THE HAMILTONIAN EVOLUTION OF YANG–MILLS FIELDS IN BOUNDED DOMAINS

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We show that there is a choice of the gauge condition such that the mixed problem for the Hamiltonian form of the evolution equations for Yang–Mills fields in spatially bounded domains (with inhomogeneous boundary conditions) admits the finite time existence and uniqueness of solutions.

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1. Introduction

In classical dynamics and field theory the Hamiltonian nature of the evolution equations is a consequence of the underlying variational principle. It plays an important role in the construction of the corresponding quantum theory. In particular, it ensures the unitarity of the quantum evolution.

For field theories with constraints the splitting of the field equations into the evolution equations and the constraint equations is somewhat arbitrary off the constraint set. One can always modify the evolution equations by

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terms which vanish on the constraint set. In the case of Yang–Mills fields such a decomposition of the field equations is usually obtained in terms of the chosen 3 + 1 splitting of the space-time into the product of the space and the time axis. It leads to the evolution equations

$$\partial_t A = E + \operatorname{grad} \Phi - [\Phi, A], \tag{1.1}$$

$$\partial_t E = -\operatorname{curl} B - [A\times, B] - [\Phi, E], \tag{1.2}$$

and the constraint equation

$$\operatorname{div} E + [A; E] = 0. \tag{1.3}$$

Here, A is the vector potential of the Yang–Mills field and Φ is the scalar potential, (both potentials have values in the Lie algebra g of the structure group G of the theory), $[\cdot,\cdot]$ denotes the bracket in g, while $\times$ and $\cdot$ denote the cross product and the dot product in $\mathbb{R}^3$, respectively.

The time evolution of the scalar potential Φ is not determined by the field equations. In order to make the evolution equations deterministic one prescribes Φ in terms of a gauge condition. If the scalar potential is chosen as a given function of the space-time variables, $\Phi = \Phi(x,t)$, then the evolution equations are Hamiltonian with the Hamiltonian

$$H_\Phi(A,E) = \frac{1}{2}\int_M (E^2 + B^2) d_3x + \int_M E\big(\operatorname{grad}\Phi - [A,\Phi]\big) d_3x. \tag{1.4}$$

and the symplectic form

$$\omega = d\theta, \tag{1.5}$$

where θ is given by

$$\langle \theta(A,E) | (\delta A, \delta E)\rangle = \int_M E \cdot \delta A \; d_3x, \tag{1.6}$$

and M is the domain in $\mathbb{R}^3$ accessible to the fields. In particular, for the temporal gauge condition

$$\Phi = 0, \tag{1.7}$$

one gets the usual Hamiltonian

$$H_0(A,E,\Psi) = \frac{1}{2}\int_M (E^2 + B^2) d_3x. \tag{1.8}$$

In order to make the above heuristic arguments more precise we have to specify the space of functions in which the evolution is taking place and to show that the evolution equations have solutions in this space. The existence and uniqueness of solutions of the Yang–Mills evolution equations has been studied in several papers, [1] through [8]. However, all of them prove the existence of solutions for non-Hamiltonian evolution equations. The methods of proof are based on the theory of perturbations of linear semigroups, [9]. In particular, it requires the elimination of the longitudinal component of E from the linearized equations; for a comprehensive discussion of this problem see [4] where this problem is solved by a non-Hamiltonian modification of the original evolution equations off the constraint. In [8] we showed that by an appropriate gauge transformation one can get Φ to be the solution of the Neumann problem in M

$$\Delta\, \Phi = -\operatorname{div} E \quad \text{and} \quad n(\operatorname{grad} \Phi) = -nE \quad \text{with} \quad \int_M \Phi \, d_3x = 0\,, \tag{1.9}$$

where $n(\operatorname{grad} \Phi)$ denotes the normal component of $\operatorname{grad} \Phi$ on the boundary ∂M of M. With this choice of Φ we proved the existence and uniqueness of solutions of the mixed problem for the evolution equations of minimally interacting Yang–Mills and Dirac fields in bounded domains with inhomogeneous boundary conditions, [8].

For the gauge condition (1.9) the scalar potential Φ depends on the dynamical variable E. Hence, the evolution equations (1.1) and (1.2) are not Hamiltonian. However, the Hamiltonian H_Φ, given by (1.4), generates the evolution equations in which (1.1) is replaced by

$$\begin{aligned}\partial_t A(x) =& E(x) + \operatorname{grad} \Phi(x) - [\Phi(x), A(x)] \\ &+ \int_M E(x')(\operatorname{grad}' \frac{\delta\Phi(x')}{\delta E(x)} - [A(x'), \frac{\delta\Phi(x')}{\delta E(x)}])d_3x'\,,\end{aligned} \tag{1.10}$$

the gradient under the integral sign is taken with respect to the variable x'.

On the constraint set, given by Eq. (1.3), all the evolution equations are equivalent and are Hamiltonian. Hence, one could argue that the form of the evolution equations off the constraint set is not important. However, the Hamiltonian nature of the evolution equations (also off the constraint set) is essential if one wants to compare the classical and the quantum reduction procedures, [10, 11].

In this paper we discuss the existence and the uniqueness of solutions of the Hamiltonian evolution equations (1.10) and (1.2), in the space

$$P(M) = \{(A, E) \in H^2(M, g) \times H^1(M, g)\}\,, \tag{1.11}$$

where M is a bounded contractible domain in $\mathbb{R}^3$, and H^k denotes the Sobolev space of fields which are square integrable over M together with their partial derivatives up to the order k, see [12], and Φ is the solution of the following Neumann problem:

$$\Delta\Phi = -\tfrac{1}{2}\operatorname{div} E \quad \text{and} \quad n(\operatorname{grad}\Phi) = -\tfrac{1}{2}\, nE \quad \text{with} \quad \int_M \Phi \, d_3x = 0\,. \tag{1.12}$$

The main result is given in the next section. Section 3 contains an outline of the proof.

2. Statement of results

We consider a bounded contractible domain M in $\mathbb{R}^3$ with smooth boundary ∂M. For the Yang–Mills vector potential A, the boundary condition is given by specifying the tangential to ∂M component of the curl of A, denoted by $t(\operatorname{curl} A)$. For $A \in H^2(M,g)$, $t(\operatorname{curl} A) \in H^{1/2}(\partial M, g)$.

Theorem

Let $\lambda(t)$ be a differentiable curve of the boundary data in $H^{1/2}(\partial M, g)$. For every $t_0 \in \mathbb{R}$ and the initial data $(A_0, E_0) \in H^2(M,g) \times H^1(M,g)$ such that

$$t(\operatorname{curl} A_0) = \lambda(t_0)\,, \tag{2.1}$$

there exists a maximal $T > 0$ and a unique curve

$$[t_0, t_0 + T) \to H^2(M,g) \times H^1(M,g) : t \to (A(t), E(t)) \tag{2.2}$$

satisfying the evolution equations

$$\partial_t A = E + \operatorname{grad}\Phi - [\Phi, A] + \int_M E\left(\operatorname{grad}' \frac{\delta\Phi}{\delta E} - [A, \frac{\delta\Phi}{\delta E}]\right) d_3x'\,. \tag{2.3}$$

$$\partial_t E = -\operatorname{curl} B - [A\times, B] - [\Phi, E]\,, \tag{2.4}$$

the initial conditions

$$A(t_0) = A_0 \quad , \quad E(t_0) = E_0\,, \tag{2.5}$$

and the boundary conditions

$$t\operatorname{curl}(A(t)) = \lambda(t)\,, \tag{2.6}$$

where Φ is the solution of the Neumann problem

$$\Delta\Phi = -\tfrac{1}{2}\operatorname{div} E \quad , \quad n(\operatorname{grad}\Phi) = -\tfrac{1}{2}\, nE \quad , \int_M \Phi \, d_3x = 0\,. \tag{2.7}$$

3. Outline of proof

Let $\lambda(t)$ be a differentiable curve of the boundary data in $H^{1/2}(\partial M, g)$. We choose a differentiable curve $A^b(t)$ of vector potentials in $H^2(M,g)$ satisfying the boundary condition

$$t(\text{curl}\, A^b(t)) = \lambda(t)\,. \tag{3.1}$$

Such a choice of a background field is always possible, in particular one can take $A^b(t)$ to be the solution of $\Delta A^b(t) = 0$ which satisfies (3.1), [13]. Then, the difference

$$a(t) = A(t) - A^b(t) \tag{3.2}$$

satisfies the homogeneous boundary condition

$$t(\text{curl}\, a(t)) = 0 \tag{3.3}$$

if and only if $A(t)$ satisfies (2.6). Moreover,

$$B = \text{curl}\, A + [A, \times A] = \text{curl}\, a + \text{curl}\, A^b + [A^b + a, \times A^b + a]\,. \tag{3.4}$$

If we rewrite the evolution equations (2.3) and (2.4) in terms of the variables $(a(t), E(t))$, and consider first the linear approximation in which the terms depending on the background field are omitted, we obtain

$$\partial_t a = E + \text{grad}\, \Phi + \int_M E(\text{grad}'\frac{\delta\Phi}{\delta E})d_3x'\,, \tag{3.5}$$

$$\partial_t E = -\text{curl}\,(\text{curl}\,(a))\,. \tag{3.6}$$

Here Φ is given by (2.7). Our aim is to prove that Eqs. (3.5) and (3.6) determine a continuous one parameter semigroup of bounded linear transformations in an appropriate function space. To this end we start with the Hodge decomposition of the Lie algebra valued vector fields E,

$$E = E^L + E^T\,, \tag{3.7}$$

where the longitudinal part is a gradient

$$E^L = \text{grad}\, \psi_E\,, \tag{3.8}$$

and the transverse part is divergence free and has vanishing normal component on the boundary

$$\text{div}\, E^T = 0 \quad , \quad nE^T = 0\,. \tag{3.9}$$

Since M is simply connected, the potential ψ_E is uniquely determined by (3.8) and the condition

$$\int_M \psi_E \, d_3x = 0 \,. \tag{3.10}$$

Moreover, if $E \in H^k(M,g)$ then $\psi_E \in H^{k+1}(M,g)$, [13]. Taking the divergence of (3.7) and taking into account (3.8) through (3.10), we see that ψ_E is the solution of the following Neumann problem:

$$\Delta \, \psi_E = \operatorname{div} E \,, \; n(\operatorname{grad} \psi_E) = nE \,, \; \int_M \psi_E d_3x = 0 \,. \tag{3.11}$$

Comparing this with (2.7) we see that

$$\Phi = -\tfrac{1}{2}\psi_E \tag{3.12}$$

so that the scalar potential satisfies

$$\operatorname{grad} \Phi = -\tfrac{1}{2} \, E^L \,. \tag{3.13}$$

Therefore,

$$\frac{\delta}{\delta E} \operatorname{grad} \Phi = \frac{\delta}{\delta E^L} \operatorname{grad} \Phi \,, \tag{3.14}$$

and the non-local term in (3.5) is

$$\begin{aligned} \int_M E(\operatorname{grad}' \frac{\delta \Phi}{\delta E}) d_3x' &= \int_M E^L(x')\{\frac{\delta}{\delta E^L(x)} \operatorname{grad}' \Phi(x')\} d_3x' \\ &= -\frac{1}{2} \int_M E^L(x')\{\delta(x - x')\} d_3x' = -\tfrac{1}{2} \, E^L(x) \,. \end{aligned} \tag{3.15}$$

Substituting (3.7), (3.13) and (3.15) into (3.5) we get

$$\partial_t a = E^T \,. \tag{3.16}$$

Hence, the linearized equations (3.5) and (3.6) decomposed into the transverse and the longitudinal components are

$$\partial_t a^T = E^T \quad , \quad \partial_t E^T = -\operatorname{curl}(\operatorname{curl}(a^T)) \,, \tag{3.17}$$

$$\partial_t a^L = 0 \quad , \quad \partial_t E^L = 0 \,. \tag{3.18}$$

Let

$$H^T = \{(a^T, E^T) \in H^1(M,g) \times L^2(M,g)\}\,, \tag{3.19}$$

$$H^L = \{(a^L, E^L) \in H^2(M,g) \times H^1(M,g)\}\,. \tag{3.20}$$

Eq. (3.17) defines a continuous one parameter semigroup $\exp(t\mathcal{T})$ of bounded linear transformations in H^T with the generator

$$\mathcal{T}(a^T, E^T) = (E^T, -\mathrm{curl}\,(\mathrm{curl}\,(a^T)) \tag{3.21}$$

defined on the domain

$$D = \{(a^T, E^T) \in H^2(M,g) \times H^1(M,g) \mid t(\mathrm{curl}\,(a^T)) = 0\}\,, \tag{3.22}$$

for details see [6]. Similarly, Eq. (3.18) defines the identity transformation in H^L.

The full evolution equations (2.3) and (2.4) can be split into the transverse and the longitudinal components, and rewritten in the form

$$\partial_t(a^T, E^T) = \mathcal{T}(a^T, E^T) + \mathcal{F}^T(a^L, E^L, a^T, E^T)\,, \tag{3.23}$$

$$\partial_t(a^L, E^L) = \mathcal{F}^L(a^L, E^L, a^T, E^T)\,, \tag{3.24}$$

where $\mathcal{F}^T$ and $\mathcal{F}^L$ denote the nonlinear and the background dependent parts of the right hand sides of (2.3) and (2.4). In order to complete the proof of the theorem it suffices to show that the following properties of the nonlinear terms $\mathcal{F}^T$ and $\mathcal{F}^L$, [14].

(a) $\mathcal{F}^L$ treated as a map from $H^L \times D$ to H^L is a continuous and locally Lipschitz map with respect to the following norm in $H^L \times D$:

$$\|(V^L, V^T)\|_1 := \|V^L\|_{H^L} + \|V^T\|_{H^T} + \|\mathcal{T}V^T\|_{H^T}\,, \tag{3.25}$$

where we have used the notation $V^L = (a^L, E^L)$ and $V^T = (a^T, E^T)$.

(b) $\mathcal{F}^T$ treated as a map from $H^L \times D$ to H^T is a continuous differentiable map with respect to the norm (3.25).

(*c*) The map $\mathcal{K} : H^L \times D \times H^T \to H^T$, given by

$$\mathcal{K}(V^L, V^T, v^T) = \mathcal{K}_1(V^L, V^T) + \mathcal{K}_2(V^L, V^T, v^T)\,, \tag{3.26}$$

where

$$\mathcal{K}_1(V^L, V^T) = D\mathcal{F}^T(V^L, V^T)(\mathcal{F}^L(V^L, V^T), 0)\,, \tag{3.27}$$

$$\mathcal{K}_2(V^L, V^T, v^T) = D\mathcal{F}^T(V^L, V^T)(0, v^T)\,, \tag{3.28}$$

is locally Lipschitz with respect to the following norm in $H^L \times D \times H^T$

$$\|(V^L, V^T, v^T)\|_2 = \|V^L\|_{H^L} + \|V^T\|_{H^T} + \|\mathcal{T}V^T\|_{H^T} + \|v^T\|_{H^T}\,. \tag{3.29}$$

Most of the estimates involved in verifying these properties are given in [8]. The only new term appearing here is

$$\int_M E(x')[A(x'), \frac{\delta\Phi(x')}{\delta E(x)}]\,d_3x' = -\int_M [E(x'), A(x')]\frac{\delta\Phi(x')}{\delta E(x)}d_3x'\,, \tag{3.30}$$

where $A = a + A^b$. The variational derivative of Φ in the direction ϵ,

$$\chi(x') = \int_M \frac{\delta\Phi(x')}{\delta E(x)}\epsilon^L(x)d_3x\,, \tag{3.31}$$

satisfies the equation

$$2\mathrm{grad}\,\chi = -\epsilon^L\,. \tag{3.32}$$

By construction,

$$\int_M \chi(x')d_3x' = 0\,, \tag{3.33}$$

and $\chi \in H^{k+1}(M, g)$ if $\epsilon \in H^k(M, g)$.

The Hodge decomposition of the 0-form (scalar function) $[E, A]$ yields

$$[E, A] = \mathrm{div}\, Z + C\,, \tag{3.34}$$

where

$$nZ = 0\ ,\ C = \mathrm{const}\,. \tag{3.35}$$

Since M is bounded and simply connected Z is unique and is in $H^{k+1}(M, g)$ if $[E, A] \in H^k(M, g)$. Hence,

$$\begin{aligned}
\int_M \Big\{ \int_M [E(x'), A(x')]\frac{\delta\Phi(x')}{\delta E(x)}\epsilon^L(x)d_3x'\Big\}d_3x &= \int_M \{\mathrm{div}'Z(x') + C\}\chi(x')d_3x' \\
&= \int_M \{-Z(x')\mathrm{grad}'\chi(x') + C\chi(x')\}d_3x' + \int_{\partial M} nZ(x')\chi(x')dS' \\
&= \frac{1}{2}\int_M Z(x')\epsilon^L(x')d_3x'\,;
\end{aligned}$$

in the last equality we have used (3.32) through (3.35). Therefore,

$$\int_M [E(x'), A(x')] \frac{\delta \Phi(x')}{\delta E(x)} d_3x' = \tfrac{1}{2} Z^L(x) . \tag{3.36}$$

For $k = 1, 2$, Eq. (3.34) implies that

$$\|[E, A]\|^2_{H^{k-1}} = \|\operatorname{div} Z\|^2_{H^{k-1}} + 2 \int_M C \operatorname{div} Z \; d_3x + \|C\|^2_{L^2} . \tag{3.37}$$

Since, $\operatorname{div} Z$ is L^2-orthogonal to the constants and $\|C\|^2_{L^2} \geq 0$, it follows that $\|\operatorname{div} Z\|^2_{H^{k-1}} \leq \|[E, A]\|^2_{H^{k-1}}$. Hence,

$$\|Z^L\|_{H^k} \leq \|\operatorname{div} Z\|_{H^{k-1}} \leq \|[E, A]\|_{H^{k-1}} \leq \|A\|_{H^2} \|E\|_{H^1} . \tag{3.38}$$

This estimate and the estimates given in [8] suffice to verify the conditions (a) through (c), which completes the proof.

4. Concluding remarks

Consider the evolution space

$$\mathcal{E} = H^2(M, g) \times H^1(M, g) \times \mathbb{R} , \tag{4.1}$$

and the 2-form

$$\Omega = \omega + dt \wedge dH_\Phi , \tag{4.2}$$

where H_Φ is given by (1.4) and Φ is given by (1.12). The boundary conditions $\lambda(t)$ determine a submanifold

$$\mathcal{E}_\lambda = \{(A, E, t) \in H^2(M, g) \times H^1(M, g) \times \mathbb{R} \mid t\operatorname{curl} A = \lambda(t)\} . \tag{4.3}$$

By construction, our solution curve $(A(t), E(t))$ gives a curve $c(t) = (A(t), E(t), t)$ in $\mathcal{E}_\lambda$. Let Ω_λ denote the pull back of Ω to S_λ. The tangent vector of the evolution curve, denoted by $\dot{c}(t)$, satisfies the Hamiltonian evolution equations

$$\dot{c}(t) \lrcorner \Omega_\lambda = 0 . \tag{4.4}$$

A choice of a background field A^b satisfying the boundary condition, cf. Eq. (3.1), leads to a reparametrization of S_λ in terms of the variables (a, E, t), where a satisfies the boundary condition

$$t(\operatorname{curl} a) = 0 . \tag{4.5}$$

These variables define in $\mathcal{E}_\lambda$ a product structure:

$$\mathcal{E}_\lambda \simeq P_0(M) \times I\!R \,, \tag{4.6}$$

where

$$P_0(M) = \{(a, E) \in H^2(M, g) \times H^1(M, g) \mid t(\mathrm{curl}\, a) = 0\} \,. \tag{4.7}$$

Since the background field is A^b is fixed, the variation δA in Eq. (1.6) can be replaced by δa. Hence, the restriction to $P_0(M)$ of the 1-form θ can be written as

$$\theta_0 = \int_M E \cdot \delta a d_3 x \,. \tag{4.8}$$

Its exterior differential

$$\omega_0 = d\theta_0 \tag{4.9}$$

is a weakly symplectic form in $P_0(M)$. The evolution equations (4.4) lead to the usual time dependent Hamiltonian equations in $P_0(M)$ with the time-dependent Hamiltonian

$$\begin{aligned} H_0(a, E, t) =& H_\Phi(a + A^b(t), E) \\ =& \frac{1}{2} \int_M \Big(E^2 + (\mathrm{curl}\, a + \mathrm{curl}\, A^b(t) + [a + A^b(t), \times a + A^b(t)])^2\Big\} d_3 x \\ &+ \int_M E\big(\mathrm{grad}\, \Phi - [a + A^b(t), \Phi]\big) d_3 x \,. \end{aligned} \tag{4.10}$$

For time independent boundary data λ, we can choose independent of time a background field A^b. In this case the above Hamiltonian is constant in time, and we have a usual time independent Hamiltonian formulation of the theory.

Using the results of [8] we could generalize our Theorem to the case of minimally interacting Yang–Mills and Dirac fields, with the Dirac field Ψ satisfying the boundary conditions

$$(i\gamma^k n_k - I)\Psi(t)|\partial M = \mu(t) \qquad \text{and} \qquad (i\gamma^k n_k - I)\mathcal{D}\Psi(t)|\partial M = \nu(t) \,. \tag{4.11}$$

Here I is the identity matrix in $V_G \otimes \mathbb{C}^4$, $|\partial M$ denotes the restriction to the boundary,

$$\mathcal{D} = -\gamma^0(\gamma^j \partial_j + im) \tag{4.12}$$

is the free Dirac operator in $I\!R^3$, and $\mu(t) \in H^{3/2}(\partial M, V_G \otimes \mathbb{C}^4)$ and $\nu(t) \in H^{1/2}(\partial M, V_G \otimes \mathbb{C}^4)$ are the boundary data satisfying the conditions

$$(i\gamma^k n_k + I)\mu(t) = 0 \quad \text{and} \quad (i\gamma^k n_k + I)\mu(t) = 0 \,. \tag{4.13}$$

In a similar way we could extend the validity of our Theorem to the minimally interacting Yang–Mills and Dirac fields in the Minkowski space-time. In this case there are no boundary conditions and Eq. (2.7) should be replaced by

$$\Delta\, \Phi = -\tfrac{1}{2}\, \mathrm{div}\, E\,, \quad \int_{\mathbb{R}^3} \Phi(x)(1+x^2)^{-2} d_3 x = 0\,, \tag{4.15}$$

for details see [15].

There are two important questions which have to be answered. The first is whether our Hamiltonian system is complete, that is if the solutions exist for all time t. Even if the system is incomplete, and solutions develop singularities in finite time, the set of Cauchy data in $P_0(M)$ for which the solutions exist up to time T is open in $P_0(M)$, [16]. Hence, they can be studied in terms of the same Hamiltonian structure. The second problem is the development of a corresponding quantum theory. This involves the physical interpretation of the boundary data.

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