ON THE RESUMMATION OF $\alpha \ln^2 x$ TERMS FOR NON–SINGLET STRUCTURE FUNCTIONS IN QED AND QCD*

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The resummation of $O(\alpha^{l+1} \ln^{2l} x)$ terms in the evolution kernels of non–singlet combinations of structure functions is investigated for both QED and QCD. Numerical results are presented for unpolarized and polarized QCD structure functions.

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1. Introduction

The resummation of leading small-x contributions to the evolution kernels of QCD singlet structure functions [1] may lead to large effects [2]. The small-x behaviour of the corresponding anomalous dimensions is dominated by the leading singularity in the N-moment plane $\sim (\alpha_s/[N - 1])^l$ and

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corrections to it. For non–singlet kernels such terms are absent both for unpolarized and polarized deep-inelastic scattering, and the most singular contributions behave like $N(\alpha_s/N^2)^l$. A resummation of these terms was derived for QCD amplitudes in Ref. [3] more than a decade ago.

Similar terms emerge also in QED. There the resummed form of these contributions may be described by the structure function method. In the present paper, after setting up our notation and recalling the standard NLO formulation in Section 2, we present in Section 3 a derivation of the resummed kernels for the case of QED and QCD [4] in parallel to relate both cases in a direct way. The asymptotic kernels are compared with those results found in complete calculations in the limit $x \to 0$ up to next-to-leading order (NLO) in QED and QCD.

Recently very sizeable corrections due to this resummation have been claimed [5] for the structure functions in both unpolarized and polarized deep inelastic scattering at small x. In this way the small-x behaviour of the structure function evolution, as *e.g.* for $xF_3^{\nu d}(x, Q^2)$, $F_2^p(x, Q^2) - F_2^n(x, Q^2)$, and $g_1^p(x, Q^2) - g_1^n(x, Q^2)$ should be considerably affected. In Section 4, we perform a detailed numerical analysis and derive the corrections to the various QCD non–singlet combinations[1].

Also in the case of the singlet anomalous dimensions for polarized deep inelastic scattering the leading singularity is expected to behave like $\sim N(\alpha_s/N^2)^l$. Corresponding equations for related QCD amplitudes have been given in [6] recently. The explicit form of the resummed anomalous dimension matrix as a function of α_s is derived in [7] where also numerical results on the behaviour of the structure function $g_1(x, Q^2)$ are presented. In the present paper we will deal with the different non–singlet cases only and refer for the singlet evolution to Ref. [7].

2. Evolution in fixed–order perturbative QED and QCD

The evolution equation for the non–singlet combinations $q_{\mathrm{NS}}^{\pm}(x, Q^2)$ of parton densities is given by

$$\frac{\partial q_{\mathrm{NS}}^{\pm}(x, Q^2)}{\partial \ln Q^2} = P_{\mathrm{NS}}^{\pm}(x, \alpha) \otimes q_{\mathrm{NS}}^{\pm}(x, Q^2) . \tag{1}$$

The corresponding splitting function combinations $P_{\mathrm{NS}}^{\pm}(x, \alpha)$ are specified below, and $\otimes$ stands for the Mellin convolution. α denotes the running coupling constant in either QED or QCD. In order to simplify the notation, we

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[1] A part of the results has been published in Ref. [4] recently.

will use the abbreviation $a \equiv \alpha/(4\pi)$ in the following. The scale dependence of the running coupling is defined by

$$\frac{da}{d \ln Q^2} = -\sum_{k=0}^{\infty} a^{k+2} \beta_k \,. \tag{2}$$

The first two coefficients of the β–function, β_0 and β_1, are independent of the renormalization scheme. For (one flavour) QED and QCD they read

$$\beta_0^{\text{QED}} = -\tfrac{4}{3} \,, \qquad\qquad \beta_1^{\text{QED}} = -4 \,,$$
$$\beta_0^{\text{QCD}} = \tfrac{11}{3} C_G - \tfrac{4}{3} T_R N_f \,, \; \beta_1^{\text{QCD}} = \tfrac{34}{3} C_G^2 - \tfrac{20}{3} C_G T_R N_f - 4 C_F T_R N_f \,, \tag{3}$$

with $C_G = N_c \equiv 3, C_F = (N_c^2 - 1)/(2N_c) \equiv 4/3$, $T_R = 1/2$, and N_f the number of flavours in the QCD case.

In what follows we drop the subscript NS wherever the non–singlet character of the considered quantity is obvious from the superscript $\pm$. The splitting functions $P^{\pm}(x, a)$ are given by the combinations

$$P^{\pm}(x, a) = P_{qq}(x, a) \pm P_{q\bar{q}}(x, \alpha) \equiv \sum_{l=0}^{\infty} a^{l+1} P_l^{\pm}(x) \,. \tag{4}$$

For the subsequent analysis we restrict ourselves to the consideration of the spacelike case $Q^2 = -q^2 > 0$. Due to fermion number conservation, the expansion coefficients P_l^- are subject to the sum rule

$$\int\limits_0^1 dx \, P_l^-(x) = 0 \,, \tag{5}$$

since a acts as an independent parameter. The non–singlet splitting functions for QCD are known up to NLO [8, 9] and read in the $\overline{\text{MS}}$ factorization scheme

$$P_{qq}(x, a) = 2a C_F \left[\frac{1 + x^2}{1 - x}\right]_+$$
$$+ a^2 \left[C_F^2 P_F(x) + \tfrac{1}{2} C_F C_G P_G(x) + C_F N_f T_R P_{N_f}(x)\right] + \mathcal{O}(a^3) \,, \tag{6}$$
$$P_{q\bar{q}}(x, a) = 4a^2 \left[C_F^2 - \tfrac{1}{2} C_F C_G\right] P_A(x) + \mathcal{O}(a^3) \,. \tag{7}$$

The functions $P_I(x)$, $I = F, G, N_f$, and A, can be found in refs. [9]. The corresponding splitting functions for QED in this scheme are obtained from

(6) and (7) by setting $C_F = T_R = 1$ and $C_G = 0$. Most QED calculations are carried out, however, in the on–mass-shell (OMS) scheme for which the NLO splitting functions are different [10]. For $x \to 0$ the leading contributions to $P^\pm(x, a)$ in the $\overline{\text{MS}}$ factorization scheme are

$$P_{x \to 0}^{+,\text{QED}}(x, a) = 2a + 2a^2 \ln^2 x + \mathcal{O}(a^3) \,,$$

$$P_{x \to 0}^{-,\text{QED}}(x, a) = 2a - 6a^2 \ln^2 x + \mathcal{O}(a^3) \,, \tag{8}$$

$$P_{x \to 0}^{+,\text{QCD}}(x, a) = 2aC_F + 2a^2 C_F^2 \ln^2 x + \mathcal{O}(a^3) \,,$$

$$P_{x \to 0}^{-,\text{QCD}}(x, a) = 2aC_F + 2a^2 \left[-3C_F^2 + 2C_F C_G \right] \ln^2 x + \mathcal{O}(a^3) \,. \tag{9}$$

Since the parton densities $q^\pm(x, Q^2)$ are scheme dependent and hence no observables beyond leading order, it is convenient to consider also the evolution equations for the related observables directly. These are given by the corresponding structure functions $F_i^\pm(x, Q^2)$, obtained by the convolution

$$F_i^\pm(x, Q^2) = c_i^\pm(x, Q^2) \otimes q_i^\pm(x, Q^2) \,. \tag{10}$$

Here $c_i^\pm(x, Q^2)$ denote the respective coefficient functions which can be expanded in a as

$$c_i^\pm(x, Q^2) = \delta(1 - x) + \sum_{l=1}^{\infty} a^l c_{i,l}^\pm(x) \,. \tag{11}$$

After transformation to an equation in a using (2), the evolution equation for $F_i^\pm(x, Q^2)$ resulting from (1) and (10) reads

$$\frac{\partial F_i^\pm(x, a)}{\partial a} = -\frac{1}{\beta_0 a^2} K_i^\pm(x, a) \otimes F_i^\pm(x, a) \,, \tag{12}$$

where in NLO the kernels can be written as

$$K_{i,1}^\pm(x, a) = a \, P_{\text{NS},0}(x) + a^2 \left[P_1^\pm(x) - \frac{\beta_1}{\beta_0} P_{\text{NS},0}(x) - \beta_0 c_{i,1}^\pm(x) \right] \,. \tag{13}$$

The terms $\propto a(a \ln^2 x)^k$ emerge in the a-expansion of the kernels $K_i^\pm(x, a)$ only in combination with the coefficient β_0. In this sense the resummation to which we turn now is of leading order.

3. Resummation of dominant terms in the limit $x \to 0$

The most singular contributions to the Mellin transforms of the structure–function evolution kernels $K^{\pm}(x, a)$ at all orders in a can be obtained from the positive and negative signature amplitudes $f_0^{\pm}(N, a)$ studied in Ref. [3] for QCD via

$$\mathcal{M}\left[K_{x \to 0}^{\pm}(a)\right](N) \equiv \int_0^1 dx \, x^{N-1} K_{x \to 0}^{\pm}(x, a)$$

$$\equiv -\tfrac{1}{2} \Gamma_{x \to 0}^{\pm}(N, a) = \frac{1}{8\pi^2} f_0^{\pm}(N, a). \qquad (14)$$

These amplitudes are subject to the quadratic equations:

$$f_0^+(N, a) = 16\pi^2 a_0 \frac{a}{N} + \frac{1}{8\pi^2} \frac{1}{N} \left[f_0^+(N, a)\right]^2, \qquad (15)$$

$$f_0^-(N, a) = 16\pi^2 a_0 \frac{a}{N} + 8b_0^- \frac{a}{N^2} f_V^+(N, a) + \frac{1}{8\pi^2} \frac{1}{N} \left[f_0^-(N, a)\right]^2. \qquad (16)$$

Here $f_V^+(N, a)$ is obtained as the solution of the Riccati differential equation

$$f_V^+(N, a) = 16\pi^2 a_V \frac{a}{N} + 2b_V \frac{a}{N} \frac{d}{dN} f_V^+(N, a) + \frac{1}{8\pi^2} \frac{1}{N} \left[f_V^+(N, a)\right]^2. \qquad (17)$$

The coefficients a_i and b_i in the above relations read for the case of QED

$$a_0 = 1, \quad b_0^- = 1, \quad a_V = 1, \quad b_V = 0, \qquad (18)$$

and for QCD [3]

$$a_0 = C_F, \quad b_0^- = C_F, \quad a_V = -\frac{1}{2N_c}, \quad b_V = N_c. \qquad (19)$$

In QED Eq. (17) further simplifies to an algebraic equation with the same coefficients as (15). The solutions of (15) and (16) were derived in Ref. [3] for the QCD case[2]. They are given by

$$\Gamma_{x \to 0}^{+, \text{QED}}(N, a) = -N \left\{1 - \sqrt{1 - \frac{8a}{N^2}}\right\},$$

$$\Gamma_{x \to 0}^{-, \text{QED}}(N, a) = -N \left\{1 - \sqrt{1 + \frac{8a}{N^2} \left[1 - 2\sqrt{1 - \frac{8a}{N^2}}\right]}\right\}, \qquad (20)$$

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[2] Note that there are a few misprints in Eq. (4.7) of Ref. [3].

$$\Gamma_{x\to 0}^{+,\mathrm{QCD}}(N,a) = -N\left\{1 - \sqrt{1 - \frac{8aC_F}{N^2}}\right\},$$

$$\Gamma_{x\to 0}^{-,\mathrm{QCD}}(N,a)$$

$$= -N\left\{1 - \sqrt{1 - \frac{8aC_F}{N^2}\left[1 - \frac{8aN_c}{N}\frac{d}{dN}\ln\left(e^{z^2/4}D_{-1/[2N_c^2]}(z)\right)\right]}\right\}, \quad (21)$$

where $z = N/\sqrt{2N_c a}$, and $D_p(z)$ denotes the function of the parabolic cylinder [11].

It is instructive to expand the resummed anomalous dimensions (20) and (21) into a series in a^k/N^{2k-1} and transform the result to x–space using

$$\mathcal{M}\left[\ln^k\left(\frac{1}{x}\right)\right](N) = \frac{k!}{N^{k+1}}. \quad (22)$$

This results in

$$K_{x\to 0}^{+,\mathrm{QED}}(x,a) = 2a + 2a^2\ln^2 x + \frac{2}{3}a^3\ln^4 x + \mathcal{O}(a^4\ln^6 x),$$

$$K_{x\to 0}^{-,\mathrm{QED}}(x,a) = 2a - 6a^2\ln^2 x - \frac{10}{3}a^3\ln^4 x + \mathcal{O}(a^4\ln^6 x), \quad (23)$$

$$K_{x\to 0}^{+,\mathrm{QCD}}(x,a) = 2aC_F + 2a^2C_F^2\ln^2 x + \frac{2}{3}a^3C_F^3\ln^4 x + \mathcal{O}(a^4\ln^6 x),$$

$$K_{x\to 0}^{-,\mathrm{QCD}}(x,a) = 2aC_F + 2a^2C_F\left[C_F + \frac{2}{N_c}\right]\ln^2 x$$

$$+ \frac{2}{3}a^3C_F\left[C_F^2 - \frac{3}{2N_c}\right]\ln^4 x + \mathcal{O}(a^4\ln^6 x). \quad (24)$$

Eqs (23)–(24) agree with the corresponding result found for $P_{NS,x\to 0}^{\pm}(x,a)$ in (8)–(9) in the complete NLO calculations of the non–singlet anomalous dimensions [9] in the most singular terms since

$$C_G - \frac{3}{2}C_F = \frac{1}{N_c} + \frac{1}{2}C_F \quad (25)$$

holds in SU(N_c). Moreover, $K_{x\to 0}^{-,\mathrm{QED}}(x,a)$ can be compared directly with a result in Ref. [10], Eqs (2.30), (2.40), (2.43), restricting to the terms $\propto a^2\ln^2 x$, where the '–'-non–singlet[3] terms were given separately in the OMS scheme. Since the corrections there refer to the initial state radiation

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[3] Note that the singlet contributions contain terms $\propto 1/x$ also in the case of QED.

in e^+e^- annihilation the NLO result for a single (massless) fermion line reads

$$K_{1,x\to 0}^{-,\text{QED}}\Big|_{\text{OMS}}(x,a) = \tfrac{1}{2}\left(\frac{\alpha}{\pi}\right)^2\left[\delta_{e^+e^-}^{\text{I}} + \delta_{e^+e^-}^{\text{II}} + \delta_{e^+e^-}^{\text{IV}}\right]$$

$$\equiv \tfrac{1}{2}\left(\frac{\alpha}{\pi}\right)^2\left[-\tfrac{1}{4} + 0 - \tfrac{1}{2}\right]\ln^2 x$$

$$= -6a^2\ln^2 x = K_{1,x\to 0}^{-,\text{QED}}\Big|_{\overline{\text{MS}}}(x,a). \tag{26}$$

The corresponding result for $K_{x\to 0}^{+,\text{QED}}$ can not be derived directly.

In the evolution equation (10) aside from the anomalous dimensions $P_l^{\pm}(x)$ also the coefficient functions $c_{i,l}^{\pm}(x)$ contribute. The latter quantities have been calculated to $O(a^2)$ (*i.e.* $l=2$) [12]–[14] for $xF_3(x,Q^2)$ and the non–singlet part of the structure functions $F_2(x,Q^2)$ and $g_1(x,Q^2)$ in the $\overline{\text{MS}}$ scheme. Expanding these coefficient functions for $x \to 0$ one finds after noticing that apparent terms $\propto 1/x^m$, $m = 1,2$ cancel in the corresponding expressions of Ref. [13, 14]

$$c_{i,1} \propto \alpha_s \ln\left(\frac{1}{x}\right), \quad c_{i,2} \propto \alpha_s^2 \ln^3\left(\frac{1}{x}\right). \tag{27}$$

Hence the terms of $O(a^2)$ and $O(a^3)$ in (23)–(24) can be identified with the parts of the $\overline{\text{MS}}$ non–singlet splitting functions $\propto a(a\ln^2 x)^l$. Thus one obtains

$$P_{2,x\to 0,\overline{\text{MS}}}^{+,\text{QED}}(x,a) = \frac{2}{3}a^3\ln^4 x,$$

$$P_{2,x\to 0,\overline{\text{MS}}}^{-,\text{QED}}(x,a) = -\frac{10}{3}a^3\ln^4 x, \tag{28}$$

$$P_{2,x\to 0,\overline{\text{MS}}}^{+,\text{QCD}}(x,a) = \frac{2}{3}C_F^3 a^3\ln^4 x,$$

$$P_{2,x\to 0,\overline{\text{MS}}}^{-,\text{QCD}}(x,a) = \left(-\frac{10}{3}C_F^3 + 4C_F^2 C_G - C_F C_G^2\right)a^3\ln^4 x. \tag{29}$$

It should be noted that the agreement of the NLO terms between (23)–(24) obtained from the above resummation and (8)–(9) holds for $q^2 < 0$ only. This is due to the violation of the Gribov–Lipatov relation in the $\ln^2 x$ term of the NLO splitting functions for $q^2 > 0$.

4. Numerical results for nucleon structure functions

We now write down the solution of the evolution equation derived in the previous sections and then study the quantitative consequences of the leading small-x resummation. In the following we will confine ourselves to the QCD case. Here we make use of the fact that the evolution equation (12) for non-singlet structure function combinations reduces to a single ordinary differential equation after transformation to Mellin moments. Including the effect of the resummed kernels (21), the corresponding solution can be written as

$$
F^{\pm}(N, a_s) = F^{\pm}(N, a_0) \left(\frac{a_s}{a_0} \right)^{\gamma_{\mathrm{NS},0}(N)/2\beta_0}
$$

$$
\times \left\{ \exp\left[\frac{1}{2\beta_0} \int\limits_{a_0}^{a_s} da \, \frac{1}{a^2} \Gamma^{\pm}(N, a_s) \right] \right.
$$

$$
\left. + \frac{a_s - a_0}{2\beta_0} \left[\tilde{\gamma}_1^{\pm}(N) - \frac{\beta_1}{\beta_0} \gamma_{\mathrm{NS},0}(N) + 2\beta_0 \hat{c}_{i,1}(N) \right] \right\} \quad (30)
$$

with

$$
\gamma_i^{\pm}(N) = -2 \int\limits_0^1 dx \, x^{N-1} P_i^{\pm}(x), \quad \hat{c}_i^{\pm}(N) = \int\limits_0^1 dx \, x^{N-1} c_i^{\pm}(x), \quad (31)
$$

and $a_0 = a_s(Q_0^2)$. In (30) $\tilde{\gamma}_1^{\pm}(N)$ stands for the two-loop anomalous dimension $\gamma_1^{\pm}(N)$ with the leading $1/N^3$ term obvious from (9) removed[4]. This latter contribution is already included in the exponential factor, which in turn is connected to (21) via the subtraction of the contribution linear in a_s,

$$
\Gamma^{\pm}(N, a_s) = \Gamma_{x \to 0}^{\pm}(N, a_s) - \frac{a_s}{N} \lim_{N \to 0} [N \gamma_{\mathrm{NS},0}(N)]
$$

$$
= \Gamma_{x \to 0}^{\pm}(N, a_s) + a_s \frac{4C_F}{N}. \quad (32)
$$

The well-known NLO evolution of $F^{\pm}(N, a_s)$ is entailed in (30) by simply expanding the exponential to first order in a_s and a_0. Finally, the transformation of the solution back to x-space at any x and Q^2 affords only one standard numerical integral in the complex N-plane [15].

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[4] In (30) and (32) we have corrected two trivial misprints in Eqs. (21) and (23) of Ref. [4].

The remaining quadrature in (30) can be performed analytically for the '+'-case, resulting in

$$
\int_{a_0}^{a_s} da\, \frac{1}{a^2}\Gamma^+(N, a_s)
$$

$$
= \frac{NA}{2}\ln\frac{a_s}{a_0} + N\left(\frac{1}{a_s} - \frac{1}{a_0}\right) - N\left\{\frac{\sqrt{1 - Aa_s}}{a_s} - \frac{\sqrt{1 - Aa_0}}{a_0}\right\}
$$

$$
-\frac{NA}{2}\ln\frac{(1 - \sqrt{1 - Aa_s})(1 + \sqrt{1 - Aa_0})}{(1 + \sqrt{1 - Aa_s})(1 - \sqrt{1 - Aa_0})}, \quad A = \frac{8C_F}{N^2}. \tag{33}
$$

On the other hand, the corresponding integration has been carried out numerically for the '−'-combinations involving the parabolic cylinder function $D_p(z = N/\sqrt{2N_c a_s})$. Alternatively, one can expand the resummed kernels $\Gamma^+(N, a_s)$ and $\Gamma^-(N, a_s)$ in the strong coupling a_s, using the Taylor series of the square root and the asymptotic expansion[5] of $D_p(z)$, respectively. In the practical applications considered below, one finds that, even at the lowest x-values considered, more than 90% of the resummation effects in (30) arise from the first two terms beyond NLO in the α_s expansion of $\Gamma^\pm(N, a_s)$.

Let us consider the quantitative consequences of the resummation (21) for two representative non-singlet combinations. For this purpose, we choose $Q_0^2 = 4\ \text{GeV}^2$ as our reference scale in (30), and employ the same initial distributions $F^\pm(N, a_0)$ and Λ_{QCD} for the NLO and the resummed calculations. The evolution is performed for $N_f = 4$ active (massless) quark flavours. Unless another value is stated explicitly, we take $\Lambda \equiv \Lambda_{\overline{\text{MS}}}(N_f = 4) = 230\ \text{MeV}$ in

$$
a_s(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)}\right]. \tag{34}
$$

We start with the unpolarized case, where we investigate the evolution of the '+'-combination[6]

$$
F_2^{ep}(x, Q_0^2) - F_2^{en}(x, Q_0^2) = c_{F_2}^+(x, Q_0^2)\otimes\tfrac{1}{3}\left[xu_v - xd_v - 2(x\bar{d} - x\bar{u})\right](x, Q_0^2), \tag{35}
$$

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[5] For $xM.10^{-7}$ the asymptotic series is no longer reliable and one has to refer to the resummed result directly.

[6] For corresponding results on the '−'-combination $xF_3(x, Q^2)$ the reader is referred to Ref. [4].

adopting the input densities from the MRS(A) [16] global fit. The small-x behaviour of the most relevant quantities is given by $xu_v(x, Q_0^2) \sim x^{0.54}$, $xd_v(x, Q_0^2) \sim x^{0.33}$. Note that these distributions are rather 'steep', *i.e.* their rightmost singularity in the complex N-plane lies about 0.5 units or more to the right of the leading singularity of the non-singlet splitting functions at $N = 0$. Studying the evolution of this F_2 difference at very small x is mainly of theoretical interest, since it is orders of magnitude smaller than F_2^{ep} and F_2^{en} there. In figure 1 the result of this investigation is depicted down to x as low as 10^{-15}. Even at these extremely low values of x, the effect of the resummed anomalous dimensions stays at the level of 1% or below, and is still dominated entirely by the first two α_s terms beyond NLO.
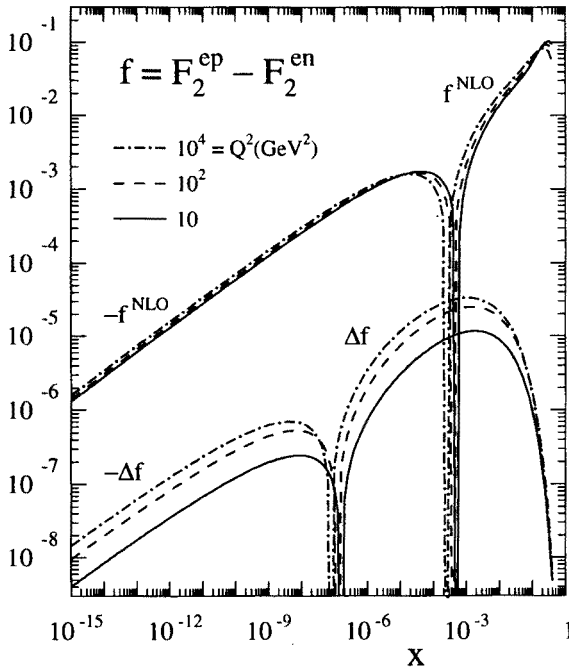

Fig. 1. The small-x Q^2–evolution of the unpolarized non–singlet structure function combination $F_2^{ep} - F_2^{en}$ in NLO and the absolute corrections to these results due to the resummed kernel derived from Ref. [3]. The initial distributions at $Q_0^2 = 4 \text{ GeV}^2$ have been adopted from [16].

In the polarized case we consider the corresponding difference

$$g_1^{ep}(x, Q_0^2) - g_1^{en}(x, Q_0^2) = c_{g_1}^-(x, Q_0^2) \otimes \tfrac{1}{6} \left(\Delta u_v - \Delta d_v \right)(x, Q_0^2). \qquad (36)$$

This case is practically much more interesting, firstly since — unlike in the unpolarized case — the non-singlet distributions are not a priori suppressed

here with respect to the singlet ones at very low x, and secondly since the shapes of the polarized initial distributions are not yet well established. We illustrate the strong dependence of the resummation effects on the latter quantities by choosing two partly rather different input sets for Δu_v and Δd_v. At first, we take those of CW [17], using $x_0 = 0.75$ in Eq. (12) of Ref. [17], which have been used in several theoretical investigations [14, 4]. The small behaviour of this input is relatively flat, $\Delta u_v \sim x^{-0.17}$ and $\Delta d_v \sim x^{+0.29}$ at small x. As an example for a more recent parametrization we adopt the 'standard' NLO set of GRSV [18] as an input, using their value of the scale parameter, $\Lambda_{\overline{MS}}(N_f = 4) = 200$ MeV. Here one has $\Delta u_v \sim x^{-0.28}$, $\Delta d_v \sim x^{-0.67}$, hence the steepness is similar to that of the unpolarized initial distributions above. We have evolved both distribution sets in NLO from their respective input scales, $Q^2 = 10$ GeV^2 in [17] and $Q^2 = 0.34$ GeV^2 in [18], to our reference scale $Q_0^2 = 4$ GeV^2.

Before we derive the quantitative results, we notice that Eq. (30) violates the fermion number conservation for the '−' non–singlet combinations. Here the conjecture is that the coefficient functions $c_{i,l}^{\pm}(x)$ do not contain terms $\propto \ln^{2l} x$ in the $\overline{MS}$ scheme. For this no proof exists yet, however, we have verified this behaviour up to 2–loop order in Section 3 for the coefficient functions of xF_3, F_2^{NS}, and g_1^{NS}. One should recall that the main resummation effect comes from that and the next order. Under this assumption fermion number conservation has to be restored for $\Gamma_{x\to 0}^{-}(N, a_s)$. We approach this problem in several ways numerically. In a first set of calculations we subtract a corresponding term $\propto \delta(1 - x)$ from the kernels K^- derived from (21), in each order in a_s. In N–space this prescription (denoted by 'A' below) leads to

$$\Gamma^-(N, a_s) \to \Gamma^-(N, a_s) - \Gamma^-(1, a_s) \, . \tag{37}$$

Another possibility is the restoration of fermion number conservation by subleading $1/N$ pole terms. An especially simple choice (denoted by 'B' in the following) is to modify Γ^- according to

$$\Gamma^-(N, a_s) \to \Gamma^-(N, a_s) \cdot (1 - N) \, . \tag{38}$$

Besides these two prescriptions, which are analogous to the procedure in the second reference in [2] with respect to energy–momentum conservation in the unpolarized singlet case, we will also show the results for two other assumptions, namely ('C')

$$\Gamma^-(N, a_s) \to \Gamma^-(N, a_s)(1 - 2N + N^2) \tag{39}$$

and ('D')

$$\Gamma^-(N, a_s) \to \Gamma^-(N, a_s)(1 - 2N + N^3) \, . \tag{40}$$

Clearly, the results of the resummed calculation are only trustworthy, and this approach is to be preferred over a fixed order calculation, if the difference of the results obtained by all these procedures is small.

The corresponding results are presented in figure 2. For the relatively flat CW input [17], the effect is up to about 15% at $x = 10^{-5}$. However, in the kinematical range accessible for polarized electron and proton scattering at HERA [19] it again amounts to about 1% or less. Note, moreover, the wide spread of the results in dependence of the employed fermion-number conservation prescription. Obviously the resummed contributions do not sufficiently dominate with respect to subleading terms at any foreseeable energy. For the steep GRSV input [18], the effect is of approximately the same marginal size as in the unpolarized case considered above. In figure 2 relative corrections are shown. Recall that the absolute values for
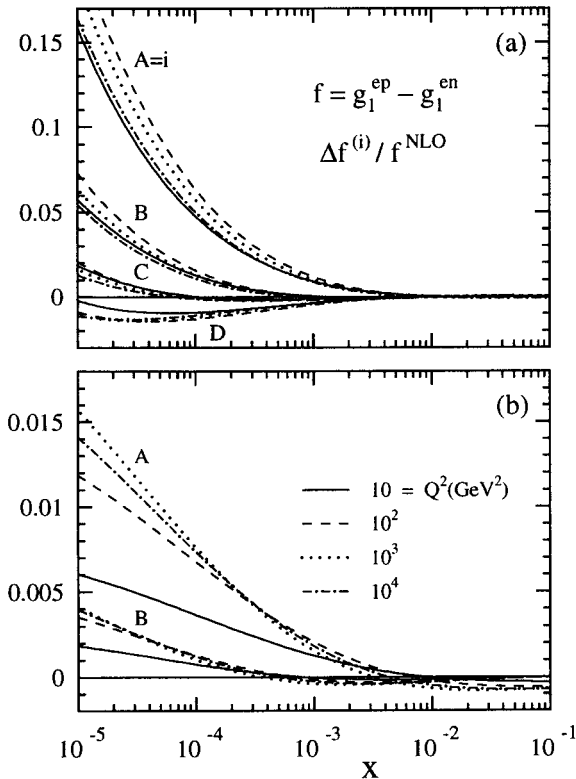

Fig. 2. The relative corrections to the NLO small-x Q^2–evolution of the polarized non–singlet structure–function difference $g_1^{ep} - g_1^{en}$ due to the resummed kernel. 'A', 'B', 'C', and 'D' denote the different prescriptions for implementing the fermion number conservation discussed in the text. The initial distribution at Q_0^2 have been taken from Ref. [17] in (a) and Ref. [18] in (b).

g_1 obtained in the different parametrizations extrapolating from the range $x \geq 10^{-2}$ of the current data down to smaller x values vary strongly [19, 20].

5. Conclusions

We have investigated the resummation of terms of order $\alpha^{l+1} \ln^{2l} x$, derived in Ref. [3] for the QCD case, on the small-x behaviour of non–singlet functions in QED and QCD. The comparison with the corresponding contributions obtained in the same order by complete NLO calculations shows the equivalence of both approaches in this limit up to order α^2 in both QED and QCD. Since the coefficient functions up to two–loop order for the non–singlet combinations considered contain only terms less singular in $\ln x$ in the $\overline{\text{MS}}$ scheme, the contributions $\propto a^3 \ln^4 x$ in the three–loop $\overline{\text{MS}}$ splitting functions $P^{\pm}(x, a)$ have been predicted on the basis of this resummation.

A numerical analysis has been performed for the QCD case of deep-inelastic (polarized) lepton scattering both off unpolarized and polarized targets. It turns out that the all–order resummation of the terms $\mathcal{O}(\alpha_s^{l+1} \ln^{2l} x)$ leads only to corrections on the level of 1% in the unpolarized case of $F_2^p - F_2^n$ even down to extremely small x values, $x = 10^{-15}$. The corrections can be larger in the polarized case, up to about 15 % at $x \simeq 10^{-5}$, depending on the presently not yet well established small-x behaviour of the polarized parton densities. In any case, the resummation effects are on the level of 1% in the kinematical range accessible experimentally at present or in the foreseeable future.

Presently unknown terms which are suppressed by powers of $\ln x$ in the splitting functions do contribute in a potentially significant way to the evolution even at the lowest x-values considered. This has been demonstrated for the '–' combination $g_1^p - g_1^n$ by applying several prescriptions to implement fermion number conservation into the resummed evolution equations. Moreover, the resummation corrections are dominated by the first two terms in an α_s expansion beyond NLO. All this indicates that fixed-order perturbation theory remains the appropriate theoretical framework for the evolution of non-singlet structure functions even at very small x.

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