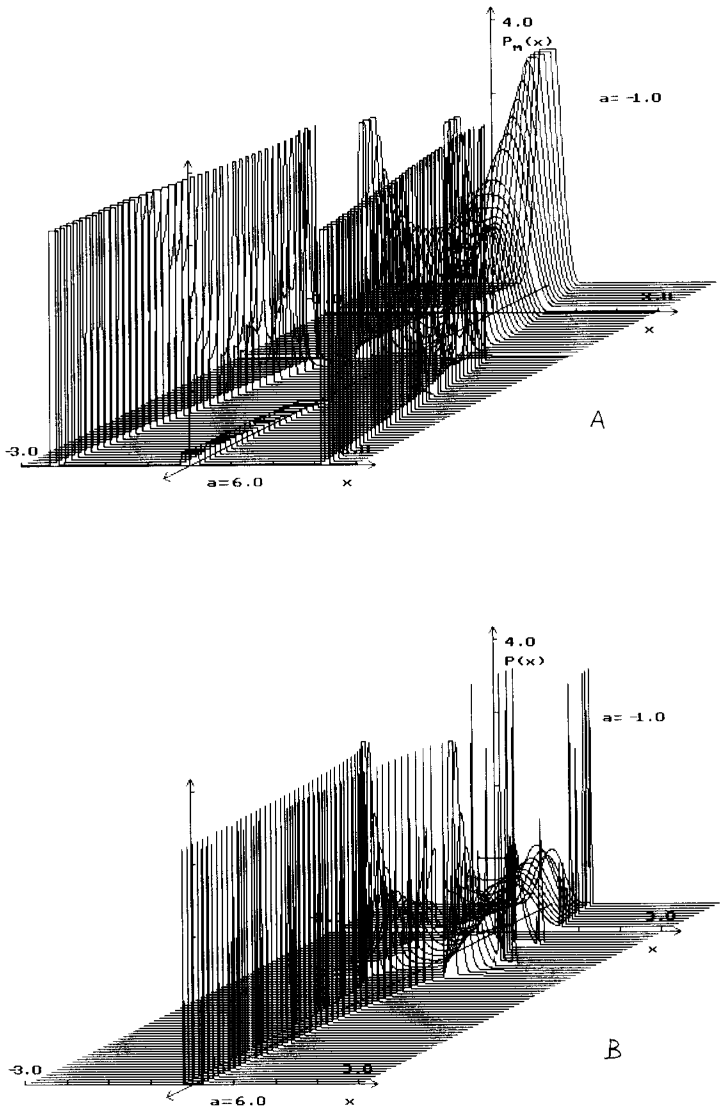


Fig. 1. Dependence of stationary probability density $P(x)$ on the deterministic bifurcation parameter a, for symmetric DN. A: Markovian case, B: non-Markovian case. $\gamma_0 = 0.0$, $\gamma_1 = 1.0$, $\Delta^2 = 1$, $\Lambda = 5$, $\nu = 0.05$.

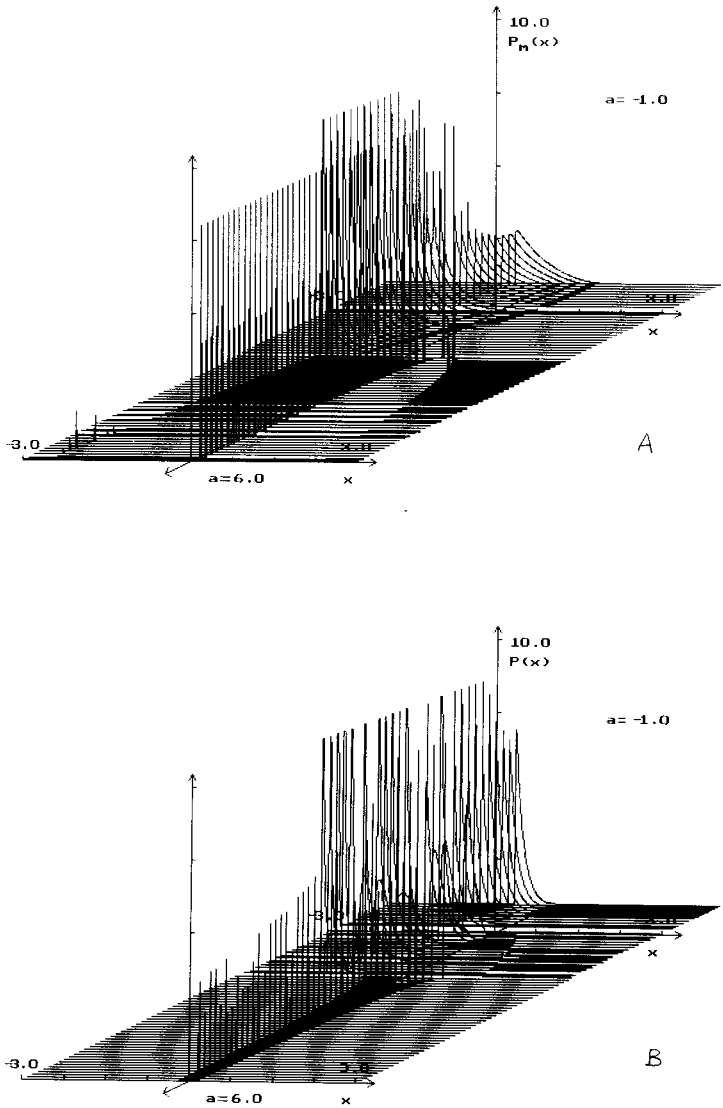
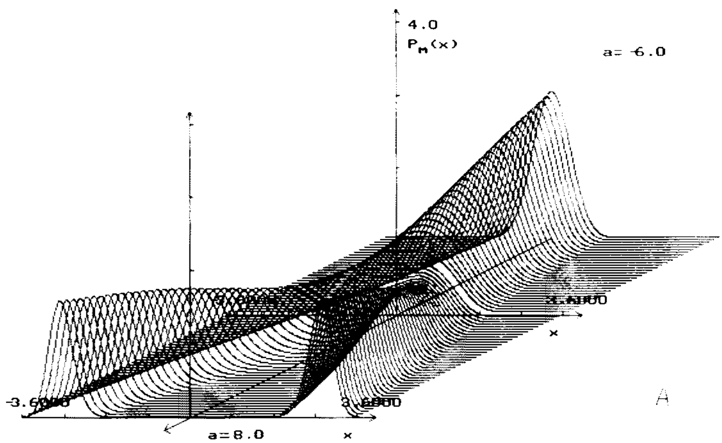


Fig. 2. The same as in Fig. 1, for WSN. $\gamma_0 = 0.0$, $\gamma_1 = 1.0$, $\Delta_2 = 1$, $w_0 = 0.2$, $\nu = 0.05$.



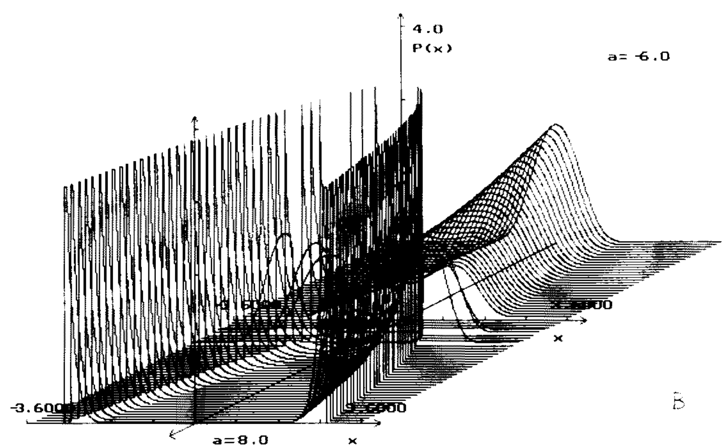
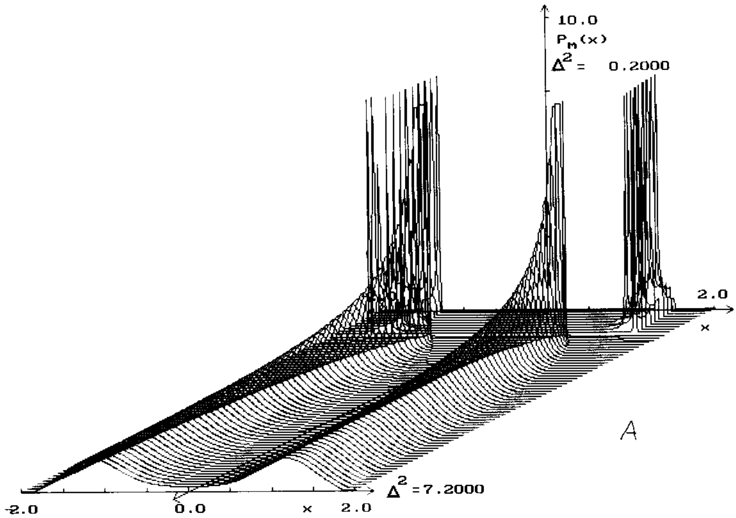


Fig. 3. The same as in Fig. 1, for GWN. $\gamma_0 = 0.0$, $\gamma_1 = 1.0$, $D_0 = 1$, $\nu = 0.05$.



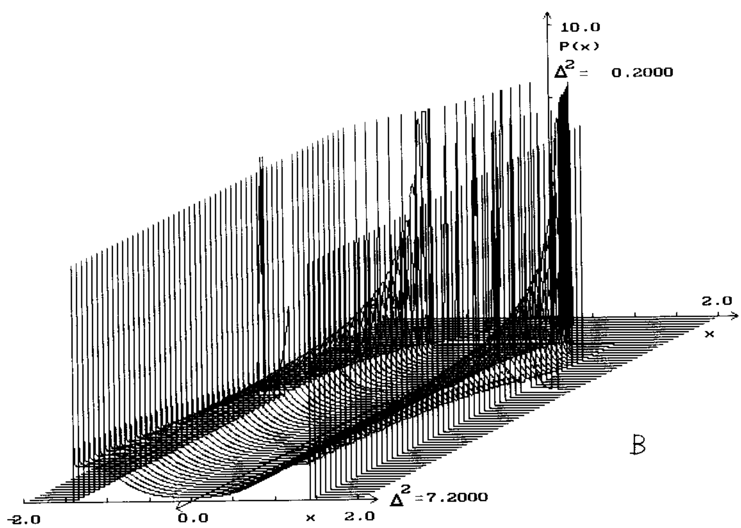


Fig. 4. The same as in Fig. 1, in dependence on the noise intensity. Δ^2. $\gamma_0 = 0.0$, $\gamma_1 = 1.0$, $\Lambda = 5.0$, $\nu = 0.05$, $a = 2.0$.

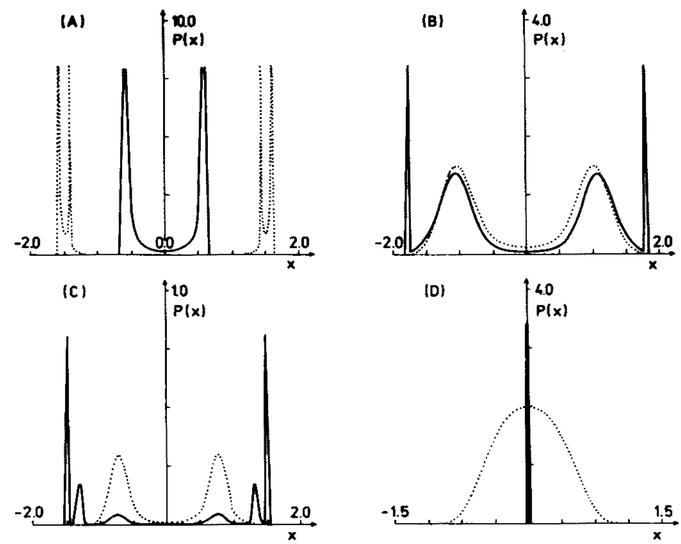


Fig. 5. Comparison of shapes of Markovian (dotted lines) and non-Markovian (full lines) $P(x)$ for a few chosen values of parameters. A: $\gamma_0 = 0$, $\gamma_1 = 1$, $\Delta^2 = 1$, $\Lambda = 5$, $\nu = 5$, $a = 2.0$. B: $\gamma_0 = 0$, $\gamma_1 = 1$, $\Delta^2 = 4$, $\Lambda = 5$, $\nu = 5$, $a = 2.0$. C: $\gamma_0 = 0$, $\gamma_1 = 1$, $\Delta^2 = 1$, $\Lambda = 5$, $\nu = 10$, $a = 1.5$. D: $\gamma_0 = 0$, $\gamma_1 = 1$, $\Delta^2 = 1$, $\Lambda = 5$, $\nu = 0.05$, $a = 10^{-6}$.

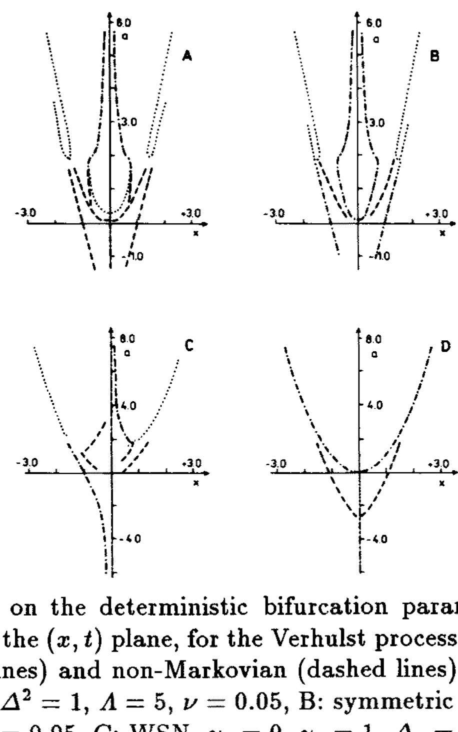


Fig. 6. Dependence on the deterministic bifurcation parameter a of the loci of maxima of $P(x,t)$ at the (x,t) plane, for the Verhulst process (3.1) driven by purely Markovian (dotted lines) and non-Markovian (dashed lines) noises. A: symmetric DN, $\gamma_0 = 0$, $\gamma_1 = 1$, $\Delta^2 = 1$, $\Lambda = 5$, $\nu = 0.05$, B: symmetric DN, $\gamma_0 = -1$, $\gamma_1 = 1$, $\Delta^2 = 1$, $\Lambda = 0.01$, $\nu = 0.05$, C: WSN, $\gamma_0 = 0$, $\gamma_1 = 1$, $\Delta_2 = 1$, $w_0 = 0.2$, $\nu = 0.05$. D: GWN, $\gamma_0 = 0$, $\gamma_1 = 1$, $D_0 = 1$, $\nu = 0.05$.

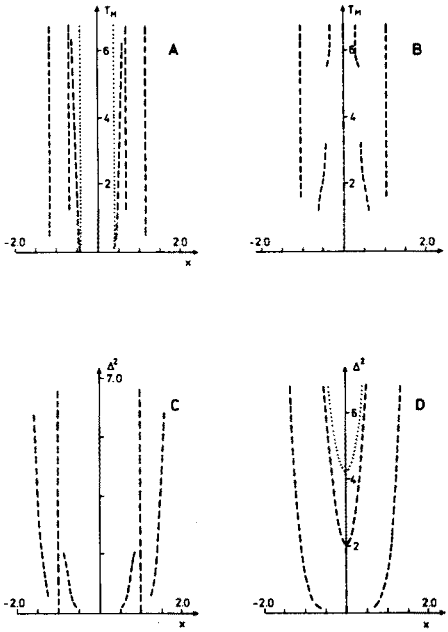


Fig. 7. The same as in Fig. 6, in dependence on the memory characteristic time $T_m = 1/\nu$ (A, B), and on noise intensity Δ^2 (C, D). $\gamma_0 = 0$, $\gamma_1 = 1$, $\Lambda = 5$. A: $\Delta^2 = 1$, $a = 0.5$. B: $\Delta^2 = 1$, $a = 0.1$. C: $\nu = 0.05$, $a = 1.0$. D: $\nu = 0.05$, $a = -0.1$.

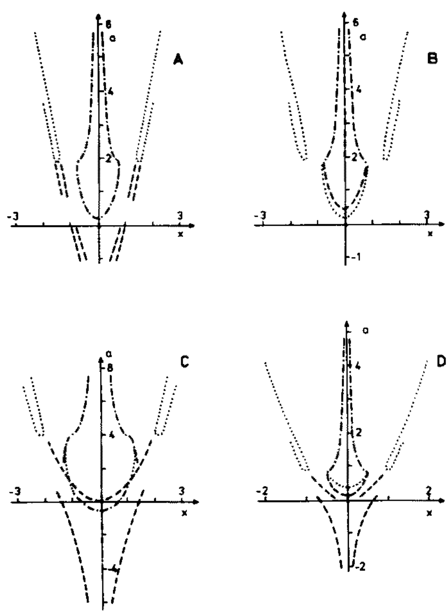


Fig. 8. The same as in Fig. 6: influence of various parameters. A: $\gamma_0 = -1$, $\gamma_1 = 1$, $\Lambda = 5$, $\Delta^2 = 1$, $\nu = 10$; B: $\gamma_0 = 1$, $\gamma_1 = -1$, $\Lambda = 5$, $\Delta^2 = 1$, $\nu = 0.01$; C: $\gamma_0 = 0$, $\gamma_1 = 1$, $\Lambda = 5$, $\Delta^2 = 9$, $\nu = 0.01$; D: $\gamma_0 = 0$, $\gamma_1 = 1$, $\Lambda = 5$, $\Delta^2 = 0.09$, $\nu = 0.01$.

Final remarks and conclusions

The general conclusions which can be drawn from these results are:

The non-Markovianity of the driving noise may result in the appearance of new noise-induced stationary states (not only transient states [2, 3]), especially for low values of the deterministic bifurcation parameter a.

In some cases, especially for higher values of a, non-Markovianity may result in the damping of Markovian noise-induced states — *cf.* Figs. 5A, 6A,B.

Non-Markovianity generally diminishes the dispersion of noise-broadened states (both noise-induced and deterministic).

For $a < 0$ non-Markovianity splits the stationary state $x = 0$. This state vanishes completely for highly negative values of a, and reappears for low negative values of a, but accompanied by two non-zero peaks.

In general, the non-Markovianity adds very narrow very strong peaks at left and right wings (extrema) of the probability distribution, both for $a < 0$ and for $a > 0$. It is to be noted that these extremal values are smaller than the extremal range of the domain $\mathcal{D}_x$ (the latter is given by the condition $D_{\rm eff} \geq 0$).

For higher values of a action of non-Markovian noises leads to the narrowing of the whole distribution: most probable values of X_{st} are much smaller than deterministic ones and than these induced by Markovian noises.

Appendix A

Markovian formulas

For the sake of completeness, we list here explicit formulas for $P_m(x)$ for Markovian noises, calculated from Eqs. (2.19), (2.24), and (2.26) for the process (3.1), *i.e.*, for $f(x) = ax - x^3$, $(b = 1)$, $g(x) = 1$.

M-GWN:

$$P_m(x) = \mathcal{N}^{-1} \exp\left\{-\frac{(x^2 - a)^2}{4D_0^2}\right\}, \tag{A.1}$$

M-WSN:

$$P_m(x) = \frac{\mathcal{N}^{-1}}{\Delta_2 - f(x)} e^{-x/w_0} J(x; \Delta_2, \Delta_2)\Theta\Big(\Delta_2 - f(x)\Big), \tag{A.2}$$

M-DN:

$$P_m(x) = \frac{\mathcal{N}^{-1}}{[\Delta_1 + f(x)]\,[\Delta_2 - f(x)]} J(x; \Delta_2, \Delta_2) J(x; \Delta_1, -\Delta_1)\Theta\Big(\Delta^2 - \Delta_0 f(x) - f^2(x)\Big), \tag{A.3}$$

with

$$J(x;\mu,w)=\exp\left\{\mu\int^{x}\frac{dx}{x^3-ax+w}\right\}=(x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2}(x-x_3)^{\alpha_3}\,, \tag{A.4}$$

where x_j are solutions of the cubic equation $x^3-ax+w=0$, and

$$\alpha_j=\frac{\mu}{(x_j-x_k)(x_j-x_l)}\,,\qquad \{j,k,l\}=\{1,2,3\}\,. \tag{A.5}$$

When two of solutions x_j are complex conjugate, *i.e.*, when

$$\tilde{D}=\frac{w^2}{4}-\frac{a^3}{27}>0\,, \tag{A.6}$$

J can be written as:

$$J(x;\mu,w)=(x-y_1)^{\alpha}\left[\left(x+\tfrac{1}{2}y_1\right)^2+\omega^2\right]^{-\alpha}\exp\left\{-3\alpha\omega y_1\arctan\left(\frac{2x+y_1}{2\omega}\right)\right\}, \tag{A.7}$$

with

$$\alpha=\frac{2\mu}{4\omega^2+9y_1^2}\,,\quad y_1=A+B\,,\quad \omega=(A-B)\sqrt{3/4}\,,$$
$$A=\left(-\tfrac{1}{2}w+\sqrt{\tilde{D}}\right)^{1/3},\quad B=\left(-\tfrac{1}{2}w-\sqrt{\tilde{D}}\right)^{1/3}. \tag{A.8}$$

For symmetric M-DN ($\Delta_1=\Delta_2$) the formula (A3) can be written in a simpler, explicitly symmetric in x form.

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