FUNDAMENTAL MASSES FROM QUANTUM SYMMETRIES*

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Firstly we show how fundamental masses (Planck mass, string mass) appear as a feature of quantum gravity as well as in fundamental string theory. Further we use the classical r-matrix approach for the description of the lowest order quantum deformations. We provide relevant examples of $D = 4$ Poincaré and $D = 4$ conformal bialgebras which introduce fundamental masses as deformation parameters.

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1. Introduction

Recently the quantum deformations [1–3] were applied also to the description of $D = 4$ space time symmetries. After proposing some particular deformations (see *e.g.* [4–14]) the classification schemes of quantum deformations of Poincaré groups [15] as well as Poincaré algebras [16, 17] were given. These mathematical classification schemes provide the deformation schemes by using the language of classical r-matrices [16] as well as Hopf algebras [15, 17]. The physical applications however should select the deformations which are more plausible from the physical point of view. We shall consider here the deformations of the following space-time Lie algebras:

a) Quantum algebras, obtained by the deformation of $D = 4$ Poincaré algebra $\mathcal{P}_{3;1}$ with generators $(P_\mu, M_{\mu\nu})$:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\tau}] &= \eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\tau} M_{\mu\rho} + \eta_{\mu\rho} M_{\nu\tau} - \eta_{\nu\rho} M_{\mu\tau}\,, \\
[M_{\mu\nu}, P_\rho] &= \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu\,, \\
[P_\mu, P_\nu] &= 0\,.
\end{aligned} \tag{1.1}$$

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b) Quantum $D = 4$ conformal algebras, obtained by the deformation of the 15-generator algebra (P_μ, $M_{\mu\nu}$, D, K_μ):

$$\begin{aligned}
&[M_{\mu\nu}, D] = 0\,, \qquad [P_\mu, D] = P_\mu\,,\\
&[M_{\mu\nu}, K_\rho] = g_{\mu\rho}K_\nu - g_{\nu\rho}K_\mu\,,\\
&[P_\mu, K_\rho] = g_{\mu\rho}D - M_{\mu\rho}\,,\\
&[K_\mu, D] = -K_\mu\,, \qquad [K_\mu, K_\nu] = 0\,,
\end{aligned} \tag{1.2}$$

supplemented by the relations (1.1). We denote the algebra (1.1)–(1.2) by $C_{3;1}$.

The generators of the Poincaré as well as the conformal algebra undergo the following transformations after the change of the length scale by a numerical factor λ

$$P'_\mu = \lambda^{-1}P_\mu\,, \qquad K'_\mu = \lambda K_\mu\,, \tag{1.3}$$

The remaining generators $M_{\mu\nu}$ and D are dimensionless ($M'_{\mu\mu} = M_{\mu\nu}$, $D' = D$). If one introduces the quantum deformations $\mathcal{U}_q(\mathcal{P}_{3;1})$, $\mathcal{U}_q(C_{3;1})$ of the classical enveloping algebras $\mathcal{U}(\mathcal{P}_{3;1}), \mathcal{U}(C_{3;1})$ one can distinguish the following three types of deformations ($\hat{g} = \mathcal{P}_{3;1}$ or $C_{3;1}$):

a) With dimensionless deformation parameter q.
In such a case there exists the isomorphism of quantum algebra

$$\mathcal{U}_q(\hat{g}) = \mathcal{U}_q(\hat{g}')\,. \tag{1.4}$$

b) With dimensionfull deformation parameter which we denote by κ, transforming under rescaling (1.4) as some fundamental mass parameter

$$\kappa' = \lambda^{-1}\kappa\,. \tag{1.5}$$

We obtain the following isomorphism

$$U_\kappa(\hat{g}) = \mathcal{U}_{\kappa'}(\hat{g}')\,. \tag{1.6}$$

c) Quantum deformations of space-time symmetries with deformation parameter not having definite scaling properties.

In this paper we would like firstly to present the arguments that at subatomic distances, comparable to Planck length, there is a place for the appearance of a new geometry describing fundamental interactions with a third fundamental constant, describing the fundamental length or mass scale (see Sect. 2). It follows therefore that the most plausible from the physical point of view are the deformations satisfying the condition (1.6). In Sect. 3 we shall consider the quantum deformations of $D = 4$ Poincaré algebras with dimensionfull mass-like deformation parameter using the language of classical r-matrices [16]. In particular we will argue that there are **three** distinguished quantum deformations of $D = 4$ Poincaré algebra,which we denote by $\mathcal{U}_\kappa^{(+)}(\mathcal{P}_{3;1})$, $\mathcal{U}_\kappa^{(-)}(\mathcal{P}_{3;1})$, $\mathcal{U}_\kappa^{(0)}(\mathcal{P}_{3;1})$, respectively with three three-dimensional $O(3)$, $O(2,1)$ and $e(2)$ classical subalgebras. In Sect. 4 we describe quantum deformation of $D = 4$ conformal algebra, with deformation parameter introducing fundamental mass (see also [18]). In Sect. 5 there are presented some conclusions.

2. The existence of fundamental length in quantum theories of gravity and strings

It is well-known that two fundamental constants — light velocity c and Planck constant $\hbar$ — are introduced respectively by relativistic kinematics (Einstein's special relativity) and quantum mechanics. In quantum mechanics the noncommutativity of the position and momentum observables

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \,, \tag{2.1}$$

implies the Heisenberg uncertainty relation

$$\Delta_\phi \hat{x}_i \Delta_\phi \hat{p}_i \stackrel{\text{df}}{=} \Delta x \, \Delta p \geq \frac{\hbar}{2} \,, \tag{2.2}$$

where ($x_i^\phi = \langle\phi|\hat{x}_i|\phi\rangle$ is a mean position)

$$\Delta_\phi \hat{x}_i = \left(\left\langle\phi|(\hat{x}_i - x_i^\phi)^2|\phi\right\rangle\right)^{1/2} . \tag{2.3}$$

In standard quantum mechanics the commutativity of the position operators $\hat{x}_i$ implies the possibility to measure the position of quantum particle with arbitrary accuracy. Due to this property the Schrődinger wave function $\psi(\vec{x}, t)$ is a classical field, with the arguments described by commuting space-time coordinates.

Recently there has been a considerable progress in the description of noncommutative or "quantum" geometry, which deals with algebra of functions on a "noncommutative manifold". The simplest example is provided by quantum phase space (2.1) and the algebra of functions $f(\hat{x}, \hat{p})$. Here we shall discuss further introduction of noncommutative structure — the case when $[\hat{x}_i, \hat{x}_j] \neq 0$. Physically, nonvanishing commutation relations of space-time coordinates could be the effects caused by quantum gravity (see *e.g.* [20–23]) or quantum string theory (see *e.g.* [24–27]). Below we shall outline some of the arguments.

2.1. Elementary Planck length and quantum gravity

It is known that quantum mechanics (Heisenberg uncertainty relation (2.2)) and relativistic kinematics put together allows to consider the concept of particle only in the space intervals larger that the Compton wave length (for simplicity we drop the three-space vector indices):

$$\Delta x > \frac{\hbar}{m_0 c} . \tag{2.4}$$

Indeed, because for relativistic particles energy $E = c(p^2 + m_0^2 c^2)^{1/2}$ we have $\Delta E = c\Delta p\, p(p^2 + m_0^2 c^2)^{-1/2}$ and for $p \gg m_0^2$ one can write $\Delta x \Delta E \sim c\Delta x \Delta p \geq \hbar c$. If ΔE is larger or equal to the rest energy $m_0 c^2$ the concept of particle with definite mass looses its meaning [19]. We see therefore that if we put $\Delta E_{\mathrm{max}} < m_0 c^2$, one gets (2.4) from (2.2) which takes the form

$$\Delta x > \frac{\hbar c}{\Delta E_{\mathrm{max}}} . \tag{2.5}$$

The uncertainty relation (2.4) would lead effectively to the existence of fundamental length, where m_0 is the rest mass of the stable particle, if the creation and annihilation processes would not take place. Because for $E > m_0 c^2$ this is not the case, therefore one should look for the universal limitations on Δx from below in another place *e.g.* in gravity theory, describing the space-time manifold as a dynamical system. The advantage of gravity is its universal nature, its coupling to any matter in the universe.

Let us consider the measurement process of the length in general relativity. Let us observe that the Eistein equations for the metric

$$\partial^2 g \sim \frac{1}{\kappa^2} \rho , \tag{2.6}$$

imply the following relation between the fluctuations of the metric Δg and the fluctuation of the energy density $\rho = \Delta E/(\Delta x)^3$

$$\frac{\Delta g}{(\Delta x)^2} \sim \frac{1}{\kappa^2} \frac{\Delta E}{(\Delta x)^3} . \tag{2.7}$$

Because photon localizing with accuracy Δx should have energy larger than $E = \hbar \nu = \hbar / \Delta x$, one gets

$$(\Delta g)(\Delta x)^2 > \frac{\hbar}{\kappa^2} = \lambda_p^2 . \tag{2.8}$$

Writing $(\Delta s)^2 = g(\Delta x)^2 > (\Delta g)(\Delta x)^2$ one can write $\Delta s > \lambda_p$ where

$$\lambda_p \simeq 1.6 \cdot 10^{-33} \text{ cm} \tag{2.9}$$

is the Planck length.

The impossibility of localizing in quantized general relativity an event with the accuracy below the Planck length follows from the creation of gravitational field by the energy necessary for the measurement process. It is known that the energy $E = 1/\lambda_p$ can create the Schwarzschild solution with the radius $R = \lambda_p$. Because the signals from the inside of Schwarzschild sphere can not be observed, effectively space-time as an object of measurement in gravity theory is transformed into a "Schwarzschild lattice", with points replaced with impenetrable spheres with Planck length radia. So operationally the notion of space-time points loses its meaning.

It should be mentioned that a similar conclusion can be reached in the framework of the lattice quantum gravity and functional integration approach to quantum gravity [28, 29].

2.2. Elementary length and string theory

The string theories (or rather superstrings theories which do not have tachyons and have consistent string loop expansions — see *e.g.* [30]) introduce the fundamental string length l_s by the dimensionfull string tension T as follows

$$l_s^2 = \frac{h}{\pi T} = \frac{\hbar^2}{M_s^2}, \tag{2.10}$$

where M_s denotes the fundamental string mass [24]. The Regge slope α' of the string trajectories is given by the formula

$$\alpha' = \frac{1}{2\pi T}. \tag{2.11}$$

The relation between the string mass and the Planck mass M_p is obtained from the description of the graviton-graviton scattering by the string tree amplitude, with dimensionless string coupling constant g. One obtains that the Newton constant $G = l_p^2$ is given by

$$G = g^2 \frac{1}{M_s^2} \tag{2.12}$$

i.e. one obtains $M_s = gM_p$. Indeed, the quantum gravity perturbative series in the coupling constant $1/M_p^2$ correspond to the string perturbative expansion with the coupling constant g^2/M_s^2.

It should be added that the quantum mechanical uncertainty relation is modified for the fundamental strings. The uncertainty in the position Δx is the sum of two terms:

a) standard term is due to Heisenberg uncertainty relation for point-like canonically quantized objects,
b) new term is related with the size of the string increasing linearly with the energy.

One obtains (see *e.g.* [26, 27])

$$\Delta x \geq \frac{\hbar}{\Delta p} + k l_p^2 \Delta p \,. \tag{2.13}$$

The minimal value of Δx is obtained for $(\Delta p)^2 \simeq \hbar / k l_p^2$ *i.e.*.

$$\Delta x \geq 2\sqrt{k\hbar}\, l_p \,. \tag{2.14}$$

We see therefore that again it follows from the fundamental string theory that the Planck fundamental length l_p describes the accuracy of the measurement of space-time distances.

3. Quantum deformations of $D = 4$ Poincaré algebra with fundamental mass scale

Quantum deformations in terms of noncommutative Hopf algebras are described infinitesimally by bialgebras. We shall consider here only the deformations described by coboundary bialgebras, *i.e.* with the coproducts described by the formula (see *e.g.* [1])

$$\Delta^{(1)}(\hat{g}) = [\hat{r}, \hat{g} \otimes 1 + 1 \otimes \hat{g}] \,, \tag{3.1}$$

where $\hat{r}$ is the classical r-matrix ($r \in \hat{g} \wedge \hat{g}$) satisfying in the general case the modified Yang–Baxter equation (MYBE) [31]:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \alpha \Omega_3 \,, \tag{3.2}$$

and Ω_3 is the $\hat{g}$-invariant three-form, *i.e.*.

$$[\Omega_3, \hat{g} \otimes 1 \otimes 1 + 1 \otimes \hat{g} \otimes 1 + 1 \otimes 1 \otimes \hat{g}] = 0 \,. \tag{3.3}$$

If $\alpha = 0$ the relation (3.1) describes classical Yang–Baxter equation (CYBE).

The partial classification of quantum deformations of $D = 4$ Poincaré algebra in the language of classical $\hat{r}$-matrices was given in [16]. The $\hat{r}$-matrices for $D = 4$ Poincaré algebra with homogeneous scaling properties are of the following three types:

1) The Lorentz algebra r-matrices depending only on the Lorentz generators, of the form
$$r_1 = r_1^{\mu\nu;\rho\tau} M_{\mu\nu} \wedge M_{\rho\tau} \,. \tag{3.4}$$
From the scale invariance of the generators $M_{\mu\nu}$ follows that the numerical coefficients $r_1^{\mu\nu;\rho\tau}$ are dimensionless. The deformations described by r_1 will not be considered here.

2) The "mixed" classical r-matrices:
$$r_2 = \frac{1}{\kappa} r_2^{\mu\nu;\rho} M_{\mu\nu} \wedge P_\rho \,. \tag{3.5}$$
From the scale invariance follows that one can factorize in front of dimensionless numerical coefficients $r_2^{\mu\nu;\rho}$ an inverse of fundamental mass parameter.

3) The r-matrices, depending only on the Abelian fourmomentum generators and describing the so-called "soft" deformations [32]:
$$r_3 = \frac{1}{\kappa^2} r_3^{\mu;\nu} P_\mu \wedge P_\nu \,. \tag{3.6}$$
where again the coefficients $r_3^{\mu;\nu}$ are dimensionless

We see that only the classical r-matrices r_2 and r_3 introduce the masslike deformation parameters and we shall discuss them in more detail.

i) **The classical r_3-matrices (see (3.6)).**
For any choice of the tensor $r_3^{\mu\nu}$ the CYBE is satisfied *i.e.* r_3 is triangular. One can write the antisymmetric 4×4 matrix $r_3^{\mu;\nu}$ in the following form:
$$r_2 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix} , \tag{3.7}$$
i.e. we obtain (see also [16]):
$$r_2 = \frac{\alpha}{\kappa^2} P_0 \wedge P_3 + \frac{\beta}{\kappa^2} P_1 \wedge P_2 \,. \tag{3.8}$$
In such a case one gets the following deformations of the classical coproducts for classical Poincaré algebra (1.1) (we recall $M_{\mu\nu} = (M_i, L_i)$)

$$\Delta(M_1) = M_1 \otimes 1 + 1 \otimes M_1 - \frac{\alpha}{\kappa^2} P_0 \wedge P_2 + \frac{\beta}{\kappa^2} P_1 \wedge P_3 \,,$$
$$\Delta(M_2) = M_2 \otimes 1 + 1 \otimes M_2 + \frac{\alpha}{\kappa^2} P_0 \wedge P_1 - \frac{\beta}{\kappa^2} P_3 \wedge P_2 \,, \tag{3.9a}$$

and

$$\Delta(L_1) = L_1 \otimes 1 + 1 \otimes L_1 + \frac{\alpha}{\kappa^2} P_1 \wedge P_2 + \frac{\beta}{\kappa^2} P_0 \wedge P_2 \,,$$
$$\Delta(L_2) = L_2 \otimes 1 + 1 \otimes L_2 + \frac{\alpha}{\kappa^2} P_2 \wedge P_3 - \frac{\beta}{\kappa^2} P_0 \wedge P_1 \,, \qquad (3.9b)$$

Remaining coproducts for M_3, L_3 and $P_\mu = (P_0, P_1, P_2, P_3)$ stay primitive.

It is easy to check that the coproducts (3.9a)–(3.9b) describe the homomorphism of classical Lie algebra.

The deformation given by the classical r-matrices (3.7) do not affect the generators $\hat{g}_{\rm cl}$ for which the lowest order deformation (3.1) of the coproduct vanishes, *i.e.*

$$[\hat{r}, \hat{g}_{\rm cl} \otimes 1 + 1 \otimes \hat{g}_{\rm cl}] = 0 \,. \qquad (3.10)$$

For the choice $\alpha\beta \neq 0$ one obtains that

$$\hat{g}_{\rm cl} = (M_3, L_3, P_\mu) \,. \qquad (3.11)$$

We see therefore that for the classical r-matrix (3.7) the classical Lorentz symmetry $O(3,1)$ is broken to $O(2) \otimes O(2)$.

ii) **The classical r_2-matrices (see (3.5)).**
There are various choices of the r_2-matrices in the still incomplete Zakrzewski classification [16]. It comprises of 15 separate cases, providing different choices of $\hat{g}_{\rm cl}$. It appears that only three choices describe these quantum deformations which leave the three-dimensional subalgebras of Lorentz symmetry classical:

1) The deformation with $O(3)$ classical symmetry.
In such a case the classical r-matrix is given by the formula ($L_i \equiv M_{i0}$)

$$r_2^{(+)} = \frac{1}{\kappa} L_i \wedge P_i \qquad (3.12)$$

and satisfies MYBE. The symmetry $\hat{g}_{\rm cl}$ as the subalgebra of $D = 4$ Poincaré symmetry is four-dimensional

$$\hat{g}_{\rm cl}^{(+)} = (M_i, P_0) \,. \qquad (3.13)$$

There is known the full quantum deformation corresponding to the classical r-matrix (3.12) in a form of noncommutative and noncocommutative Hopf algebra [4, 6]. Such a quantum deformation is known as κ-deformation of Poincaré algebra and has been recently formulated in different bases [9, 33].

2) The deformation with $O(2,1)$ classical symmetry.
In such a case one chooses

$$r_2^{(-)} = \frac{1}{\kappa}(M_1 \wedge P_2 - M_2 \wedge P_1 + L_3 \wedge P_0) \tag{3.14}$$

and the classical r-matrix (3.14) satisfies MYBE. The classical symmetry generators $\hat{g}_{\rm cl}$ satisfying for the choice (3.14) the relation (3.10) are given by the formula

$$\hat{g}_{\rm cl}^{(-)} = (L_1, L_2, M_3, P_3) \tag{3.15}$$

and describes $O(2,1)$ classical subalgebra supplemented by one spatial momentum. In the theory of representations of classical Poincaré symmetry the choice (3.15) of the generators of the stability group describes the tachyonic particles with imaginary masses. Further we shall call the deformations generated by (3.14) the tachyonic κ-deformations of $D = 4$ Poincaré algebra. The "full" tachyonic quantum deformation has been firstly considered in [34] as one of the examples in the discussion of deformations of $D = 4$ inhomogeneous rotation algebras with various signatures.

3) The deformation with $E(2)$ classical symmetry.
There exists only one classical r-matrix for $D = 4$ Poincaré algebra which satisfies two conditions

α) It satisfies CYBE.
β) It leaves classical three-dimensional subalgebra of Lorentz algebra $O(3,1)$.

This classical r-matrix takes the form

$$r_2^{(0)} = \frac{1}{\kappa}(L_3 \wedge P_+ + \tilde{E}_1 \wedge P_1 + \tilde{E}_2 \wedge P_2)\,, \tag{3.16}$$

where $P_\pm = P_0 \pm P_3$ and

$$\tilde{E}_1 = L_1 + M_2\,, \qquad \tilde{E}_2 = -L_2 + M_1\,. \tag{3.17}$$

The generators satisfying (3.10) for the choice (3.16) of the classical $\hat{r}$-matrix are described by the following four-dimensional algebra [18].

$$g_{\rm cl} = (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, P_+)\,, \tag{3.18}$$

where $\tilde{E}_3 = M$. One obtains the known relation of $D = 2$ Euclidean algebra $e(2)$:

$$[\tilde{E}_1, \tilde{E}_2] = 0\,, \qquad [\tilde{E}_2, \tilde{E}_3] = -\tilde{E}_1\,, \qquad [\tilde{E}_1, \tilde{E}_3] = \tilde{E}_2\,. \tag{3.19}$$

The symmetries generated by (3.18) correspond to the stability group for the massless particle representations of the classical Poincaré algebra. We shall call it conformal κ-deformation of $D = 4$ Poincaré algebra, because as a triangular deformation it has the extension to $D = 4$ conformal algebra (see Sect. 4). It should be pointed out also that the "full" quantum deformation corresponding to the classical $\hat{r}$-matrix (3.16) has been presented firstly in [35].

In such a way we have obtained three different κ-deformations of $D = 4$ Poincaré algebra which correspond to three classes of the representations of classical Poincaré algebra. It should be stressed that these choices of the quantum deformations should be relevant for the discussion of the deformation of three classes (with positive mass, tachyonic and massless) of induced representations of $D = 4$ Poincaré symmetries.

4. Quantum deformations of $D = 4$ conformal algebra with fundamental mass

Contrary to the case of $D = 4$ Poincaré algebra at present it does not exist in explicite form the classification of infinitesimal deformations (classical r-matrices) for $O(4,2)$ Lie algebra, or for its complexified version — $sl(4)$ Lie algebra. It should be stressed however that the general principles how to construct such a classification can be found in the literature (see *e.g.* [36–38]).

The $O(4,2)$ Lie algebra is simple and for any simple Lie algebras the solutions of MYBE are known [39]. We shall use the $O(4,2)$ Cartan–Weyl basis (h_1, h_2, h_3, $e_{\pm1}$, $e_{\pm2}$, $e_{\pm3}$, $e_{\pm4}$, $e_{\pm5}$, $e_{\pm6}$) satisfying the reality conditions (see [40, 18])

$$\begin{aligned} h_1^+ &= -h_3\,, & h_2^+ &= -h_2\,, \\ e_{\pm1}^+ &= e_{\pm3}\,, & e_{\pm2}^+ &= -e_{\pm2}\,, \\ e_{\pm4}^+ &= -e_{\pm5}\,, & e_{\pm6}^+ &= -e_{\pm6}\,. \end{aligned} \tag{4.1}$$

The physical antihermitean generators of $D = 4$ conformal algebra are given by the formulae:

$$h_1 = L_3 - iM_3\,, \qquad h_2 = -(D + L_3) \qquad h_3 = L_3 + iM_3\,,$$

$$e_1 = \tfrac{1}{2}(M_+ + iL_+)\,, \qquad e_3 = -\tfrac{1}{2}(M_- - iL_-)\,,$$

$$e_2 = \tfrac{1}{2}(P_0 - P_3)\,, \qquad e_6 = \tfrac{1}{2}(P_0 + P_3)\,,$$

$$e_4 = \tfrac{i}{2}(P_1 + iP_2)\,, \qquad e_5 = -\tfrac{i}{2}(P_1 - iP_2)\,, \tag{4.2a}$$

$$e_{-1} = \tfrac{1}{2}(M_- + iL_-)\,, \qquad e_{-3} = -\tfrac{1}{2}(M_+ - iL_+)\,,$$

$$e_{-2} = \tfrac{1}{2}(K_0 + K_3)\,, \qquad e_{-6} = \tfrac{1}{2}(K_0 - K_3)\,,$$

$$e_{-4} = -\tfrac{i}{2}(K_1 - iK_2)\,, \qquad e_{-5} = \tfrac{i}{2}(K_1 + iK_2)\,. \tag{4.2b}$$

The standard Drinfeld–Belavin classical r-matrix is given as follows

$$r^{DB} = \sum_{i,j=1}^{3} d_{ij}\, h_i \wedge h_j + \sum_{A=1}^{6} e_A \wedge e_{-A}\,, \tag{4.3}$$

where d_{ij} is the inverse symmetric Cartan matrix for $sl(4)$:

$$d_{ij} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}. \tag{4.4}$$

It is easy to see from (4.2a)–(4.2b) that the classical r-matrix (4.3) describes the quantum deformation with dimensionless deformation parameter. It can be shown that after adding suitably chosen symmetric part it leads to the well-known Drinfeld–Jimbo deformation of complexified $D = 4$ conformal algebra $sl(4)$ (see *e.g.* [1, 3]).

Our aim here is to provide the deformations of $D = 4$ conformal algebra with the deformation parameter introducing fundamental masses. We shall consider here one example of such a deformation — the κ-deformation of $D = 4$ conformal algebra obtained from conformal κ-deformation of $D = 4$ Poincaré subalgebra (see [18]).

Let us observe that every solution of CYBE spanned by the generators of a Lie subalgebra $\hat{g}' \subset \hat{g}$ can be treated as the solutions of CYBE for the full Lie algebra $\hat{g}$. In Sect. 3 we found the classical $\hat{r}$-matrix (3.16) satisfying CYBE. Because the Poincaré algebra is a subalgebra of the conformal one ($\hat{g}' = \mathcal{P}_{3;1}$, $\hat{g} = C_{3;1}$) the classical $\hat{r}$-matrix (3.16) is also a classical $\hat{r}$-matrix for $D = 4$ conformal algebra and can be written in terms of the Cartan–Weyl generators for $sl(4)$ (see [7, 40]) as follows

$$r_2^{(0)} = \tfrac{1}{2}(h_1 + h_3) \wedge e_6 + e_1 \wedge e_5 - e_3 \wedge e_4\,. \tag{4.5}$$

Using the formula (3.1) one obtains that:

$$\Delta^{(1)}(g_{\rm cl}) = 0\,, \qquad g_{\rm cl} = \tilde{E}_1,\, \tilde{E}_2,\, M_3,\, P_+\,,$$

$$\begin{aligned}
\Delta^{(1)}(P_r) &= \frac{1}{\kappa} P_r \wedge P_+ \,, \qquad r = 1,2\,, \\
\Delta^{(1)}(P_-) &= \frac{2}{\kappa} P_0 \wedge P_3 \;=\; \frac{1}{\kappa} P_- \wedge P_+ \,, \\
\Delta^{(1)}(\tilde{F}_1) &= \frac{1}{\kappa} (\tilde{F}_1 \wedge P_+ + \tilde{E}_1 \wedge P_- + M_3 \wedge P_2 \,, \\
\Delta^{(1)}(\tilde{F}_2) &= \frac{1}{\kappa} (\tilde{F}_2 \wedge P_+ + \tilde{E}_2 \wedge P_- + M_3 \wedge P_1 \,, \\
\Delta^{(1)}(L_3) &= \frac{1}{\kappa} (L_3 \wedge P_+ + \tilde{E}_r \wedge P_r)
\end{aligned} \tag{4.6a}$$

and

$$\Delta^{(1)}(D) = -\frac{1}{\kappa}(L_3 \wedge P_+ + M_1 \wedge P_2 - M_2 \wedge P_1)\,. \tag{4.6b}$$

Remaining coproducts for K_μ are obtained by the following isometry of the conformal algebra

$$M_{\mu\nu} \longrightarrow M_{\mu\nu}\,, \qquad D \longrightarrow -D\,, \qquad K_\mu \longrightarrow P_\mu\,. \tag{4.7}$$

It should be mentioned that the quantum deformation od 11-parameter $D = 4$ Weyl algebra (Poincaré + dilatations) in the form of Hopf algebra with lowest order coproducts given by (4.6a)–(4.6b) was given recently in [41].

The problem of classifying the κ-deformations of $D = 4$ conformal algebra is an open question. In particular it would be interesting to describe a deformation depending on all 9 generators of the Borel subalgebra of $D = 4$ conformal algebra, consisting of the generators (e_{+A} ($A = 1,\ldots,6$), h_i ($i = 1,2,3$)). Comparing with (4.5) we should find in physical basis a solution of CYBE depending additionally on h_2, e_2. Such a solution would describe a classical $\hat{r}$-matrix for 11-parameter $D = 4$ Weyl algebra. This problem in now under consideration.

5. Conclusions

The existence of fundamental length or fundamental mass manifests itself in the modification of the canonical Poisson brackets for the phase space variables (see *e.g.* [42, 43]); in particular one obtains the modification of vanishing Poisson brackets for the space coordinates. After quantization one gets the noncommutative space coordinates:

$$[\hat{x}_i, \hat{x}_j] \neq 0\,, \qquad i,j = 1,2,3\,. \tag{5.1}$$

If we assume that the introduction of fundamental mass parameter does not modify the classical nonrelativistic $O(3)$ symmetries, one should keep the relation $[\hat{x}_i, \hat{x}_j] = 0$, and from $O(3)$-covariance follows that

$$[\hat{x}_i,\, \hat{x}_j] = \frac{i}{\kappa}\hat{x}_i\,. \tag{5.2}$$

The relations (5.2) describe the translation sector of the κ-deformed $D = 4$ Poincaré group, obtained by the quantization od Drinfeld–Sklyanin Poisson–Lie bracket with the classical r-matrix (3.12) [11]. It can be added that respectively for the tachyonic κ-deformation, described by the classical r-matrix (3.14), and conformal κ-deformation (see (3.16)) the relation (5.2) is modified as follows:

$$[\hat{x}_s,\, \hat{t}] = \frac{i}{\kappa}\hat{x}_s\,, \tag{5.3}$$

where $\hat{x}_s = (\hat{x}_1,\, \hat{x}_2,\, \hat{x}_0)$, $\hat{t} = \hat{x}_3$ for the tachyonic κ-deformation, and $\hat{x}_s = (\hat{x}_1, \hat{x}_2, \hat{x}_+ = \frac{1}{2}(\hat{x}_0 - \hat{x}_3))$, $\hat{t} = \frac{1}{2}(\hat{x}_0 + \hat{x}_3)$ for conformal κ-deformation.

The relations (5.2)–(5.3) are of Lie algebra type. More general class of noncommutative space-time coordinates obtained after their identification with the translation sector of quantum Poincaré group was considered in [15]. One gets the following algebraic relations:

$$(R - 1)_{\mu\nu}{}^{\rho\tau}(\hat{x}_\rho\hat{x}_\tau + \frac{1}{\kappa}T_{\rho\tau}{}^{\lambda}\hat{x}_\lambda + \frac{1}{\kappa^2}C_{\rho\tau}) = 0\,, \tag{5.4}$$

where the matrix R describes the quantum R-matrix for the Lorentz group satisfying the condition $R^2 = 1$, and $T_{\mu\nu}{}^{\rho}$, $C_{\mu\nu}$ are the numerical coefficients (for details see [15]) which are dimensionless. The condition $R^2=1$ can be removed if we consider quantum Poincaré groups belonging to larger class of so called braided Hopf algebras (see *e.g.* [44]). The relations (5.2)–(5.3) follow as a special case of the relations (5.4), with $R = \tau$ (classical Lorentz symmetry), $C_{\mu\nu} = 0$ and a particular choice of $T_{\mu\nu}{}^{\rho}$ (see also [41]).

The noncommuting space-time coordinates is an old idea (see *e.g.* [45, 46]) which was also considered as a cure of the divergence problems in quantum field theories (see *e.g.* [47]). In particular the ultraviolet pathologies of quantum gravity led to the development of the fundamental string theory and the replacement of the perturbation theory with gravitons by the perturbation theory with virtual string states. In the quantum group approach the property that the space-time coordinates do not commute and can not be measured with arbitrary accuracy are introduced in kinematic, purely algebraic way. The algebraic structure of space-time coordinates is described by the algebraic sector of the quantum Poincaré group; the coalgebra sector determines the noncommutativity in the fourmomentum sector. The momentum counterpart of the relations (5.4) written as an algebraic ansatz looks as follows:

$$(R-1)_{\mu\nu}{}^{\rho\tau}(\hat{p}_\rho\hat{p}_\tau + \kappa\tilde{T}_{\rho\tau}{}^{\lambda}\hat{p}_\lambda + \kappa^2\tilde{C}_{\rho\tau}) = 0\,. \tag{5.5}$$

We see that for $\kappa^2 \longrightarrow \infty$ finite limit requires $\tilde{T}_{\mu\nu}{}^{\rho} = \tilde{C}_{\mu\nu} = 0$. In particular for classical Lorentz symmetry ($R = \tau$) one obtains the commuting fourmomenta. This result is confirmed by the formulas in the fourmomentum sector for κ-deformed Poincaré algebras [4, 6].

Finally, we would like to mention here that the kinematic description of generalized uncertainty relation which in quantum gravity is of dynamical origin has been recently also considered outside of the algebraic framework of quantum groups [48]. In such an approach the space-time coordinates are not commuting, but the Poincaré symmetry remains classical. Such as approach limits the noncommutativity (5.4) only to the last term in (5.4) ($R = 1$, $T_{\mu\nu}{}^{\rho} = 0$). It should be stressed however that the approach presented in [48] breaks necessarily the manifest nonrelativistic $O(3)$ symmetry — the noncommutativity introduces the distinguished 2-plane in fourdimensional space-time.

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