A "NO GO" THEOREM RELATED TO EXTENSIONS OF RELATIVISTIC SYMMETRY GROUP

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Dedicated to Wojciech Królikowski in honour of his 70th birthday

We begin with the presentation of the well known method used in classical mechanics for purpose of extending the Galilei group centrally. We show then that it is not possible to carry over this method *mutatis mutandis* to the case of classical scalar fields and relativistic transformations like Poincaré, Lorentz and translational groups.

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1. Introduction

It is known from mathematical studies that some Lie groups admit an extension of the set of generators by elements which commute with themselves and all the rest. These generators are called central and the extension itself — a central extension. A criterion for a non trivial inequivalent central extension of a Lie algebra is that the elements of the second scalar cohomology group are not trivial. Moreover, for a semisimple Lie algebra the second cohomology group is trivial (theorems of Whitehead, see *e.g.* [1]). For example the cohomology, considered above, for the Galilei group is not trivial and therefore the algebra of the latter can be centrally extended by a scalar generator. The same applies to the algebra of the translational group. On the other hand the Lorentz group has a trivial cohomology and does not allow any central extension. It is possible to incorporate also some super-algebras in the scheme mentioned above. In this case, for $N \geq 2$, the supersymmetric Poincaré transformation admits a nontrivial central extension [2].

This all does not mean that the algebra of the Lorentz group does not admit any extension. It can be extended not only to the Poincaré group algebra, *i.e.* to a group comprising the Lorentz as well as the 4-dimensional translation group, but even to the 4–dimensional conformal group algebra, consisting of the special conformal, dilatational and Poincaré groups.

In case of the Galilei group the central extension of its algebra can be most easily obtained and physically interpreted by modifying the Lagrangian usually used to describe a classical free nonrelativistic massive particle. It is well known that this Lagrangian is weakly invariant under the Galilei transformations of the three position and one time coordinates. We may, however, enlarge the original representations space by one additional degree of freedom in such a way that a properly modified Lagrangian, depending now on all five variables, stays invariant under the Galilei transformation. The presence of the additional coordinate causes that the number of generators of the group becomes larger too, namely it grows by one. This new generator commutes with all the others. This is just the central extension of the original Galilei group, playing an important role in quantum mechanics.

It seems natural to apply a similar method in case of Lorentz or Poincaré transformation. In classical relativistic case, however, the Lagrangian for one free particle is invariant under this transformation and therefore there is no point to apply this method in this particular case. But the situation is different in case of a model of a classical relativistic field. Here the Lagrangian is weakly invariant under the Lorentz or Poincaré transformations, it means that the transformed Lagrangian differs from the original one by a 4-dimensional divergence of a vector. Thus one may try to apply the method outlined above to this case *mutatis mutandis*. To test the possibility of carrying over the method used in classical mechanics to the case of classical fields we concentrate upon the case of scalar fields. As the divergence concerns a 4-dimensional vector we do not expect that the extension of the groups under consideration will be central.

Unfortunately, it turns out that it is not possible to use this method successfully in this case. The reason is the incompatibility of the standard transformation properties of the 4-vector appearing in the divergence under the relativistic groups with the transformation properties, resulting from the requirement to keep the modified Lagrangian invariant under these group transformations. Thus in case of translations, Lorentz as well as Poincaré transformations our construction explained in detail below leads inevitably to a statement of a type of a "no go" theorem.

2. Nonrelativistic free massive classical particle and the central extension of the Galilei group

Let us start by recapitulating some well known facts about the Galilei transformation of a system consisting of a scalar massive particle moving freely in a 3-dimensional Euclidean space [3].

The Galilean transformation reads

$$x \to x' = g(t, x) \equiv Rx - vt + a\,, \tag{2.1}$$

$$t \to t' = h(t) \equiv t + b\,, \tag{2.2}$$

where $x = (x_1, x_2, x_3)$, $v = (v_1, v_2, v_3)$, $a = (a_1, a_2, a_3)$ *etc.* All quantities appearing in (2.1) and (2.2) are real. The 3-dimensional matrix $R = (R^{-1})^T$ as well as v, a and b are constants. Since

$$Det \begin{pmatrix} R & -v \\ 0 & 1 \end{pmatrix} \neq 0\,,$$

we have

$$x' \to x = R^T x' + R^T v t' - R^T a - R^T v b\,,$$

$$t' \to t = t' - b\,,$$

We assume not that x is a function of some continuous parameter τ which we identify with t. Thus

$$x = f(\tau = t)\,,$$

and

$$x' = g(t, x(t)) = f'(t) = Rf(t) - v(t) + a\,.$$

The Lagrangian density of a nonrelativistic free particle of mass $m \neq 0$

$$\mathcal{L} \equiv \frac{m}{2}\left(\dot{f}(t)\right)^2 \qquad \dot{f}(t) \equiv \frac{df(t)}{dt}$$

is not invariant under the Galilei transformation, namely we have

$$\mathcal{L}' = \mathcal{L} + \dot{W} \tag{2.3a}$$

$$W \equiv m\left\{\frac{1}{2}\int_0^b \left(\dot{f}(t+u)\right)^2 du - vR\,f(t+b) + \frac{1}{2}t + c\right\}\,. \tag{2.3b}$$

i.e. the Lagrangian is weakly invariant.

Although we are dealing here exclusively with a classical theory our final goal is quantum mechanics. It is well known that the Schrödinger wavefunctions are given in the latter case not as vector but as ray representations of the Galilei group. The quantum mechanical group is a central extensions of the classical group [4]. We may also perform this extension in the classical case by introducing an additional variable, say, s in addition to x and t. This variable will be defined in such a way that the modified Lagrangian should stay invariant under the Galilei transformation.

The new Lagrangian reads

$$\tilde{\mathcal{L}} = \mathcal{L} = m\dot{s}\,.$$

This Lagrangian is singular as $\dot{s}$ enters here linearly. We require that

$$\tilde{\mathcal{L}}' = \tilde{\mathcal{L}}$$

under the Galilei transformation. Therefore,

$$s \to s' = s + \frac{1}{m} W\,.$$

Notice, that the transform of s depends on the prescription how x changes with $\tau = t$ due to the term

$$\int_t^{t+b} \left(\dot{f}(u)\right)^2 du\,.$$

in (2.3).

For infinitesimal transformations we have

$$R = \mathbf{1} + \lambda + o(\lambda)\,, \qquad \lambda^T = -\lambda\,.$$

We may also write

$$\lambda_{ij} = \sum_k \epsilon_{ijk} l_k\,, \qquad i,j = 1,2,3\,.$$

In case of pure rotations we have up to terms of higher order of smallness.

$$x_l \to x_l' = x_l + \lambda_{lm} x_m = \left(1 + \lambda_{jm} x_m \partial_j\right) x_l = \left(1 + l_k L_k\right) x_l\,,$$

with

$$L_k \equiv -\varepsilon_{kmj} x_m \partial_j\,,$$

as well as

$$s \to s' = s + c^{(1)} = s + l_i c_i^{(1)} = \left(1 + l_i L_i^{(s)}\right) s\,,$$

with

$$L_i^{(s)} \equiv c_i^{(1)} \frac{d}{ds}\,.$$

In case of boosting we obtain

$$x_l \to x_l' = x_l - t v_l = (1 - v_i G_i)\, x_l\,,$$

$$G_i \equiv -t\partial_i\,,$$

$$s \to s' = s - v_i x_i + v_i c_i^{(2)} = \left(1 + v_i G_i^{(s)}\right) s\,,$$

$$G_i^{(s)} \equiv \left(x_i - c_i^{(2)}\right) \frac{d}{ds}\,.$$

In case of spatial translation

$$x_i \to x_i' = x_i + a_i = \left(1 + a_j P_j\right) x_i\,,$$

$$P_j \equiv \partial_j\,,$$

$$s \to s' = s + c^{(3)} = (1 + a_i P_i^s)\, \dot{s}\,,$$

$$P_i^s \equiv c_i^{(3)} \frac{d}{ds}\,.$$

Finally for time translations

$$t \to t' = t + b = (1 + bP_0)\, t\,,$$

$$P_0 \equiv \frac{d}{dt}\,,$$

$$s \to s' = s + \frac{1}{2} b \left(\dot{f}(t)\right)^2 + bc^{(4)} = \left(1 + bP_0^{(s)}\right) s\,,$$

$$P_0^{(s)} \equiv \left[\frac{1}{2}\left(\dot{f}(t)\right)^2 + c^{(4)}\right] \frac{d}{ds}\,.$$

We have

$$c \equiv l_i c_i^{(1)} + v_i c_i^{(2)} + a_i c_i^{(3)} + bc^{(4)}\,.$$

We may introduce modified generators of the Galilei group, corresponding to the central extension, *viz.*

$$\begin{aligned}
\widetilde{L}_i &\equiv L_i + c_i^{(1)} P^{(s)} , \\
\widetilde{G}_i &\equiv G_i - \left(x_i - c_i^{(2)}\right) P^{(s)} , \\
\widetilde{P}_i &\equiv P_i + c_i^{(3)} P^{(s)} , \\
\widetilde{P}_0 &\equiv P_0 + \left[\frac{1}{2}\left(\dot{f}(t)\right)^2 c^{(4)}\right] P^{(s)} ,
\end{aligned}$$

where

$$P^{(s)} \equiv \frac{d}{ds} .$$

Some of the standard commutation relations of the Lie algebra of the Galilei group undergo now some changes, *viz.*

$$\begin{aligned}
\left[\widetilde{L}_i, \widetilde{L}_j\right] &= \varepsilon_{ijk}\left(\widetilde{L}_k - c_k^{(1)} P^{(s)}\right) , \\
\left[\widetilde{L}_i, \widetilde{G}_j\right] &= \varepsilon_{ijk}\left(\widetilde{G}_k - c_k^{(2)} P^{(s)}\right) , \\
\left[\widetilde{L}_i, \widetilde{P}_j\right] &= \varepsilon_{ijk}\left(\widetilde{P}_k - c_k^{(3)} P^{(s)}\right) , \\
\left[\widetilde{G}_i, \widetilde{P}_j\right] &= \delta_{ijk} P^{(s)}
\end{aligned}$$

other commutators stay unchanged.

This abstract relations of the Lie algebra of the Galilei group, extended centrally by the operator $P^{(s)}$, have to be supplemented by a constraint [5]

$$P^{(s)} \equiv \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{s}} = m . \tag{2.4}$$

This constraint is physically closely linked to quantal effects namely to the super selection rule, preserving the validity of the superposition principle [4]. The eigenstate of $iP^{(s)}$ corresponding to the eigenvalue μ is proportional to $\exp\{i\mu s\}$. Under the Galilei transformation this state goes over into

$$e^{i\mu s} \exp\left\{-i\frac{\mu}{m} W\right\} .$$

For $\mu_1 \neq \mu_2$ we are not able to observe the validity of the superposition principle as

$$e^{i\mu_1 s} + e^{i\mu_2 s} \to e^{i\mu_1 s'} + e^{i\mu_2 s'} \neq e^{i\alpha}\left(e^{i\mu_1 s} + e^{i\mu_2 s}\right),$$

unless $\mu_1 = \mu_2 = m$. The latter is insured by the constraint (2.4).

3. Testing the extension of relativistic groups on the model of a relativistic classical scalar field

Our goal is to transfer the method presented in Section 2 to systems consisting of classical relativistic fields and investigate the corresponding symmetries, namely Lorentz, translation or Poincaré transformations. This method cannot be transferred automatically and has to be adjusted to new circumstances. The Lagrangian density of the fields is, as we know, weakly invariant under these transformations. We are going hereafter to restrict ourselves to the case of scalar fields putting off the investigations of fields of different tensor character to a later stage.

Unfortunately, the results obtained by us in case of scalar fields have partly the nature of a "no go" theorem.

In case of classical relativistic scalar fields no *a priori* restriction upon the shape of the Lagrangian density with respect to the relativistic symmetries will be imposed except that it should be a scalar density.

Our metric will be $\eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$.

Let us consider a scalar field $\phi(x)$ obeying the following transformation prescription with respect to the Poincaré group [6],

$$\phi'(x) = \phi(x'), \qquad x' \equiv \Lambda x + a,$$

$$x \equiv (x_0, x_1, x_2, x_3), \qquad a \equiv (a_0, a_1, a_2, a_3), \qquad \Lambda \in L_+^\dagger \; etc..$$

For the Lagrangian

$$\mathcal{L}(x) \equiv \mathcal{L}\left(\phi(x), \partial\phi(x)\right),$$

$$\partial\phi(x) \equiv \left(\partial_0\phi(x), \partial_1\phi(x), \partial_2\phi(x), \partial_3(x)\right),$$

and $a = 0$ (proper Lorentz transformation) we have

$$\mathcal{L}(\Lambda x) - \mathcal{L}(x) = \int_0^1 \frac{d}{du}\mathcal{L}(\tilde{x})du = \int_0^1 \frac{\partial\mathcal{L}(\tilde{x})}{\partial x_\mu}\left(\tilde{\Lambda}^{-1}\right)_\mu^{\ \nu} \frac{d}{du}\tilde{\Lambda}_\nu^{\ \kappa} x_\kappa du, \quad (3.1)$$

where

$$\tilde{x} \equiv \tilde{\Lambda}(u)x$$

$\tilde{\Lambda}(u) \in L_+^\dagger$, $0 \leq u \leq 1$, and $\tilde{\Lambda}(u)$ describe a path for which $\tilde{\Lambda}(0) = \mathbf{1}$ and $\tilde{\Lambda}(1) = \Lambda$.

From (3.1) we obtain

$$\mathcal{L}(\Lambda x) - \mathcal{L}(x) = \frac{\partial}{\partial x_\mu}\left(\int_0^1 \mathcal{L}(\widetilde{x})(\widetilde{\Lambda}^{-1})_\mu{}^\nu \frac{d\widetilde{\Lambda}_\nu{}^\kappa}{du}\, x_\nu du\right) - \int_0^1 \mathcal{L}(\widetilde{x})\,\mathrm{Tr}\left(\widetilde{\Lambda}^{-1}\frac{d\widetilde{\Lambda}}{du}\right) du\,. \tag{3.2}$$

The last term vanishes. To see that notice that

$$\frac{\widetilde{\Lambda}^{-1}}{du}\,\widetilde{\Lambda} = -\widetilde{\Lambda}^{-1}\frac{d\widetilde{\Lambda}}{du},$$

and

$$\eta\widetilde{\Lambda}^T\eta = \widetilde{\Lambda}^{-1}\,.$$

The left hand side of (3.2) is a divergence, which, in general, does not vanish and therefore $\mathcal{L}(x)$ is weakly invariant under Lorentz transformations.

In a similar way we may handle the case $a \neq 0$ (proper Poincaré transformation). We get

$$\mathcal{L}(x') - \mathcal{L}(x) = \frac{\partial}{\partial x_\mu} G_\mu(x, \Lambda, a)\,, \tag{3.3a}$$

where

$$G_\mu(x,\Lambda,a) \equiv \int_0^1 \widetilde{\mathcal{L}}(u)\left(\widetilde{\Lambda}^{-1}\right)_\mu{}^\nu\left(\frac{d\widetilde{\Lambda}_\nu{}^\kappa}{du}\, x_\kappa + \frac{d\widetilde{a}_\nu}{du}\right) du\,, \tag{3.3b}$$

$$\widetilde{\mathcal{L}}(u) = \mathcal{L}\left(\widetilde{\Lambda}(u)x + \widetilde{a}(u)\right)\,, \tag{3.3c}$$

$$\widetilde{a}(0) = 0 \qquad \widetilde{a}(1) = a \qquad\qquad 0 \le u \le 1\,. \tag{3.3d}$$

Let us introduce four new fields, S_ν, $\nu = 1,2,3$, and require that they transforms under Poincaré transformations as follows

$$S_\nu(x,\Lambda,a) = S_\nu(x) + \frac{1}{g}\, G_\nu(x,\Lambda,a) + \frac{1}{g}\,\widehat{h}_\nu(x,\Lambda,a)\,, \tag{3.4a}$$

where

$$\partial^\mu \widehat{h}_\nu = 0\,. \tag{3.4b}$$

We may define now a new Lagrangian, using the fields S_ν, *viz.*

$$\widehat{\mathcal{L}}(x) \equiv \mathcal{L}(x) - g\partial^\nu S_\nu(x) . \tag{3.5}$$

We require, as in case of a nonrelativistic particle, that [6]

$$\widehat{\mathcal{L}}(x') = \widehat{\mathcal{L}}(x) , \qquad x' = \Lambda x + a . \tag{3.6}$$

By virtue of (3.3), (3.4) and (3.5) we conclude from (3.6) that

$$\left(\Lambda^{-1}\right)_\nu{}^\lambda \frac{\partial S_\lambda(x')}{\partial x_\nu} = \frac{\partial S_\nu(x,\Lambda,a)}{\partial x_\nu} ,$$

or

$$\left(\Lambda^{-1}\right)_\nu{}^\lambda S_\lambda(\Lambda x + a) = S_\nu(x,\Lambda,a) + \frac{1}{g}\widetilde{h}_\nu(x,\Lambda,a) , \tag{3.7a}$$

with

$$\partial^\nu \widetilde{h}_\nu = 0 . \tag{3.7b}$$

If we insert (3.7) in (3.4) we get

$$\left(\Lambda^{-1}\right)_\nu{}^\lambda S_\lambda(\Lambda x + a) - S_\nu(x) = \frac{1}{g}G_\nu(x,\Lambda,a) + \frac{1}{g}h_\nu(x,\Lambda,a) , \tag{3.8a}$$

with

$$\partial^\nu h_\nu = 0 . \tag{3.8b}$$

4. The case of translational and Poincaré transformations

Let us first concentrate upon the case of translations, *i.e.* when in (3.8) $\Lambda = 1$. If functions S_ν would exist in this case relations (3.5) and (3.6) imply

$$\mathcal{L}(x+a) - g\frac{\partial}{\partial x_\mu}S_\mu(x+a) = \mathcal{L}(x) - g\frac{\partial}{\partial x_\mu}S_\mu(x) .$$

for each real a. We conclude immediately that in this case

$$\mathcal{L}(x) = \frac{\partial}{\partial x_\mu}\left(g_\mu S_\mu\right)$$

up to a constant. This renders the theory trivial.

It follows from the latter statement that also in case of Poincaré transformations ($\Lambda \not\equiv 1$, $a \not\equiv 0$) we are not able to introduce the variable S_ν unless the theory becomes trivial.

5. The case of Lorentz transformations

It seems to be clear, looking at (3.5), that in case of Lorentz transformations $\widehat{\mathcal{L}}$ should be a function of x^2 only. This follows also from the equations obtained in case of infinitesimal Lorentz transformations, *viz.*,

$$\begin{aligned} &\eta^{\mu\nu}S^{\lambda} - \eta^{\lambda\nu}S^{\mu} - x^{\lambda}\partial^{\mu}S^{\nu} + x^{\mu}\partial^{\lambda}S^{\nu} \\ &= -\frac{1}{g}\mathcal{L}\left(\eta^{\mu\nu}x^{\lambda} - \eta^{\lambda\nu}x^{\mu}\right) + \frac{1}{g}h^{\nu\mu\lambda}, \end{aligned} \tag{5.1}$$

where $h^{\nu\mu\lambda} = -h^{\nu\lambda\mu}$ and $\partial_\nu h^{\nu\mu\lambda} = 0$. By taking the divergence with respect to ∂_ν on both sides of (5.1) we get

$$x^{\lambda}\partial^{\mu}\widehat{\mathcal{L}} = x^{\mu}\partial^{\mu}\widehat{\mathcal{L}}. \tag{5.2}$$

From (5.2) follows then

$$\widehat{\mathcal{L}} = F(x^2), \tag{5.3}$$

or constant.

Since $\widehat{\mathcal{L}}$ depends on $\phi(x)$, $\partial_\mu\phi(x)$ as well as on $\partial_\mu S^\mu(x)$, where these fields are not yet specified, (5.3) implies that all of them have to be functions of x^2 only or constants. However, this solution of the problem is of no interest in a standard classical field theory.

This renders the theory trivial again.

6. Final remark

What is the reason that the procedure applied successfully in case of a nonrelativistic freely moving scalar massive particle fails when simulated in case of relativistic scalar field theory?

In our opinion, both cases differ essentially from each other. Notice, that the counterpart of the 3-vector (x_1, x_2, x_3) in the classical one-particle model is now the scalar field ϕ and the counterpart of $\tau = t$ is the 4-vector (x_0, x_1, x_2, x_3). In case of the Galilei group t as well as x transform under a common transformation. In case of relativistic groups the only changes arise from the transformation of x, while ϕ does not undergo any variation. Even if we take into consideration a tensor field, which transforms under the Lorentz transformation according to its rank, this field and x transform separately under two different representations of the group.

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