PERSPECTIVES IN LATTICE GRAVITY

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Dedicated to Wojciech Królikowski in honour of his 70th birthday

We briefly overview the development of Euclidean quantum gravity in four dimensions regarded as a branch of statistical mechanics of discretized random manifolds.

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1. Introduction

The absence of a fully consistent quantum theory of gravity is felt by many theorists as a challenge. The corresponding research is not motivated by present phenomenology. But it is likely that the difficulty one encounters trying to merge together general relativity and quantum mechanics reflects our misunderstanding of some basic issues and the feeling that it might indeed be so is sufficient to trigger activity. There exist several approaches to the problem. In this paper we shall discuss only one of them.

It is well known that the formulation of the classical theory of gravity can start with the introduction of interacting tensor fields living in a flat auxiliary space, provided one imposes appropriate constraints, in the first place the gauge invariance. The identification of the field with the metric of the physical space-time is then done at the next stage. It might appear that proceeding that way is the best strategy in the quantization program: one has the correct classical theory at the tree level, computing loops will give quantum corrections. Unfortunately one gets a theory plagued with infinities. Upgrading the symmetry to super-symmetry does not resolve the

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problem. It is by now commonly admitted that the basic objects should be extended, which complicates the story considerably. In spite of the appeal of super-string theories and of their potential ability to unify interactions, it is fair to say that the progress in this direction is slow if not uncertain.

The approach to be discussed starts from an attempt to give a precise meaning to Feynman's path integral over all metrics of a manifold. This is, perhaps, closer to Einstein's geometrical intuition, since the metric remains the central concept of the theory. The price to pay is that one has to adopt the Euclidean version of the latter [1]. Someone may object at this point that the equivalence of the Euclidean and Minkowskian theories is questionable in the case of gravity, which renders the whole approach suspect. Notice, however, that Euclidean and Minkowskian gravities have in common several of their salient features: gauge symmetry, perturbative non-renormalizability, bottomless action. A successful quantization of the former, which is doubtlessly more tractable, is likely to be a prerequisite for the understanding of the latter.

Because of lack of space we shall leave aside most of the developments concerning quantum gravity in less than four dimensions ($4d$). We would like to stress, however, that the study of $2d$ gravity (random surfaces) triggered by Polyakov's famous paper [2] has to a large extent inspired the research reviewed below. The following sections express a personal view of the subject. Are left aside, in particular, the papers where the dynamical triangulation recipe (see Section 2) has *not* been adopted (see *e.g.* [3, 4]). We apologize to all those whose work has not been given due attention.

2. Discrete theory

The pure gravity theory is defined formally by the path integral

$$Z = \int [\mathcal{D}g_{ab}] e^{-S} . \tag{1}$$

Here S is the action, which is usually assumed to have the Einstein–Hilbert form

$$S = \int_{\mathcal{M}} d^d x \sqrt{g}\, (\Lambda - \frac{1}{16\pi G} R\,) , \tag{2}$$

and $\mathcal{M}$ is a compact closed manifold. The integration involves, in principle, a summation over all topologies and, for a given topology, the integration over all metrics that can be obtained one from another by a continuous deformation. Actually, for $4d$, the summation over topologies is ill defined, and should be restricted, say, to smooth manifolds. In $2d$, where the classification of topologies is simple, the number of manifolds grows like a factorial of the genus [5], so that the sum is not Borel summable. The situation in $4d$ is certainly not simpler. In practice, most studies at $d > 2$ use a discrete formulation of the theory and assume the topology is fixed.

The discretization of a theory invariant under general coordinate transformations is not a trivial matter. The first basic idea is due to Regge [6], who suggested to replace the manifold $\mathcal{M}$ by a collection of flat d-simplexes forming a simplicial complex. The curvature then resides on $(d-2)$-dimensional hinges. The second important idea is that of *dynamical triangulations* [7–10]: the simplexes are assumed to be equilateral and the sum over metrics is replaced by the sum over all possible manners to glue them together. For an ensemble of exactly solvable models in $2d$ one can show that the continuum and the discrete version belong to the same universality class, when the dynamical triangulation recipe is adopted.

Let us denote by N_k the number of k-simplexes in the complex. For $d =$ 3 and 4 only two of these quantities are independent. When the simplexes are equilateral, the RHS of (2) discretized à la Regge [11] becomes a linear combination of these two independent numbers, which is remarkably simple. For $4d$ one can write

$$S = \kappa_4 N_4 - \kappa_2 N_2 \,, \tag{3}$$

where $\kappa_2 \sim a^2/G$ and a denotes the lattice unit. The partition function (1) takes the form

$$Z(\kappa_2, \kappa_4) = \sum_{N_2, N_4} Z_{N_2,N_4} e^{-S} \,, \tag{4}$$

where

$$Z_{N_2,N_4} = \sum_{T(N_2,N_4)} W(T) \,. \tag{5}$$

The sum is over all $4d$ closed manifolds $T(N_2, N_4)$ of, say, spherical topology, with fixed N_2 and N_4. The symmetry factor W(T) equals the number of distinct labelings of the vertices of T divided by $N_0!$. The model is now defined precisely enough to be converted into a computer code.

The lattice can be further decorated with matter fields, for example with Ising spins. This does not present any conceptual difficulty. We limit ourselves here to pure gravity, since for $d > 2$ the study of models involving matter fields has not been pushed far enough to warrant discussion in a short review.

3. Numerical algorithms

Any two combinatorially equivalent simplicial complexes[1] can be connected by a series of moves introduced long ago by Alexander [12]. A smaller set of local moves has been proposed in the context of lattice gravity [13, 14]. All these moves can be introduced as follows: let a collection of d-simplexes in a d-dimensional manifold be a part of the boundary of a $(d+1)$-simplex. A move consists in replacing the collection in question by the rest of the boundary. Since $(d+1)$-simplex has $d+2$ d-dimensional faces, there are $d+1$ possible moves of this type. Each move has a reciprocal and, when d is even, there is one self-reciprocal move. It has been pointed out in [15] that for $d = 3$ all Alexander moves can be constructed using these simple ones. The formal proof valid for all $d < 5$ can be found in Ref. [16]. The ergodicity of the simple moves follows from the fact that all the Alexander moves are reducible to them[2].

A word of caution is in order at this point: although any two simplicial complexes can be deformed one into another by a finite number of simple local moves, the number of steps needed to connect any two lattice configurations might grow so fast with the volume that the ergodicity would not be insured in practice. The possibility of this unpleasant scenario has to be kept in mind.

The further implementation of these ideas in computer software has been greatly facilitated by the experience gained in developing algorithms appropriate for random surfaces [10, 17, 18]. In this respect the techniques worked out to simulate the so-called grand-canonical ensemble[3] of random surfaces [17, 18] are particularly instructive. Anyone wishing to participate in the numerical studies of quantum gravity is advised to start by getting conversant with them.

The efficiency of the algorithms is considerably improved when the ergodic local moves are supplemented by the global ones, where entire baby-universes are cut out at one place and glued elsewhere [19, 20] [4]. We shall come to baby-universes later on.

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[1] For $d < 7$ triangulated smooth manifolds of the same topology are combinatorially equivalent.

[2] The moves used earlier in the context of $2d$ gravity are identical or reducible to these ones. However, the situation in $2d$ is particularly simple and the ergodicity is proved by rather elementary methods.

[3] That is the ensemble where the number of nodes is fluctuating.

[4] These global moves are not ergodic by themselves.

4. Entropy of random manifolds

In order for the theory to make sense the entropy of manifolds should be an extensive quantity. In other words, the number $Z(N_d)$ of distinct d-dimensional simplicial complexes, made up of N_d d-simplexes and with fixed topology, should be bounded by $\exp(cN_d)$, with c being some finite positive number. It has been a surprise for the physicists who got interested in the problem to learn that their colleagues from the maths department have no idea how $Z(N_d)$ does behave when $d > 2$ and $N_d \to \infty$.

In $2d$ the exponential bound has been proved analytically [21]. A numerical evidence for such a bound in $3d$ has been given first in Ref. [22] for spherical topology, and confirmed by later studies. A similar result has been obtained for $d = 4$ in [23]. There has been a controversy concerning the validity of this result, but the present consensus is that it is correct (see the review in [24]). Of course, numerical evidence is not a proof. Hence, several people presented analytic arguments to the effect that the exponential bound does hold. It seems that these claims rest on too restrictive assumptions, but we do not feel expert enough in topology to develop this point.

5. Phase diagram

Let us take the existence of the exponential bound discussed in the preceding section for granted:

$$\log Z(\kappa_2, N_4) \sim \kappa_{4\mathrm{crit}}(\kappa_2) N_4 + \dots . \tag{6}$$

where

$$Z(\kappa_2, N_4) = \sum_{N_2} Z_{N_2,N_4} e^{\kappa_2 N_2} , \tag{7}$$

and the subleading terms have not been written explicitly for simplicity. It is clear from (4) that the theory does not exist for $\kappa_4 < \kappa_{4\mathrm{crit}}(\kappa_2)$. As κ_4 approaches the critical line $\kappa_4 = \kappa_{4\mathrm{crit}}(\kappa_2)$ from above the partition function $Z(\kappa_2, \kappa_4)$ develops a singularity.

Notice, that in pure quantum gravity it does not make much sense to attach physical significance to Λ and G separately. The content of the theory, as defined by (1) remains unchanged under the rescaling of the metric $g_{ab} \to s g_{ab}$, which corresponds to $\Lambda \to s^{d/2}\Lambda$ and $G \to s^{-d/2+1}G$. Only the invariant combination $\Lambda G^{d/(d-2)}$ is relevant (see the discussion in [25]). In other words one has to tune both κ_4 and κ_2 in order to define the continuum theory. One needs for that a critical point on the line $\kappa_4 = \kappa_{4\mathrm{crit}}(\kappa_2)$.

Such a point has first been discovered in Ref. [15], in the context of $3d$ gravity (which in this respect resembles the $4d$ one, except for the order of the transition, see later). It has been found that below that point the system is in a crumpled phase, where the average number of nodes per simplex tends to zero when the number of simplexed is sent to infinity. Above the critical point this ratio tends to a finite limit, so that at least a sensible thermodynamical limit can be defined. An analogous critical point has subsequently been found for $d = 4$ [26, 27]. Contrary to $3d$, where it is of first order [28], the transition in $4d$ appears to be continuous [26, 29, 20]. This is what one might hope, since in $4d$ there should be a place for the graviton, absent in $3d$. It is now customary to refer to the crumpled phase as to the *hot* one. The phase above the critical point is called *cold*. A careful analysis [20] of the phase structure in $4d$ further demonstrates that the *internal* fractal dimension d_H of the manifolds is close to 2 in the cold phase and presumably infinite in the hot one.

It is worth mentioning at this place that the lattice theory always has a well defined most probable configuration (vacuum state). It appears that this vacuum is not just the state with largest curvature, which on a dynamically triangulated manifold is necessarily finite. The vacuum seems to be nontrivial and is stabilized by the entropy od manifolds, which in this formulation of the discrete theory is defined unambiguously [15, 28].

6. Baby universes

Baby universes (BU) are sub-universes connected to the rest of the universe by a narrow neck. They are Euclidean analogs of black holes and early speculations concerning Euclidean quantum gravity have already introduced this concept [1]. The numerical simulations of random manifolds have revealed that the emergence of BU is an extremely common phenomenon, in all dimensions that have been considered. It is very unlikely that a random manifold remains more or less smooth (in the intuitive sense of the word). If one starts a simulation with a smooth manifold, soon there are BU growing out of it. Further, there are BU growing on BU and so on. The final structure is tree-like. It can be demonstrated analytically in $2d$ [30] that this tree is a fractal. In $4d$ the tree-like structure is especially manifest near and above the critical point. Actually, in the cold phase, the tree resembles a branched polymer [20]. The tree has the topology of a sphere because the algorithm keeps the topology fixed by construction. It is very likely that the typical configuration would remain a collection of sub-universes connected by wormholes if one succeeded to upgrade the algorithm so as to allow the topology to change. The possible relevance of such a geometry for the cosmological constant problem has been pointed out long ago by Hawking, Coleman and others [1].

The discovery of the tree-like geometry of typical random manifolds with fixed topology is a very important and a very intriguing finding. A quantized manifold does not resemble at all the familiar systems making small quantum fluctuations around a smooth classical configuration. It is perhaps not surprising that the construction of the quantum space-time starting with interacting elementary entities (see the Introduction) is not a simple matter.

The average number $n(N_B, N_4)$ of BU with a given volume N_B can be found using a combinatorial argument [30][5]. The result is particularly simple when one assumes that

$$Z(\kappa_2, N_4) \sim N_4^{\gamma-3} e^{\kappa_{4\text{crit}} N_4}\,, \tag{8}$$

which is true in $2d$ and is likely to hold in $4d$ in the vicinity of the critical point:

$$n(N_B, N_4) \sim N_4 \left[\left(1 - \frac{N_B}{N_4}\right) N_B\right]^{\gamma-2}, \quad N_B < N_4/2\,. \tag{9}$$

There exist general arguments [32][6] to the effect that generically $\gamma \leq 0$ or else the manifolds degenerate into branched polymers with $\gamma = \frac{1}{2}$. Thus the educated guess is that in the sensible sector of the theory the number of BU carrying a *finite* fraction of the total volume is $\sim N_4^{\gamma}$ and tends to a constant or vanishes when $N_4 \to \infty$, *i.e.* in the continuum limit.

7. Scaling and renormalization group

Recently, much activity has concentrated on the behavior of the discrete theory in the neighborhood of the critical point. We have no place here to give justice to all this effort and, in particular, to all the facets of the particularly thorough work by Ambjørn and Jurkiewicz [20] (we have already referred to it on several occasions).

The geometry of the ensemble of manifolds can be characterized by invariant correlations between local operators $O(x)$. The simplest correlator is the two-point function with $O(x) = 1$. On a lattice it takes the form

$$G(r, N_4, \kappa_2) = N_4^{-2} \langle \sum_{A,B} \delta(r - \mid x_A - x_B \mid) \rangle_{N_4}\,, \tag{10}$$

[5] The neck of a BU can be regarded as a puncture on each of the two parts of the manifold it connects.

[6] Strictly speaking, this paper deals with random surfaces. However the geometrical arguments employed are certainly of more general validity.

where $\mid x_A - x_B \mid$ is the geodesic distance between simplexes A and B. The large distance behavior at fixed κ_2 is [33, 20]

$$G(r, N_4, \kappa_2) \sim e^{-c(r/N_4^{1/d_H})^{\frac{d_H}{d_H-1}}}, \tag{11}$$

where c is some constant and d_H is the internal Hausdorff dimension. Both can depend on κ_2. For large enough N_4

$$\langle r \rangle_{N_4} \sim N_4^{1/d_H} . \tag{12}$$

The reciprocal relation

$$\langle N_4 \rangle_r \sim r^{d_H} \tag{13}$$

also holds [7]. The finiteness of d_H has been assumed. The scaling manifest in (11) has been earlier observed empirically [34] in the full range of r: $G(r, N_4)$ is mostly a function of $r/\langle r \rangle$. It has been further claimed in [34] that this function has an approximately constant shape along trajectories in (N_4, κ_2) plane.

It is tempting to attack the problem of scaling using the techniques of the real space renormalization group. The very definition of a blocking procedure is non-trivial in this context: ideally, the blocking should be a self-similarity transformation, a constraint difficult to satisfy when one deals with a random lattice. It has been proposed in [35] to define the renormalization group (RG) transformation as the process of cutting the last generation of baby universes, that is those BU which have no further BU growing on them [8]. Under this operation the tree gets smaller, in lattice units, and less branched, which is interpreted as reflecting the loss of the resolving power.

Let us keep fixed the *physical* volume $V = N_4 a^4$ of the manifold. Consider the moments $\langle r^k \rangle$ of the correlator (10). They transform under RG: $\langle r \rangle \to \langle r \rangle - \delta r$, *etc.* Assuming that κ_2 is the only coupling relevant for the in-large geometry one has along the RG flow

$$\delta r = r_N \delta \ln N_4 + r_\kappa \delta \kappa_2 , \tag{14}$$

where r_N and r_κ are the partial derivatives of $\langle r \rangle$ with respect to $\ln N_4$ and κ_2, respectively. Furthermore,

$$\delta \ln \frac{1}{a} = \frac{1}{4} \delta \ln N_4 . \tag{15}$$

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[7] The LHS is the average number of simplexes in manifolds with two boundaries separated by invariant geodesic distance r.

[8] In practice, one cuts only the minimum-neck BU (minBU), which are easy to identify.

From (14) and (15) and using computer data one can calculate the β-function [36, 38]

$$\beta(\kappa_2) = \frac{d\kappa_2}{d\ln\frac{1}{a}}\,. \tag{16}$$

It is found that the theory possesses an ultraviolet stable fixed point $\kappa_2 = \kappa_{2\text{crit}}$. The value of $\kappa_{2\text{crit}}$ obtained from RG is close to that found by other methods. Thus, in the neighborhood of the critical point

$$\beta(\kappa_2) = \beta_0(\kappa_{2\text{crit}} - \kappa_2), \quad \beta_0 > 0\,. \tag{17}$$

Integrating (16) one gets

$$a = a_0 \mid \kappa_{2\text{crit}} - \kappa_2 \mid^{\frac{1}{\beta_0}}, \tag{18}$$

where a_0 is an integration constant, which should be given a value, in physical units, in order to define the theory. The RG flow lines are

$$N_4^{\beta_0/4} \mid \kappa_{2\text{crit}} - \kappa_2 \mid \equiv t = [V/a_0^4]^{\beta_0/4}\,. \tag{19}$$

The continuum limit is

$$\begin{aligned} N_4 &\to \infty\,, \\ \kappa_2 &\to \kappa_{2\text{crit}}\,, \\ t &= \text{const}\,. \end{aligned} \tag{20}$$

These results are closely analogous to those obtained in $2+\epsilon$ dimensions in the continuum framework (see [25] and references therein). It follows from the above discussion that one should be careful in interpreting results obtained at fixed κ_2: the line $\kappa_2 = \text{const}$ intersects an infinity of RG trajectories, each representing a distinct version of the theory.

Other interesting correlators are those obtained setting $O(x) = R(x)$, where $R(x)$ is the scalar curvature. Their integrals are unambiguously defined:

$$m_k(N_4) = \frac{\partial^k \ln Z(\kappa_2, N_4)}{\partial \kappa_2^k}\,, \tag{21}$$

and are the cumulants of the node distribution[9]. Computer data [26, 29, 37, 38], are compatible with simple finite-size scaling

$$m_2(N_4) \sim N_4^b f[(\kappa_2 - \kappa_{2\text{crit}})N_4^c]\,, \tag{22}$$

$$m_3(N_4) \sim N_4^{b+c} f'[(\kappa_2 - \kappa_{2\text{crit}})N_4^c]\,. \tag{23}$$

[9] One has $N_2 = 2(N_4 + N_0 - 2)$.

This suggests the existence of a finite mass gap scaling to zero when $\kappa_2 \to \kappa_{2\mathrm{crit}}$. As long as one is on the lattice the mass gap *is expected* to be finite, since the continuum gauge symmetry responsible for the existence of the graviton is absent. According to the preliminary data [38] $\beta_0/4 > c$. This seems to indicate that the mass gap vanishes in the continuum limit (20) as it should. Much more work will be necessary to check the spin of the corresponding particle.

8. Conclusion

Let us conclude with a few words about the open questions. There are, of course, the fundamental questions relative to the summation over topologies, the continuation to real time *etc.*, which require very bright new ideas. There are also more accessible problems, within the lattice formalism. The central one is the nature of the continuum limit and, in particular, the search of an evidence for the graviton. The next one is a careful study of the interaction of matter fields with geometry. Is all this sensible and does it for sure correspond to a genuine gravity theory, let it be Euclidean ? What is certain is that the progress in this field is rapid and that people involved have a lot of fun!

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