

QCD COUPLING CONSTANT AT FINITE TEMPERATURE

M. CHAICHIAN

High Energy Physics Laboratory, Department of Physics
and Research Institute for High Energy Physics, University of Helsinki
P.O.Box 9, Siltavuorenpenger 20 C, FIN-00014 Helsinki, Finland

AND

M. HAYASHI

School of Life Science, Tokyo University of Pharmacy and Life Science
1432-1 Horinouchi, Hachioji, Tokyo 192-03, Japan

(Received March 25, 1996)

Dedicated to Andrzej Bialas in honour of his 60th birthday

We work out the method for evaluating the QCD coupling constant at finite temperature (T) by making use of the finite T renormalization group equation up to the one-loop order on the basis of the background field method with the imaginary time formalism. The background field method, which maintains the explicit gauge invariance, provides notorious simplifications since one has to calculate only the renormalization constant of the background field gluon propagator. The results for the evolution of the QCD coupling constant at finite T reproduce partially the ones obtained in the literature. We discuss, in particular, the origin of the discrepancies between different calculations, such as the choice of gauge, the break-down of Lorentz invariance, imaginary versus real time formalism and the applicability of the Ward identities at finite T .

PACS numbers: 12.38. Aw, 12.38. Lg, 11.10. Wx

1. Introduction

One of the application of field theory at finite T [1-4] is to find the behaviour of coupling constant as a function of energy, temperature, and

the chemical potential using the renormalization group (RG) equation. The knowledge of coupling constant at finite T environment then can be used, for instance, in perturbative calculations for quark-gluon plasma created in ion-ion collisions at high energies, in the evaluation of the grand unification scale in a cosmological context and the shift of the energy levels in a hydrogen plasma.

A great deal of attention has been paid to the study of the behaviour of the QCD coupling constant at finite T [5–19]. The resulting formula for the temperature and scale dependent part of the coupling constant has proved to be very sensitive to the prescription chosen. In this paper we wish to reexamine this and clarify the origin of the discrepancies between different calculations in the literature. We employ the background field method (hereafter abbreviated as BFM) [20,21], which is based on a manifestly gauge invariant generating functional. The BFM provides notorious simplifications since we have to calculate only the renormalization constant of the background field (BF) gluon propagator. We discuss this method in Section 2.1. Since the coupling constant depends not only on energy (μ) but also on T , we derive a pair of RG equations — one for μ and one for T . This is done in Section 2.2 where we also derive its solution.

In order to get the renormalization constant of the BF gluon propagator we have to calculate the polarization tensor. At finite T due to the lack of the Lorentz invariance the structure of the polarization tensor is not equivalent to the one at $T = 0$. In Section 3.1 we give a prescription how to define the renormalization constant at finite T . In Section 3.2 we calculate the polarization tensor in the one-loop approximation. In Section 3.3 calculating the vacuum part of the polarization tensor we reproduce the well known formula for $T = 0$ [22, 23], which reads as

$$[g^2(\mu)]^{-1} = [g_0^2(\mu_0)]^{-1} [1 + 2g_0^2(\mu_0)K_0 \ln(\mu/\mu_0)] , \quad (1.1)$$

where

$$K_0 = (11N/3 - 2n_F/3)/(4\pi)^2 . \quad (1.2)$$

Here N is for $SU(N)$, n_F is the number of flavours, μ is the energy scale and μ_0 is the reference scale.

We calculate the matter part of the polarization tensor in Section 4.1 and, in Section 4.2, we derive the formulae for T and μ dependent part of the coupling constant.

In Section 5.1 we review the results of the previous works and in Section 5.2 we compare the asymptotic expansion formulae at $a = \mu/T \ll 1$ derived in different schemes. We discuss, in particular, the origin of the discrepancies between different calculations and give a few comments.

2. Formal methods

2.1. Background field method

Let us start with the generating functional for QCD:

$$Z(J) = \int DAD\Psi D\bar{\Psi} D\eta D\bar{\eta} \exp \left\{ i \int d^4x [\mathcal{L}_{\text{QCD}}(A, \Psi, \bar{\Psi}) - (G^a)^2/(2\xi) + J^{\mu a} A_{\mu a} + \bar{\eta} \partial G^a / \partial \omega^a \eta] \right\}, \quad (2.1)$$

where A is the gluon field, Ψ and $\bar{\Psi}$ are the fermion fields, η and $\bar{\eta}$ are the ghost fields, $(G^a)^2$ is a gauge fixing term with ξ being a gauge parameter, ω^a is a SU(3) group parameter, and $J^{\mu a} A_{\mu a}$ is a source term. In SU(3) the gauge fields transform as:

$$A_\mu \rightarrow U(A_\mu - i\partial_\mu/g)U^\dagger, \quad (2.2)$$

$$\Psi \rightarrow U\Psi, \bar{\Psi} \rightarrow \bar{\Psi}U^\dagger, \quad (2.3)$$

$$\eta \rightarrow U\eta, \bar{\eta} \rightarrow \bar{\eta}U^\dagger, \quad (2.4)$$

where $U(x) = \exp[i\omega^a(x)T_a]$ is a unitary transformation with T_a being a generator for SU(3). By using transformations (2.2)–(2.4) one can show that the gauge invariance of $Z(J)$ is lost in commonly used gauges such as $\partial_\mu A_\mu = 0$. Thus we see that the explicit gauge invariance, which is present at the classical level in gauge field theories, is lost at the quantum level.

The advantage of the BFM [20, 21] is that it can maintain the explicit gauge invariance. For this purpose we divide the gluon field A_μ into a sum of a classical BF B_μ and a quantum field Q_μ as

$$A_\mu = B_\mu + Q_\mu, \quad (2.5)$$

and choose for the gauge fixing term G^a the BF gauge condition

$$G^a = (D_\mu(B))^{ab} Q_b^\mu, \quad (2.6)$$

where D_μ is the covariant derivative:

$$(D_\mu)_{ab} = \partial_\mu \delta_{ab} + g f_{abc} B_\mu^c. \quad (2.7)$$

Then the generating functional reads

$$Z(J, B) = \int DQD\Psi D\bar{\Psi} D\eta D\bar{\eta} \int \exp \left\{ i \int d^4x [\mathcal{L}_{\text{QCD}}(B + Q, \Psi, \bar{\Psi}) - (G^a)^2/(2\xi) + J^{\mu a} A_{\mu a} + \bar{\eta} \partial G^a / \partial \omega^a \eta] \right\}. \quad (2.8)$$

This functional can be shown to be gauge invariant by using the transformations:

$$B_\mu \rightarrow U(B_\mu - i\partial_\mu/g)U^\dagger, \quad (2.9)$$

$$Q_\mu \rightarrow UQ_\mu U^\dagger, \quad (2.10)$$

$$D_\mu(B) \rightarrow UD_\mu(B)U^\dagger, \quad (2.11)$$

$$J_\mu(B) \rightarrow UJ_\mu(B)U^\dagger, \quad (2.12)$$

in addition to Eqs (2.3) and (2.4). Actually we are only changing the integration variables in Eq. (2.8).

In the BFM the quantum gauge fields and the ghost field need not be renormalized since they appear only inside loops. Thus only renormalizations

$$g_0 = Z_g g_R, \quad (2.13)$$

$$B_0 = (Z_B)^{1/2} B_R, \quad (2.14)$$

$$\xi_0 = Z_\xi \xi_R, \quad (2.15)$$

are needed.

The explicit gauge invariance of $Z(J, B)$ implies that the perturbation series is gauge invariant in every order in g_R and that Z_B and Z_g are related with each other. In order that the field tensor $F_{\mu\nu}^a$ in \mathcal{L}_{QCD} :

$$F_{\mu\nu}^a = (Z_B)^{1/2} \left[\partial_\mu (B_\nu^a)_R - \partial_\nu (B_\mu^a)_R + g_R Z_g (Z_B)^{1/2} f^{abc} (B_\mu^b)_R (B_\nu^c)_R \right] \quad (2.16)$$

be gauge covariant one has to have

$$Z_g = (Z_B)^{-1/2}, \quad (2.17)$$

which enables us to calculate the evolution of coupling constant g_R in a simple way. We assume this relation to be valid also at finite T .

2.2. Coupled RG equations

Equations (2.13) and (2.17) lead us to:

$$g_0 = g_R (Z_B)^{-1/2}. \quad (2.18)$$

Using the dimensional regularization [24, 23] let us perform our calculations in $(4 - 2\varepsilon)$ dimensions. We notice from the QCD Lagrangian that the

dimension of a gluon (quark) field is $\mu^{1-\epsilon}$ ($\mu^{3/2-\epsilon}$, resp.). We redefine the bare coupling g_0 so that it becomes dimensionless and rewrite Eq. (2.18) as

$$g_0 = g_R(Z_B)^{-1/2} \mu^\epsilon, \tag{2.19}$$

where g_0 does not depend on T and μ . Then a pair of RG equations results upon taking the derivative of Eq. (2.19) with respect to T and μ

$$T \partial / \partial T [g_R(Z_B)^{-1/2} \mu^\epsilon] = 0, \tag{2.20}$$

$$\mu \partial / \partial \mu [g_R(Z_B)^{-1/2} \mu^\epsilon] = 0. \tag{2.21}$$

A pair of these equations should determine the behaviour of g_R with respect to the changes in T and μ . Generally Z_B has the form [22, 23]

$$Z_B = 1 + \sum_{i=1}^{\infty} g_R^{2i} \left[\sum_{j=1}^i A_i^{(j)} / \epsilon^j + f^{(i)}(\mu, T) \right], \tag{2.22}$$

where $A_j^{(j)} / \epsilon^j$ are the divergent contributions independent on T and μ , while $f^{(j)}(\mu, T)$ are convergent functions depending on T and μ which vanish in the low- T limit.

Writing hereafter $g = g_R$, for simplicity, one can derive the following RG equation from Eqs (2.21) and (2.22) by taking the limit $\epsilon \rightarrow 0$:

$$\mu \partial g / \partial \mu = g^3 \{ -A^{(1)} + (\mu/2) \partial / \partial \mu [f^{(1)}(\mu, T)] \} + O(g^5). \tag{2.23}$$

This equation is reduced to the ordinary one-loop RG equation in the limit $T \rightarrow 0$ since the function $f^{(1)}(\mu, T = 0)$ vanishes.

Similarly one can derive the RG equation for T from Eqs (2.20) and (2.22) as:

$$T \partial g / \partial T = (g^3/2) T \partial / \partial T [f^{(1)}(\mu, T)] + O(g^5). \tag{2.24}$$

Equations (2.23) and (2.24) constitute the coupled RG equations. One can easily obtain the solution to these coupled RG equations by integrating Eq. (2.23) from μ_0 to μ and Eq. (2.24) from T_0 to T . The desired solution reads as

$$[g^2(\mu, T)]^{-1} = [g^2(\mu_0, T_0)]^{-1} + 2A^{(1)} \ln(\mu/\mu_0) - [f^{(1)}(\mu, T) - f^{(1)}(\mu_0, T_0)]. \tag{2.25}$$

This equation describes the evolution of the QCD coupling constant as a function of T and μ [5–10]. (μ_0, T_0) denotes the reference point.

3. Polarization tensor

3.1. Structure of the polarization tensor

At $T = 0$ environment the polarization tensor is Lorentz invariant and can be expressed as [22]

$$\Pi_{\mu\nu} = \Pi(k^2 g_{\mu\nu} - k_\mu k_\nu). \quad (3.1)$$

At $T = 0$ it satisfies the transversality condition (current conservation):

$$k_\mu \Pi_{\mu\nu} = 0. \quad (3.2)$$

At finite T , in the presence of matter, the Lorentz invariance is lost and the polarization tensor can only be $O(3)$ rotational invariant [27, 17]. Then it can generally depend only on 4 independent quantities, which we can choose, for example, Π_{00} , $k_i \Pi_{0i}$ and the two scalars Π_L and Π_T appearing in

$$\Pi_{ij} = \Pi_T(\delta_{ij} - k_i k_j / k^2) + \Pi_L k_i k_j / k^2. \quad (3.3)$$

At finite T whether the transversality condition is satisfied or not depends on the gauge used. In the Coulomb gauge ($\partial_i A_i = 0$), for example, it is not transversal but is transversal at the one-loop level in the temporal axial gauge ($A_0 = 0$), and in every order of the perturbative calculations in the BF gauge [17].

From Eq. (3.2) we have

$$k_0 \Pi_{00} = k_i \Pi_{i0}, \quad (3.4)$$

and from $k_\mu k_\nu \Pi_{\mu\nu} = 0$, we have

$$k_0^2 \Pi_{00} = k_i k_j \Pi_{ij}. \quad (3.5)$$

Using Eqs (3.3) and (3.5) we obtain

$$\Pi_L = k_0^2 \Pi_{00} / k^2. \quad (3.6)$$

For the coefficient of the transversal part of the polarization tensor one can derive from Eqs (3.3) and (3.6) an expression

$$\Pi_T = [\Pi_{\mu\mu} - \Pi_{00}(1 + k_0^2/k^2)]/2. \quad (3.7)$$

Thus we have derived in Eqs (3.3)–(3.7) the general form of the T dependent $O(3)$ symmetric polarization tensor in the BFM.

The polarization tensor can be split into a sum of a T independent (vacuum) part and a T and μ dependent (matter) part [4]:

$$\Pi_{\mu\nu} = \Pi_{\mu\nu}|_{\text{vac}} + \Pi_{\mu\nu}|_{\text{matt}} . \quad (3.8)$$

Since the polarization tensor at $T = 0$ is related to the gluon propagator, $D_{\mu\nu}$, as

$$\Pi_{\mu\nu} = D_{\mathbf{R}\mu\nu}^{-1} - D_{0\mu\nu}^{-1} = \Pi D_{\mathbf{R}\mu\nu}^{-1} , \quad (3.9)$$

and the relation between the bare and the renormalized propagators (see Eq. (2.14)) is

$$D_{0\mu\nu} = Z_{\mathbf{B}}^{-1} D_{\mathbf{R}\mu\nu} , \quad (3.10)$$

one is led to

$$Z_{\mathbf{B}} = 1 - \Pi . \quad (3.11)$$

Thus the renormalization constant $Z_{\mathbf{B}}$ can be obtained from the BF gluon self-energy tensor Π .

In investigating the behaviour of the coupling constant at finite T one has to include the T dependent parts of Π in $Z_{\mathbf{B}}$ via Eq. (3.11). In determining $Z_{\mathbf{B}}$ we encounter another type of ambiguity which is caused by the lack of the Lorentz invariance. In order to define Π at finite T we generalize Eq. (3.11) as:

$$Z_{\mathbf{B}} = 1 - \Pi|_{\text{vac}} - \Pi|_{\text{matt}} , \quad (3.12)$$

where we have either

$$\Pi|_{\text{matt}} = \Pi_{00}/\mathbf{k}^2 , \quad (3.13)$$

or

$$\Pi|_{\text{matt}} = \Pi_T/\mathbf{k}^2 . \quad (3.14)$$

Π_{00} and Π_T are not connected with each other and in general there is no a priori way to decide which one is more natural.

3.2. The polarization tensor at the one-loop level

Our working formulae, which allow to analyze the evolution of QCD coupling constant at finite T , are: Eq. (2.22), Eq. (2.25) and Eqs (3.12)–(3.14). Accordingly we have to evaluate the self-energy diagrams in Fig. 1 to get $Z_{\mathbf{B}}$ up to the one-loop order. The Feynman rules for the interaction vertices are the same as in the $T = 0$ case, and therefore identical to those given in [21]. Evaluating the one-loop diagrams (1a)–(1d) in the Feynman gauge, we find for the boson contributions in the polarization tensor

$$\begin{aligned} \Pi_{\mu\nu}|_{\text{boson}} = & ig^2 N \int d^4 p / (2\pi)^4 [4g_{\mu\nu} k^2 + 2(k_\mu p_\nu + k_\nu p_\mu) \\ & + 4p_\mu p_\nu - 3k_\mu k_\nu - 2(k+p)^2 g_{\mu\nu}] / \{(k+p)^2 p^2\} . \end{aligned} \quad (3.15)$$

For the fermion loop, neglecting the quark masses compared to μ and T , we have from the diagram (1e):

$$\begin{aligned} \Pi_{\mu\nu}|_{\text{fermion}} = & -4ig^2 T_F n_F \int d^4 p / (2\pi)^4 \{k_\mu p_\nu + k_\nu p_\mu \\ & + 2p_\mu p_\nu - g_{\mu\nu}[(kp) + p^2]\} / \{(k+p)^2 p^2\}, \end{aligned} \quad (3.16)$$

where $T_F = 1/2$ for SU(3).

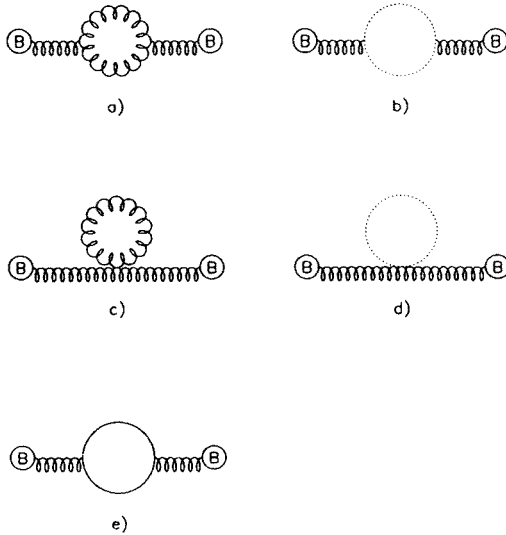


Fig. 1. Diagrams for the one-loop calculation of the BF renormalization factor Z_B : a) Gluon loop; b) Ghost loop; c) Gluon loop from the 4-gluon vertex; d) Ghost loop from the 2-gluon 2-ghost vertex; e) Fermion loop. Wavy lines are quantum gauge field propagators. Wavy lines terminating in a "B" represent external gauge particles. Solid lines are fermion propagators and dashed lines represent ghost propagators.

Our polarization tensors satisfy the transversality condition

$$k_\mu \Pi_{\mu\nu}|_{\text{boson}} = k_\mu \Pi_{\mu\nu}|_{\text{fermion}} = 0. \quad (3.17)$$

3.3. Vacuum part of the polarization tensor

Using a standard technique and introducing the Feynman parametrization in Eq. (3.15) we get the formula for the boson contributions:

$$\begin{aligned} \Pi_{\mu\nu}|_{\text{boson}} = & ig^2 N \int d^4 p / (2\pi)^4 \int_0^1 dx [4g_{\mu\nu} k^2 + 2(1-2x)(k_\mu p_\nu + k_\nu p_\mu) \\ & + 4p_\mu p_\nu + (-3-4x+4x^2)k_\mu k_\nu] / [p^2 + k^2 x(1-x)]^2. \end{aligned} \quad (3.18)$$

Similarly we get the formula for the fermion contributions from Eq. (3.16):

$$\begin{aligned} \Pi_{\mu\nu}|_{\text{fermion}} &= -2ig^2 n_F \int d^4p/(2\pi)^4 \int_0^1 dx [2(-x + x^2)k_\mu k_\nu + (1 - 2x) \\ &\times (k_\mu p_\nu + k_\nu p_\mu) - g_{\mu\nu}(kp) + 2p_\mu p_\nu + g_{\mu\nu}k^2 x]/[p^2 + k^2 x(1 - x)]^2. \end{aligned} \quad (3.19)$$

Notice that all of the integrals are ultraviolet divergent and thus have to be regularized. For this purpose we employ the dimensional regularization [24, 23], which preserves gauge symmetries explicitly. The integrals for the vacuum parts become Euclidean if we change $ip_0 \rightarrow p_4$ and thus can be easily evaluated. We obtain the following results:

$$\Pi_{\mu\nu}|_{\text{boson}} = -11g^2 N/(3\alpha)(g_{\mu\nu}k^2 - k_\mu k_\nu)/\varepsilon + O(1), \quad (3.20)$$

and

$$\Pi_{\mu\nu}|_{\text{fermion}} = 2g^2 n_F/(3\alpha)(g_{\mu\nu}k^2 - k_\mu k_\nu)/\varepsilon + O(1), \quad (3.21)$$

where

$$\alpha = (4\pi)^2. \quad (3.22)$$

Combining Eqs (3.20) and (3.21) and taking into account Eqs (2.22), (3.1) and (3.11), we reproduce the standard formula for $A^{(1)}$ defined in Eq. (2.25), *i.e.*, Eq. (1.2) for $T = 0$. We end this subsection by emphasizing the simplicity of the calculation by the BFM in contrast to the conventional methods, *e.g.*, in the covariant gauge [7–9].

4. Temperature dependent parts of the polarization tensor

4.1. Calculation of Π_{00} and $\Pi_{\mu\mu}$

To specify the subtraction point in calculating Π_{00} and $\Pi_{\mu\mu}$ we employ the static limit of zero external energy, which is commonly used in the literature [5]. In this prescription the momentum k is specified to be space-like $k = (0, \mathbf{k})$ with $k^2 = -\mu^2$. Such a choice enables us to determine the static properties. Making use of the imaginary time formalism [25, 26] from Eqs (3.15) and (3.16) we obtain for Π_{00}

$$\begin{aligned} \Pi_{00} &= g^2 T^2 (N/6 + n_F/12) - 2g^2 N \mu^2 [4F_{B0}(a) - F_{B2}(a)] \\ &\quad - 2g^2 n_F \mu^2 [F_{F0}(a) - F_{F2}(a)], \end{aligned} \quad (4.1)$$

with $a = \beta\mu$. Here the boson and fermion functions $F_{in}(\beta\mu)$ ($i = B, F$) are defined as:

$$F_{in}(\beta\mu) = (1/\alpha) \int_0^\infty dx x^n N_i(\mu x/2) L, \quad (4.2)$$

with

$$L = \ln |(1+x)/(1-x)|. \quad (4.3)$$

Similarly we derive an expression for $\Pi_{\mu\mu}$:

$$\Pi_{\mu\mu} = g^2 T^2 (N/3 + n_F/6) - 2g^2 \mu^2 [11NF_{B0}(a) + 2n_FF_{F0}(a)]. \quad (4.4)$$

4.2. Temperature dependent part of the coupling constant

As pointed out in Section 3.1 we encounter an ambiguity in determining the renormalization constant Z_B as a direct consequence of the lack of Lorentz invariance. Here we write two formulae, one derived from Π_{00} and the other one from Π_T . In the prescription with $k = (0, \mathbf{k})$, one has from Eq. (3.7):

$$\Pi_T = (\Pi_{\mu\mu} - \Pi_{00})/2. \quad (4.5)$$

Thus from Eq. (3.13) with Π_{00} we have

$$\begin{aligned} -f^{(1)}(\mu, T) = & (N/6 + n_F/12)/a^2 - 2N[4F_{B0}(a) - F_{B2}(a)] \\ & - 2n_F[F_{F0}(a) - F_{F2}(a)], \end{aligned} \quad (4.6)$$

and the other one from Eq. (3.14) with Π_T

$$\begin{aligned} -f^{(1)}(\mu, T) = & (N/12 + n_F/24)/a^2 - N[7F_{B0}(a) + F_{B2}(a)] \\ & - n_F[F_{F0}(a) + F_{F2}(a)]. \end{aligned} \quad (4.7)$$

5. Discussion

5.1. Results of previous works

Gendenshtein [5] calculated the QCD coupling constant at finite T in the one-loop approximation by using the RG equation, the dimensional regularization, and the covariant gauge with a space-like normalization momentum $p^\mu = (0, \mu)$ and obtained

$$-f^{(1)}(\mu, T) = (N/3 + n_F/6)/a^2. \quad (5.1)$$

Kajantie *et al.* [6] studied the gauge field part of QCD with N colors. They used the $A_0 = 0$ gauge and defined two renormalization schemes by writing

$$D_{\mu\nu}^{-1} = D_{0\mu\nu}^{-1}(Z_A - G/p^2) - (F - G)P_{\mu\nu}^L, \quad (5.2)$$

where $D_{\mu\nu}$ is the gluon propagator and F, G and $P_{\mu\nu}^L$ come from the polarization tensor

$$\Pi_{\mu\nu} = FP_{\mu\nu}^L + GP_{\mu\nu}^T. \quad (5.3)$$

In the “magnetic prescription” they fixed the propagator at the point $p^\mu = (0, \mu)$ and had for the T depending function the expansions (without quarks)

$$- f^{(1)}(\mu, T) = 5N/(16a), \text{ for } a \ll 1, \quad (5.4)$$

$$- f^{(1)}(\mu, T) = N[17/(90a^2) + 83\alpha/(6300a^4)], \text{ for } a \gg 1. \quad (5.5)$$

In the so-called “electric prescription” they derived the coupling constant from F as

$$- f^{(1)}(\mu, T) = N[1/(3a^2) - 1/(4a) - 22(\ln a)/(3\alpha)], \text{ for } a \ll 1, \quad (5.6)$$

$$- f^{(1)}(\mu, T) = -N[1/(18a^2) - 11\alpha/(900a^4)], \text{ for } a \gg 1. \quad (5.7)$$

Notice here the change of sign in the high- T behaviour. They concluded that the “magnetic prescription” is more natural than the “electric” one because the former uses the physical part of the gluon propagator.

Nakagawa *et al.* [7] used the real-time formalism and studied the scale-parameter ratios $\Lambda(a)/\Lambda$ derived from different vertices in 4-flavour QCD. They used the covariant gauge and found that the ratios derived from the three-gluon vertex and the gluon-ghost vertex show just the opposite behaviour than the one derived from the gluon-quark vertex.

Fujimoto and Yamada [8] used the real-time formalism and derived the T depending coupling constant from the gauge-independent Wilson loop. At $a \ll 1$, it reads as (without quark contributions)

$$- f^{(1)}(\mu, T) = C[1/(3a^2) - 1/(4a) - 22(\ln a)/(3\alpha)]. \quad (5.8)$$

The same authors have discussed the finite T RG equations [9] in the one-loop approximation, using the real-time formalism, the covariant gauge and the dimensional regularization. Their results are summarized as follows. From the gluon propagator and three-gluon vertex:

$$\begin{aligned} - f^{(1)}(\mu, T) &= (C + T_f)/(4a^2) + C[-23/3F_{B0}(a) - 3F_{B2}(a) \\ &- 14/3G_{B0}(a) + 32G_{B2}(a)] + 2T_f[F_{F0}(a) - 3F_{F2}(a) + 152/9G_{F0}(a)], \end{aligned} \quad (5.9)$$

where $T_f = 1/2$, and $C = N$ for $SU(N)$.

From the fermion propagator and fermion-gluon vertex:

$$\begin{aligned} -f^{(1)}(\mu, T) &= (C + 3C_f + 2T_f)/(12a^2) - 2(C_f + 11C)F_{\mathbf{B0}}(a)/3 \\ &- 16(C_f + 10C)G_{\mathbf{B0}}(a)/9 - 32CG_{\mathbf{B2}}(a) + 2(-5C/6 + C_f/3 + T_f) \\ &\times F_{\mathbf{F0}}(a) - 4(8C_f + 17C)G_{\mathbf{F0}}(a)/9 + 16CG_{\mathbf{F2}}(a), \end{aligned} \quad (5.10)$$

where $C_f = (N^2 - 1)/(2N)$ for $SU(N)$.

From the ghost propagator and ghost-gluon vertex:

$$\begin{aligned} -f^{(1)}(\mu, T) &= (C + T_f)/(12a^2) - C[7F_{\mathbf{B0}}(a) + F_{\mathbf{B2}}(a) \\ &+ 2G_{\mathbf{B0}}(a)] - 2T_f[F_{\mathbf{F0}}(a) + F_{\mathbf{F2}}(a)]. \end{aligned} \quad (5.11)$$

The boson and fermion functions $G_{in}(a)$ ($i = \mathbf{B}, \mathbf{F}$) are defined as

$$G_{in}(a) = (2/\alpha) \int_0^\infty dx \int_0^1 dy x^{n+1} N_i(\mu x/2) / [x^2(y^2 + 3) - 1]. \quad (5.12)$$

The functions $G_{in}(a)$ ($i = \mathbf{B}, \mathbf{F}$) do not appear in our results in Eqs (4.6) and (4.7), since they come from the trigluon renormalization only.

Stephens *et al.* [19] performed a BF one-loop calculation of gauge invariant beta functions at finite T , using the retarded/advanced formalism developed by Aurenche and Becherrawy [28] and derived:

$$\begin{aligned} -f^{(1)}(\mu, T) &= (N/12 + n_{\mathbf{F}}/24)/a^2 - N[21/4F_{\mathbf{B0}}(a) \\ &+ F_{\mathbf{B2}}(a) + 7/2G_{\mathbf{B1}}(a)] - n_{\mathbf{F}}[F_{\mathbf{F0}}(a) + F_{\mathbf{F2}}(a)]. \end{aligned} \quad (5.13)$$

The numerical coefficients of the leading terms of the high- T expansion and the fermion parts are in complete agreement with our result in Eq. (4.7). In the boson parts, however, there are some numerical discrepancies with our result.

5.2. Comparison of asymptotic expansions

In order to compare our results with those mentioned in Section 5.1 we derive the asymptotic expansions for $-f^{(1)}(\mu, T)$. Equations (5.9)–(5.11) at $a = \mu/T \ll 1$ in the high- T regime read as follows:

$$\begin{aligned} -f^{(1)}(\mu, T) &= \pi C/(9\sqrt{3}a^2) - 23C/(48a) + [(-22 \\ &+ \pi/3\sqrt{3})C + (18 - 38\pi/9\sqrt{3})T_f](\ln a)/(3\alpha), \end{aligned} \quad (5.14)$$

$$\begin{aligned}
 -f^{(1)}(\mu, T) &= [(1 - \pi/\sqrt{3})C + 3C_f]/(12a^2) - (11C + C_f)/(24a) \\
 &- [(23 + 82\pi/9\sqrt{3})C + (4 - 8\pi/9\sqrt{3})C_f + 4T_f](\ln a)/(3\alpha), \quad (5.15)
 \end{aligned}$$

$$-f^{(1)}(\mu, T) = -7C/(16a) - [C(22 + \pi/\sqrt{3}) - 8T_f](\ln a)/(3\alpha). \quad (5.16)$$

Next we derive the asymptotic expansion for Eqs (4.6) at $a \ll 1$.

$$\begin{aligned}
 -f^{(1)}(\mu, T) &= (N/3 + n_F/6)/a^2 - N/(2a) \\
 &- [N(22\ell_1 - 71/3) - 4n_f(\ell_2 - 2/3)]/(3\alpha), \quad (5.17)
 \end{aligned}$$

where

$$\ell_1 = \ln(a/4\pi) + \gamma, \quad (5.18)$$

$$\ell_2 = \ln(a/\pi) + \gamma. \quad (5.19)$$

The T^2 term coincides with the one of Gendenshtein [5], *i.e.*, Eq. (5.1). The gluon part in this asymptotic formula is consistent with the result of Elze *et al.* [17] derived in the BFM:

$$-f^{(1)}(\mu, T) = N[1/(3a^2) - 1/(2a) - 22(\ln a)/(3\alpha) + \dots], \text{ for } a \ll 1. \quad (5.20)$$

This coincides with the result of Nadkharni [13], who has the terms up to $O(1/a)$. Next from Eq. (4.7) we find the asymptotic formula:

$$-f^{(1)}(\mu, T) = -7N/(16a) - [N(22\ell_1 - 127/6) - 4n_F(\ell_2 - 5/12)]/(3\alpha). \quad (5.21)$$

The behaviour of Π_T in the infrared region is known to have a form [17]

$$\lim \Pi_T(0, \mathbf{k})/\mu^2 = g^2[-cN/a + O(\ln a)]. \quad (5.22)$$

The factor c has been calculated in different gauges. In the covariant ξ -gauge its value is [27]

$$c = (9 + 2\xi + \xi^2)/64, \quad (5.23)$$

whereas in the temporal axial gauge it is [6]

$$c = 5/16. \quad (5.24)$$

Our formula (5.21) at small a behaves as Eq. (5.22) with

$$c = -7/16, \text{ for bosons,} \quad (5.25)$$

and

$$c = 0, \text{ for fermions.} \quad (5.26)$$

These results are consistent with the results of Refs [9] (see Eq. (5.17)) and [13]. The relation (5.26) was also noticed by Elze *et al.* [17]. We note here that our c is negative and hence no spurious pole appears in the transversal propagator

$$D_{Tij}(0, \mathbf{k}) = -(\delta_{ij} - k_i k_j / k^2) / [k^2 - \Pi_T(0, \mathbf{k})], \quad (5.27)$$

in contrast to the covariant and the temporal axial gauge cases.

It is a known fact that the T^2 terms are gauge independent and the gauge parameter dependence starts at $\sim T$ order [14]. Thus the fact that the T^2 terms in Eqs (5.14) and (5.15) are different from others, *e.g.* Eq. (5.1), should not originate from the gauge choices.

5.3. Concluding remarks

We end the paper by giving a few comments:

1) In Sections 5.1 and 5.2 we have seen that the T dependence of the QCD coupling constant is very sensitive to the prescription chosen. This is not a trivial issue, because all the results obtained hitherto also heavily depend on the vertex chosen (*i.e.*, the trigluon, the ghost-gluon, or the quark-gluon vertex) to satisfy the renormalization condition of the QCD coupling constant. Furthermore in some gauges, *e.g.*, in the Coulomb gauge, the transversality condition (3.2) on the polarization tensor does not hold and hence the structure of the polarization tensor in such gauges is different from the one which satisfies the condition.

One of the reasons why we encounter different results in the literature is that the broken Lorentz invariance has not been treated as we have done in Eqs (3.13) and (3.14). For example in deriving Eqs (5.9)–(5.11) the authors of Ref. [9] have extracted the gluon and the vertex renormalization constants in front of Lorentz invariant structures not paying attention to the break-down of Lorentz invariance.

Another possible source for discrepancies is that the Ward identities are used at finite T for different renormalization constants. The derivation of the Ward identity is based on the gauge invariance and also on the Lorentz invariance at $T = 0$. Thus it is not a surprise that results from different gauges or even (within the same gauge) from different vertices are totally different.

The correspondence between the imaginary and real time formalisms has been investigated in detail [7, 15–16]. The choice of the formalism can also be the source of the discrepancies under study, as it was pointed out in these references.

To clarify the issue of choosing a suitable renormalization prescription, one would need to compute the two-loop contributions to the coupling constant at finite T . It was shown, in particular, in the massive $O(N)$ scalar

model that the one-loop result is drastically changed by two-loop contributions at high T and in zero momentum limit [29]. Recently Elmfors and Kobes have also argued that the inclusion of higher-loop effects is necessary to consistently obtain the leading term in the calculation of the thermal beta function in a hot Yang-Mills gas [30].

2) As we have seen the function $f^{(1)}(\mu, T)$ shows a power-like T -dependence instead of a logarithmic fall-off as a function of μ . Thus we have

$$-[f^{(1)}(\mu, T) - f^{(1)}(\mu_0, T_0)] = \sum_{n=1,2} c_n [(T/\mu)^n - (T_0/\mu_0)^n] + \dots \quad (5.28)$$

However, if a relation $T = (\text{const})\mu$ holds, where $(\text{const}) \simeq 1/3$ [6], then one would have a logarithmic T -dependence like in $T = 0$ environment. Such a relation holds if one is dealing with the thermal equilibrium.

We have derived Eq. (2.25) using two essentially different RG equations (2.23) and (2.24). Suppose we had used only one RG equation for T , *i.e.*, Eq. (2.24), then we would have

$$-[f^{(1)}(\mu, T) - f^{(1)}(\mu_0, T)] = \sum_{n=1,2} c_n [(T/\mu)^n - (T/\mu_0)^n] + \dots, \quad (5.29)$$

instead of Eq. (5.28). In this case we would have a power-like dependence even with the relation $T = (\text{const})\mu$.

We would like also to mention that all the machinery for the evolution of the running coupling constant at finite T can be analogously applied for the case of a quantum field theory (such as QCD) at finite energy as formulated in [31].

In conclusion, we note in accordance with the previous observations that there is in fact no unique way to define a T depending QCD coupling constant and the issue of finding its reasonable prescription is left as a subject of further investigation.

REFERENCES

- [1] L. Dolan, R. Jackiw, *Phys. Rev.* **D9**, 3320 (1974).
- [2] S. Weinberg, *Phys. Rev.* **D9**, 3357 (1974).
- [3] For a review: N.P. Landsman, Ch.G. van Weert, *Phys. Rep.* **145**, 141 (1987).
- [4] J. Kapusta, *Field Theory at Finite Temperature and Density*, Cambridge University Press, Cambridge 1988.
- [5] L.E. Gendenshtein, *Sov. J. Nucl. Phys.* **29**, 841 (1979).
- [6] K. Kajantie, J. Kapusta, *Ann. Phys.* **160**, 477 (1985); K. Enqvist, K. Kajantie, *Mod. Phys. Lett.* **A2**, 479 (1987).

- [7] H. Nakkagawa, A. Niégawa, *Phys. Lett.* **B193**, 263 (1987); **B196**, 571(E) (1987); H. Nakkagawa, A. Niégawa, H. Yokota, *Phys. Rev.* **D38**, 2566 (1988); *Phys. Lett.* **B244**, 63 (1990); H. Nakkagawa, H. Yokota, A. Niégawa, *Phys. Rev.* **D38**, 3211 (1988).
- [8] Y. Fujimoto, H. Yamada, *Phys. Lett.* **B212**, 77 (1988).
- [9] Y. Fujimoto, H. Yamada, *Phys. Lett.* **B200**, 167 (1988); *Phys. Lett.* **B195**, 231 (1987).
- [10] N.P. Landsman, *Phys. Lett.* **B232**, 240 (1989).
- [11] R. Baier, B. Pire, D. Schiff, *Phys. Lett.* **B238**, 367 (1990).
- [12] K. Enqvist, K. Kainulainen, *Z. Phys.* **C53**, 87 (1992).
- [13] S. Nadkharni, *Phys. Lett.* **B232**, 362 (1989).
- [14] Y. Fujimoto, H. Yamada, *Z. Phys.* **C40**, 365 (1988).
- [15] R. Kobes, *Phys. Rev.* **D42**, 562 (1990).
- [16] R. Baier, B. Pire, D. Schiff, *Z. Phys.* **C51**, 581 (1991).
- [17] H. Elze, U. Heinz, K. Kajantie, T. Toimela, *Z. Phys.* **C37**, 305 (1988); U. Heinz, K. Kajantie, T. Toimela, *Ann. Phys.* **176**, 218 (1986); H. Elze, K. Kajantie, T. Toimela, *Z. Phys.* **C37**, 601 (1988).
- [18] J. Antikainen, M. Chaichian, N.R. Pantoja, J.J. Salazar, *Phys. Lett.* **B242**, 412 (1990).
- [19] M.A. van Eijck, C.R. Stephens, Ch. G. van Weert, *Mod. Phys. Lett.* **A9**, 309 (1994); D. O'Connor, C.R. Stephens, F. Freire, *Mod. Phys. Lett.* **A8**, 1779 (1993).
- [20] B.S. DeWitt, *Phys. Rev.* **162**, 1195 (1967).
- [21] L.F. Abbott, *Nucl. Phys.* **B185**, 189 (1981).
- [22] F.J. Yndurain, *Quantum Chromodynamics*, Springer-Verlag, Berlin 1983.
- [23] P. Pascual, R. Tarrach, *QCD: Renormalization for the Practitioner*, Springer-Verlag, Berlin 1984.
- [24] G. 't Hooft, M. Veltman, *Nucl. Phys.* **B44**, 189 (1972).
- [25] C.W. Bernard, *Phys. Rev.* **D9**, 3312 (1974).
- [26] K. Kinslinger, M. Morley, *Phys. Rev.* **D13**, 2771 (1976).
- [27] O.K. Kalashnikov, V.V. Klimov, *Sov. J. Nucl. Phys.* **31**, 699 (1980); *Sov. J. Nucl. Phys.* **33**, 443 (1981).
- [28] P. Aurenche, T. Becherrawy, *Nucl. Phys.* **B379**, 259 (1992).
- [29] K. Funakubo, M. Sakamoto, *Phys. Lett.* **B186**, 205 (1987).
- [30] P. Elmfors, R. Kobes, *Phys. Rev.* **D51**, 774 (1995).
- [31] M. Chaichian, I. Senda, *Nucl. Phys.* **B396**, 737 (1993); M. Chaichian, H. Satz, I. Senda, *Phys. Rev.* **D49**, 1566 (1994).