A METAPHYSICAL REMARK ON VARIATIONAL PRINCIPLES

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In theoretical physics, one often considers symmetries which change a Lagrangian by a total divergence. Such transformations preserve the equations of motion and lead to conservation laws. It is argued here, on the basis of a few examples, that the appearance of such a divergence is an indication that one is dealing with some approximation or a limiting case of a 'better' theory, in which the corresponding, possibly modified, symmetries fully preserve the action integral. These suggestions are 'metaphysical' in the sense that they cannot be tested by physical experiments.

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Lex perpetua naturae est ut agat minimo labore, mediis et modis simplicissimis, facilissimis, certis et tutis.

Giovanni Borelli (1608 – 1679)

1. Introduction

The Noether theorems [1] on the relations between conservation laws and symmetries of variational principles constitute a fundamental, beautiful and often used part of mathematical physics. They are presented in many texts [2–4] and papers [5–7]. For the purposes of this introduction it is enough to recall that the theorems refer to *equations of motion* which can be derived from a *principle of stationary action*,

$$\delta W(f,U) = 0, \quad \text{where} \quad W(f,U) = \int_U L(x, f(x), f'(x))\mathrm{d}x^1 \cdots \mathrm{d}x^n = 0, \tag{1}$$

and L is the *Lagrangian* depending on the coordinates $x = (x^\mu)$, $\mu = 1, \ldots, n$, explicitly and through the functions f and their derivatives, the functions and the coordinates being defined on an n-dimensional (relatively compact) domain of integration $U \subset E = \boldsymbol{R}^n$. I assume here, for the sake of simplicity of exposition, that L does not contain derivatives of order higher than the first, even though the classical Hilbert Lagrangian of Einstein's theory depends on the second derivatives of the metric tensor; see Sec. 3.4. Under symmetries of W — to be defined precisely below — the Lagrangian behaves like a scalar density; it may also change by the addition of a total divergence. Symmetries transform solutions of the Euler–Lagrange (EL) equations into solutions of the same equations. Continuous (Lie) groups of such symmetries lead to conservation laws (first Noether's theorem, Sec. 2.2). Groups of symmetries, depending on arbitrary functions of the coordinates lead to (differential) identities linear in the EL expressions (second theorem, Sec. 2.3). The subject of conservation laws and identities has a rich literature; in this paper some of the basic results are recalled in order to make the paper self-contained. From the point of view of direct applications of continuous groups of symmetries, it is irrelevant whether a total divergence appears or not in the transformation law of L. For example, in classical, Newtonian mechanics, the Lagrangian of a particle moving in a spherically symmetric field of force is strictly invariant under rotations, but the kinetic energy of a particle changes under Galilean transformations. The Hilbert Lagrangian $\sqrt{-g}\,R$ is a proper scalar density, but the often used in general relativity theory (GRT) Lagrangian quadratic in Christoffel symbols, changes by a divergence, when acted upon by a coordinate transformation. The purpose of this paper is to present a few similar examples in order to argue, on their basis, that the appearance of such divergences is an indication that one has to do with a theory with symmetries that can be 'improved' by more or less radical changes in its structure.

2. The Noether theorems revisited

2.1. Invariant transformations

To recall the derivation of conservation laws of classical physics, it is convenient to consider the Lagrangian appearing in (1) as depending on x, the *values* of the functions f describing *histories*, and on their derivatives.

Let $F = \boldsymbol{R}^N$, F' and F'' be the spaces of values of the functions, of their first and second derivatives, respectively. The coordinates in F, F' and F'' are, respectively, y^i, y^i_μ and $y^i_{\mu\nu}$, where $i = 1, \ldots, N$, and $\mu, \nu = 1, \ldots, n$. One can write $f = (f^i)$, $f' = (f^i_{,\mu})$, where $f^i_{,\mu} = \partial f^i / \partial x^\mu$, etc. Putting $M = E \times F$, $\bar{M} = E \times F \times F'$ and $\bar{\bar{M}} = E \times F \times F' \times F''$ one sees that the Lagrangian is a function on $\bar{M}$. With every history $f : E \to F$ one associates its (graph) extensions $\varphi : E \to M$, $\bar{\varphi} : E \to \bar{M}$ and $\bar{\bar{\varphi}} : E \to \bar{\bar{M}}$, given by $\varphi(x) = (x, f(x))$, $\bar{\varphi}(x) = (x, f(x), f'(x))$ and $\bar{\bar{\varphi}}(x) = (x, f(x), f'(x), f''(x))$, respectively [1]. One has $L(x, f(x), f'(x)) = L \circ \bar{\varphi}(x)$. Denoting by F^* the dual of the vector space F and introducing the differential operator (a collection of n vector fields on $\bar{\bar{M}}$)

$$D_\mu = \frac{\partial}{\partial x^\mu} + y^i_\mu \frac{\partial}{\partial y^i} + y^i_{\mu\nu} \frac{\partial}{\partial y^i_\nu} \tag{2}$$

one can express the *Euler–Lagrange map*

$$[L] : \bar{\bar{M}} \to F^* \quad \text{as} \quad [L]_i = \frac{\partial L}{\partial y^i} - D_\mu \frac{\partial L}{\partial y^i_\mu}. \tag{3}$$

The EL equations for the history f, resulting from (1), are

$$[L] \circ \bar{\bar{\varphi}} = 0. \tag{4}$$

In many cases, transformations of histories occurring in physics are given by a diffeomorphism

$$\omega : M \to M \quad \text{of the form} \quad \omega(x, y) = (\xi(x), \eta(x, y)). \tag{5}$$

The diffeomorphism ξ is a transformation of coordinates in E. For every $x \in E$, the map $y \mapsto \eta(x, y)$ is a diffeomorphism of F onto itself. For example, if f is a tensor field on E of type ρ, *i.e.* transforming with the representation $\rho = (\rho^i{}_j) : \mathrm{GL}(n, \boldsymbol{R}) \to \mathrm{GL}(N, \boldsymbol{R})$, then

$$\eta^i(x, y) = \rho^i{}_j(\xi'(x)) y^j, \quad \text{where} \quad \xi' : E \to \mathrm{GL}(n, \boldsymbol{R}) \tag{6}$$

is the Jacobian map, $\xi'^\mu_\nu(x) = \xi^\mu_{,\nu}(x)$. Its determinant J is assumed from now on to be positive. Another example is provided by *gauge transformations* acting on one-forms with values in the Lie algebra of a Lie group G.

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[1] The proper mathematical setting involves here differentiable fibre bundles and their first and second jet extensions [6, 7].

Given a function $S : E \to G$, one defines the corresponding gauge transformation by putting $\xi = \mathrm{id}_E$ and $\eta(x,y) = S^{-1}(x)yS(x) + S^{-1}(x)(\mathrm{d}S)(x)$.

If φ is the graph map of a history f, then the composition $\omega \circ \varphi \circ \xi^{-1} = \varphi_\omega$ is the graph map of the history f_ω,

$$f_\omega(x) = \eta(\xi^{-1}(x)), f(\xi^{-1}(x)).$$

One extends ω to diffeomorphisms $\bar{\omega} : \bar{M} \to \bar{M}$ and $\bar{\bar{\omega}} : \bar{\bar{M}} \to \bar{\bar{M}}$ by requiring that, for every graph map φ, one should have $\bar{\omega} \circ \bar{\varphi} \circ \xi^{-1} = \overline{\omega \circ \varphi \circ \xi^{-1}}$ and similarly for $\bar{\bar{\omega}}$.

One says that ω is an *invariant transformation* for the action W given by (1) if, for every history f and (relatively compact) domain of integration U, one has $W(f_\omega, \xi(U)) = W(f, U)$. This is equivalent to

$$JL \circ \bar{\omega} = L. \tag{7}$$

Without assuming that ω is an invariant transformation, one obtains, by a direct computation,

$$[JL \circ \bar{\omega}]_i = J \frac{\partial \eta^j}{\partial y^i} [L]_j \circ \bar{\bar{\omega}}. \tag{8}$$

If ω is an invariant transformation, then (7) and (8) give

$$[L]_i = J \frac{\partial \eta^j}{\partial y^i} [L]_j \circ \bar{\bar{\omega}} \tag{9}$$

and eq. (4) implies that the transformed history f_ω is also a solution of the EL equations, $[L] \circ \bar{\bar{\varphi}}_\omega = 0$.

One says that ω is a *generalized invariant transformation* for W if (9) holds. By comparing the identity (8) with the condition (9), one obtains that ω is a generalized invariant transformation if, and only if,

$$[L - JL \circ \bar{\omega}] = 0.$$

According to a classical result (see pp. 193–196 in [2]), the last condition is equivalent to the existence of a map $\kappa : \bar{M} \to E$ such that

$$L - JL \circ \bar{\omega} = D_\mu \kappa^\mu. \tag{10}$$

This generalization of (7), and its application to conservation laws, is due to Bessel-Hagen [8].

2.2. Conservation laws

Consider now a *one-parameter group* (ω_t) of transformations of M of the form (5) so that $\omega_t(x,y) = (\xi_t(x), \eta_t(x,y))$, $\omega_t \circ \omega_s = \omega_{t+s}$ for $t, s \in \boldsymbol{R}$ and $\omega_0 = \mathrm{id}_M$. Let

$$Z(x,y) = (X(x), Y(x,y))$$

be the vector field on M induced by the group (ω_t). Putting $J_t = \det(\xi^\mu_{t,\nu})$, one obtains

$$(\mathrm{d}J_t/\mathrm{d}t)_{t=0} = \operatorname{div} X. \tag{11}$$

The extension $(\bar\omega_t)$ of the group to $\bar M$ induces a vector field $\bar Z$ on $\bar M$. Let (ω_t) be a one-parameter group of generalized invariant transformations of W so that there is a curve $t \mapsto \kappa_t$ in the space of maps from $\bar M$ to E such that

$$L - J_t L \circ \bar\omega_t = D_\mu \kappa_t^\mu$$

for every $t \in \boldsymbol{R}$. Evaluating the derivative at $t = 0$ of both sides of the last equation, one obtains

$$\bar Z(L) + L \operatorname{div} X + D_\mu K^\mu = 0, \tag{12}$$

where $K^\mu = (\mathrm{d}\kappa_t^\mu/\mathrm{d}t)_{t=0}$. Using the explicit form of the differential operator

$$\bar Z(x,y,y') = X^\mu(x)\frac{\partial}{\partial x^\mu} + Y^i(x,y)\frac{\partial}{\partial y^i} + Y^i_\mu(x,y,y')\frac{\partial}{\partial y^i_\mu},$$

where

$$Y^i_\mu(x,y,y') = (D_\mu Y^i)(x,y,y') - y^i_\nu X^\nu_{,\mu}(x)$$

one can transform Eq. (12) into the *Noether–Bessel-Hagen equation* [1, 6, 8] on $\bar{\bar M}$:

$$(Y^i - y^i_\nu X^\nu)[L]_i + D_\mu\Big(LX^\mu + (Y^i - y^i_\nu X^\nu)\frac{\partial L}{\partial y^i_\mu} + K^\mu\Big) = 0. \tag{13}$$

For every solution of (4) there holds the *conservation law* in differential form

$$\frac{\partial}{\partial x^\mu}(t^\mu \circ \bar\varphi) = 0, \quad \text{where} \quad t^\mu = LX^\mu + (Y^i - y^i_\nu X^\nu)\frac{\partial L}{\partial y^i_\mu} + K^\mu.$$

If Z generates an invariant transformation, then $K^\mu = 0$, but the term LX^μ is always present: it reflects the fact that for the action to be invariant under coordinate changes, the Lagrangian should be a *scalar density*.

2.3. Identities

This section contains, as an example, the derivation of the identity, satisfied by the EL expression, as a result of 'general invariance'. Let f be a tensor field of type ρ. One says that W is *generally invariant* if (10) holds for every diffeomorphism ξ with η given by (6). Consider a one-parameter group of diffeomorphisms (ξ_t) generated by X and denote by ξ'_t the Jacobian map associated with ξ_t. One can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho^i_j(\xi'_t)_{|t=0} = \rho^{i\mu}_{j\nu}\frac{\partial X^\nu}{\partial x^\mu} \quad \text{so that} \quad Y^i(x,y) = \rho^{i\mu}_{j\nu}\frac{\partial X^\nu}{\partial x^\mu}(x)y^j \tag{14}$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}t}f^i_{\omega_t}{}_{|t=0} = X^\mu\frac{\partial f^i}{\partial x^\mu} - \rho^{i\mu}_{j\nu}f^j\frac{\partial X^\nu}{\partial x^\mu}$$

is the Lie derivative of f in the direction of X. Assuming that K is a linear and homogeneous function of the vector field X and its first derivatives, and substituting Y given in (14) into Eq. (13), one can write that equation in the form

$$a_\mu X^\mu + D_\mu(b^\mu_\nu X^\nu + c^{\mu\nu}_\rho D_\nu X^\rho) = 0, \tag{15}$$

where

$$a_\mu = D_\nu(\rho^{i\nu}_{j\mu}y^j[L]_i) + y^i_\mu[L]_i \tag{16}$$

and b^μ_ν and $c^{\mu\nu}_\rho$ are functions on $\bar{\bar{M}}$. General invariance of W implies that (15) holds for every X. At every single point of E, the values of the components of X, and of its first and second derivatives, can be chosen at will. Taking this into account, one obtains from Eq. (15): $c^{\mu\nu}_\rho + c^{\nu\mu}_\rho = 0$, $b^\mu_\nu + D_\rho c^{\rho\mu}_\nu = 0$ and $a_\mu + D_\nu b^\nu_\mu = 0$. Therefore, $a_\mu = -D_\nu b^\nu_\mu = D_\nu D_\rho c^{\rho\nu}_\mu = 0$ because $[D_\mu, D_\nu] = 0$. The vanishing of the right side of (16) is a special case of a *Noether identity* [1, 9]. If one considers a theory based on such a Lagrangian and assumes that the field f has a *source* described by a tensor field T of type ρ so that the field equations are $[L]\circ\bar{\bar{\varphi}} = T$, then the identity (16) implies a generalized conservation law to be satisfied by T for the field equations to have a solution. In particular, if the Lie derivative of f in the direction of X vanishes, then (16) gives an 'ordinary' conservation law,

$$\frac{\partial}{\partial x^\mu}(\rho^{i\mu}_{j\nu}f^jT_iX^\nu) = 0.$$

3. The examples

3.1. Mechanics of point particles

The Lagrangian description of classical point particles in the Newtonian theory, their symmetries and conservation laws, are so well known that no equations need be written here. For an isolated system, Galilean transformations induce generalized invariant transformations and lead to the *centre-of-mass theorem* [5]. In the theory of special relativity, the classical action for a free particle is proportional to the integral of the proper time along the world line of the particle. This quantity is strictly invariant with respect to the full Poincaré group.

3.2. Classical and quantum theories of charged particles

In this and the following section, it is assumed that E is the Minkowski spacetime of special relativity theory with its flat metric tensor $\eta_{\mu\nu}$, ($\mu, \nu = 1, \ldots, 4$) referred to Cartesian coordinates (x^μ) and $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. The metric tensor, and its inverse $\eta^{\mu\nu}$, serve to lower and raise indices.

To a point particle of mass m and charge e, with world-line $x^\mu(s)$, moving in an electromagnetic field of potential A_μ, there corresponds, in the theory of relativity, the action $\int L\mathrm{d}x^4 = \int(-m\mathrm{d}s + eA_\mu \mathrm{d}x^\mu)$. The gauge transformation $A_\mu \mapsto A_\mu + \chi_{,\mu}$ is a generalized symmetry transformation: it induces the change $L\mathrm{d}x^4 \mapsto L\mathrm{d}x^4 + e\mathrm{d}\chi$. A similar remark holds for a classical, charged fluid described phenomenologically by a conserved current j^μ: the interaction term $j^\mu A_\mu$ also changes by a divergence. Field (first quantized) theories of charged particles are better in this respect. For example, let ϕ be a complex scalar field describing particles of charge e and mass m. The corresponding Lagrangian, taking into account a minimal coupling of the particles with electromagnetism,

$$\eta^{\mu\nu}(\bar{\phi}_{,\mu} + \mathrm{i}eA_\mu\bar{\phi})(\phi_{,\nu} - \mathrm{i}eA_\nu\phi) + m^2\bar{\phi}\phi,$$

is strictly invariant under the simultaneous changes $A_\mu \mapsto A_\mu + \chi_{,\mu}$ and $\phi \mapsto \phi\exp(\mathrm{i}e\chi)$.

3.3. Linearized gravitation

The linearized theory of gravitation is often considered, either as a simple, *toy model*, or as a first step in approximate calculations in GRT. It assumes the Minkowski space with its flat metric and a symmetric tensor field $h_{\mu\nu}$ to describe gravitation. The 'gravitational potentials' h satisfy differential equations obtained by linearization of Einstein's equations. Using square brackets to denote antisymmetrization over the enclosed indices, introducing the linearized curvature tensor,

$$S_{\mu\nu\rho\sigma} = h_{\rho[\mu,\nu]\sigma} + h_{\sigma[\nu,\mu]\rho}, \tag{17}$$

and its contractions $S_{\mu\nu} = \eta^{\rho\sigma} S_{\mu\rho\sigma\nu}$ and $S = \eta^{\mu\nu} S_{\mu\nu}$, one can write the field equations as

$$S^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu} S = T^{\mu\nu}, \tag{18}$$

where $T^{\mu\nu}$ describes the sources of the field. The equations (18) can be derived from the principle of stationary action with a Lagrangian of the form $L(h, h'') - h_{\mu\nu}T^{\mu\nu}$, where

$$L(h, h'') = \tfrac{1}{2}h_{\mu\nu}(S^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu} S).$$

The tensor (17) is left unchanged by the 'linearized coordinate transformation'

$$h_{\mu\nu} \mapsto h_{\mu\nu} + a_{\mu,\nu} + a_{\nu,\mu}, \tag{19}$$

where the functions a_μ are arbitrary. The replacement (19) is a generalized invariant transformation for the Lagrangian given above; the invariance property leads to the (linearized, contracted Bianchi) identity $(S^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} S)_{,\nu} = 0$ and implies a conservation equation for the sources. The second order Lagrangian L can be replaced by the equivalent Lagrangian $L - (h_{\mu\nu,\rho}\partial L/\partial h_{\mu\nu,\rho\sigma})_{,\sigma}$ which is quadratic in the first derivatives of h. None of the Lagrangians that lead to the equations (18) are strictly invariant with respect to (19): they all change by a divergence.

3.4. General relativity

The principle of stationary action giving the correct Einstein equations,

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu} R = T^{\mu\nu},$$

was found in 1915 by Hilbert. The left side of the equations is obtained by varying $\sqrt{-g}\,R$ with respect to the metric tensor $g_{\mu\nu}$. This Lagrangian is a proper scalar density; it depends (linearly) on the second derivatives of the metric tensor. Similarly as in the linearized theory, it can be replaced by the 'Einstein Lagrangian' $\sqrt{-g}\,R - (g_{\mu\nu,\rho}\partial\sqrt{-g}\,R/\partial g_{\mu\nu,\rho\sigma})_{,\sigma}$ which is quadratic in the first derivatives of the metric tensor. Under coordinate transformations, the Einstein Lagrangian changes by a divergence. It is used to derive a non-tensorial conservation law of gravitational energy and momentum, applicable to isolated systems (see [3, 10] and the references given there).

4. Concluding remarks

Variational and related principles fascinated scientists already in the XVII and XVIII centuries. The early ideas of Fermat, Borelli and Maupertuis had a teleological — and, in some cases, even a theological — overtone. The first proper variational problem — that of the brachistochrone — was solved by J. Bernoulli in 1696. At about that time, Leibniz used the expression *actio formalis* which probably led to the present use of the word *action* in physics. The principles of least — or rather stationary — action, and the related canonical formalism were put forward and developed in the XVIII and XIX centuries by Euler, Lagrange, Poisson, Hamilton, Jacobi, Poincaré and other scientists; see [11] for a historical survey of the subject. This formalism was applied, for example, to perturbation computations in celestial mechanics; later it was used to derive equations and conservation laws in field theories. Hilbert put down a variational principle as a basis of his *Foundations of Physics* [12]. A certain mystery surrounded the variational principles: *why* should the fundamental laws of nature be so derivable; *what* is the physical meaning of the action W which, unlike energy, momentum or electric charge, depends on the history of a system, rather than on its state, and does not satisfy a conservation law? The situation changed for the better with the advent of quantum physics: it began with the introduction by Planck of the *universal unit of action* $\hbar$. Bohr and Sommerfeld have shown how one can obtain the quantized (approximate) values of the *energy* of simple, periodic systems, by assuming that the *reduced action* $\int p\,dq$, corresponding to one period, is an integer multiple of $2\pi\hbar$. Further clarification was brought about by the deep ideas of Dirac (see §32 in [13]) and Feynman [14]. According to Feynman, the basic quantum amplitude determining the probability for a system to go from one state to another is proportional to the sum $\sum \exp(iW/\hbar)$ taken over *all classical histories* satisfying suitable boundary conditions, depending on the initial and final states of the system. For a generic 'classical' system, the action W is very large as compared to $\hbar$; in this case, according to the *saddle point method*, the main contribution to the sum comes from histories rendering the action stationary, *i.e.* from histories satisfying the classical equations of motion. In this manner, quantum theory gives the action a central place in physics and 'explains' why classical histories follow equations resulting from $\delta W = 0$. In a sense, the action integral, as it appeared in early physics, had been a forerunner of the quantum ideas that came much later.

Since we now know that the *value* of the action integral — and not only its *variation* — is physically relevant, one can argue that there is a preference for theories with symmetries described by proper, as opposed to generalized, invariant transformations. The examples given above support this belief. They also show that there is no simple rule how to construct the improved theory. Going from Galilean to relativistic mechanics required changing the group and the Lagrangian. To find an action, for charged particles, properly invariant under gauge transformations, one has to replace a phenomenological description of the charges by a field-theoretic one: this requires the introduction of new degrees of freedom, such as a scalar or a Dirac field. The theory of gravitation provides the least trivial example: to go from the theory of spin 2 mass zero particles to GRT, one has to introduce a new geometry and very essential non-linearities. *Supersymmetries* [15] provide another example of transformations preserving Lagrangians up to a divergence. In my opinion, supersymmetric theories such as supergravity would gain much if they were reformulated so as to make the supersymmetries into proper invariant transformations.

REFERENCES

[1] E. Noether, *Nachr. Ges. Göttingen* (math.-phys. Kl.), 235 (1918).
[2] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, vol. I, transl. from German, Interscience, New York 1953.
[3] A. Trautman, in: *Gravitation: An Introduction to Current Research*, ed. L. Witten, Wiley, New York 1962, pp. 169–198.
[4] W. Thirring, *Classical Field Theory*, Springer-Verlag, New York 1978.
[5] E.L. Hill, *Rev. Mod. Phys.* **23**, 253 (1951).
[6] A. Trautman, *Commun Math. Phys.* **6**, 248 (1967).
[7] A. Trautman, in: *General Relativity* (Papers in honour of J.L. Synge), ed. L. O'Raifeartaigh, Clarendon Press, Oxford 1972, pp. 85–99.
[8] E. Bessel-Hagen, *Math. Ann.* **84**, 258 (1921).
[9] P.G. Bergmann, *Phys. Rev.* **75**, 680 (1949).
[10] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Freeman, San Francisco 1970.
[11] L.S. Polak, *Variatsionnye Printsipy Mekhaniki*, Gosud. Izdat. Fiziko-Matem. Literatury, Moskva 1960 (in Russian).
[12] D. Hilbert, *Math. Ann.* **92**, 1 (1924).
[13] P.A.M. Dirac, *Principles of Quantum Mechanics*, 4th ed., Clarendon Press, Oxford 1958.
[14] R.P. Feynman, *Rev. Mod. Phys.* **20**, 367 (1948).
[15] P.G.O. Freund, *Introduction to Supersymmetry*, Cambridge U. P., Cambridge 1986.