BEYOND INTERMITTENCY: ERRATICITY

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Dedicated to Andrzej Białas in honour of his 60th birthday

Erraticity analysis of multiparticle production data is introduced as a way of extracting the maximum amount of information on self-similar fluctuations. It is presented as the next logical step to take beyond the intermittency analysis. An erraticity spectrum $e(\alpha)$ can be determined analogous to the multifractal spectrum $f(\alpha)$. An analytical example is presented to elucidate the method of analysis and the type of results that can be obtained.

PACS numbers: 12.40. Ee

1. A historical overview

Andrzej Białas has played an influential role in the physics of hadronic and nuclear collisions throughout his career. The work that he did with Robi Peschanski on intermittency has dominated the attention of physicists working on multiparticle production in the last ten years. It is fitting at this point to review the significance of intermittency and ask where we can go from here.

When many particles are produced in high-energy collisions, the very natural quantities to study theoretically and experimentally are averages, such as the mean multiplicity $\langle n \rangle$, the first few moments of the multiplicity distribution P_n, and the rapidity distribution dn/dy. Indeed, those were the quantities investigated intensively in the beginning of the era of multiparticle production.

Then as the collision energy was increased, the total rapidity range Y became large enough to permit meaningful partitioning of Y into smaller bins of various sizes δ. It was found that the distributions $P_n(\delta)$ can be

well fitted by negative binomial distributions with the normalized width increasing, as δ decreases [1, 2]. Thus began the interest in the study of multiplicity fluctuation as a function of the bin size. Such studies did not catch fire until the significance of intermittency, proposed by Białas and Peschanski [3], was fully appreciated.

In particle physics intermittency refers to the power-law behavior of the normalized factorial moments F_q, as the bin size is decreased. The observation of that behavior [4] therefore suggests that the mechanism for particle production has a self-similar property. It means that the occurrence of a large burst of particles in a small bin is rare, but possible if one waits long enough for such an event to take place. Any model that does not possess such intermittent features is thus ruled out.

In recent years it was found that much of the intermittency phenomenon can be attributed to Bose-Einstein correlation among like-sign charged particles. The bunching of particles in small bins cannot be distinguished from the interference effect due to the coherent emission of same type particles from an extended source. While this is an important experimental finding, one should not let the BE correlation effect completely obscure the intermittency behavior, which is still seen in the unlike-sign charged-particle F_2 [5, 6]. If it exists in the unlike-sign sector, then it must also exist in the like-sign sector, though small in comparison to the BE effect. An effect that is small is not necessarily unimportant. In this case it is our only clue to an important aspect of the dynamics of soft interaction: self-similarity. Hence, in my view the study of intermittency should go on.

If the dynamics is self-similar, it is natural to ask about the multifractal properties of multiplicity distributions. The G moments were constructed to exhibit those properties through the multifractal spectrum $f(\alpha)$ [7]. The advantage is that the order q of the moments G_q is continuous, and can be negative. That makes possible the determination of α which is a derivative in q, and facilitates the study of dips in addition to spikes in the rapidity distribution. The disadvantage is that, unlike F_q, the G_q moments do not filter out the statistical fluctuations automatically and therefore require explicit elimination by "subtraction" [8, 9]. When that is done, the dynamical Renyi dimensions $D_q^{\mathbf{dyn}}$ can be determined. For the Monte Carlo code ECCO [10] that simulates hadronic collisions with intermittency, it is found that $D_q^{\mathbf{dyn}}/d$ is independent of the dimension d in which the G-moments analysis is done [9]. A way to continue F_q to noninteger values of q, while maintaining the attribute $F_q = 1$ for all q when P_n is Poissonian, has been devised [11]. Its application to real data has recently been attempted [12].

At this point there is a slow-down in the acquisition and analysis of multiparticle data of hadronic collisions. New methods of analysis have been proposed, notably by means of correlation integrals [13] and wavelets

[14]. They are more efficient and powerful than studying F_q in discrete bins, and can extract more information on self-similiarity. The application of the wavelet analysis to real data has not yet been done, and the reward for such an improved analysis remains to be realized.

While the progress in phenomenology is slow, one nevertheless can ask the theoretical question: what is next? Is intermittency analysis the most that one can do to extract information of self-similarity of the particle production process? In the following section a suggestion is made to carry the study to yet another level where more information on fluctuations can be obtained.

2. Erraticity

Let us examine in detail the normalized factorial moments F_q. The horizontally averaged vertical moments are

$$F_q^{(v)} = \frac{1}{M} \sum_{j=1}^{M} \frac{\langle n(n-1)\cdots(n-q+1)\rangle_j}{\langle n\rangle_j^q} \,, \tag{1}$$

where $\langle \cdots \rangle_j$ is the (vertical) average over all events of the quantity bracketed at the jth bin, n being the multiplicity in that bin, and M is the total number of bins (*e.g.*, $M = Y/\delta$ in the 1-dimensional case). If the space in which the partition into M bins is done is made to have a flat single-particle distribution by use of the cumulative variable [15, 16], one can also meaningfully define the vertically averaged horizontal moments

$$F_q^{(h)} = \frac{1}{\mathcal{N}} \sum_{e=1}^{\mathcal{N}} \frac{\langle n(n-1)\cdots(n-q+1)\rangle_e}{\langle n\rangle_e^q} \,, \tag{2}$$

where $\langle \cdots \rangle_e$ is now the (horizontal) average over all bins for the eth event, and $\mathcal{N}$ is the total number of events. It is clear that the two definitions are complementary and in most instances they behave the same way. In either case intermittency refers to the scaling behavior

$$F_q \propto M^{\varphi_q} \,, \tag{3}$$

when M is increased in a fixed portion of the phase space, *i.e.*, when the bin size δ is decreased.

The numerator of (1) and (2) are nonzero only when the bin multiplicity n is $\geq q$. Thus they pick out events and bins with large fluctuations, $n \gg \langle n\rangle_{j,e}$, when δ is small, since $\langle n\rangle_{j,e} \propto \delta$. It is possible that, when q is

large and δ is small, one may have to wait for many non-contributing events to go by before finding a spike that contributes. That is why Białas and Peschanski have coined the word intermittency for the phenomenon. The emphasis on bin multiplicity fluctuations marked a significant advance that intermittency generated in the subject of multiparticle production.

However, intermittency as studied so far has not fully exhausted the characterization of fluctuations that the system can exhibit. Let us focus on $F_q^{(h)}$ in (2) to be definite. The summand is a quantity that characterizes the "spatial" fluctuations (in phase space or any other space) in an event. Since it plays a central role in the discussion to follow, let us denote it by F_q^e so that

$$F_q^e = \frac{\langle n(n-1)\cdots(n-q+1)\rangle_e}{\langle n\rangle_e^q} . \tag{4}$$

We then see from (2) that $F_q^{(h)}$ is an average of F_q^e over all events. We know that F_q^e fluctuates greatly from event to event. Those fluctuations are ignored by the study of $F_q^{(h)}$, so intermittency in $F_q^{(h)}$ does not fully account for all the fluctuations that the system exhibits. To capture the nature of those fluctuations and to find the associated scaling behavior constitute what can be called the erraticity analysis, which I now describe.

It should be remarked that the problem to be addressed is not removed by upgrading F_q^e to correlation integrals or wavelets. We shall use F_q^e as defined in (4) as one possible, but simple, characterization of the spatial pattern of an event. Other descriptions can be chosen, and can be denoted by F_q^e, used as a generic symbol. Indeed, F_q^e need not refer to multiparticle production. Any system that involves repeated samplings whose outcome can fluctuate from event to event can be investigated in the erraticity analysis. To emphasize the generality of the method, let us simplify the symbol F_q^e to F_e, when the order q is immaterial to the discussion.

With F_e describing the spatial pattern of an event, there should exist a distribution $P(F)$ of F after many events. Let $P(F)$ be normalized

$$\int_0^\infty P(F)dF = 1 . \tag{5}$$

Clearly, $F_q^{(h)}$ in (2) is the average

$$\langle F\rangle = \frac{1}{\mathcal{N}}\sum_{e=1}^{\mathcal{N}} F_e = \int_0^\infty FP(F)dF , \tag{6}$$

which conveys only a small piece of the information about $P(F)$. Experimentally, the whole distribution $P(F)$ should be determined. However, that may provide too much information, if the q and M dependences are fully explored. Thus a few moments of $P(F)$ may be sufficient. Define the standard normalized moments

$$C_p = \frac{\langle F^p \rangle}{\langle F \rangle^p},\tag{7}$$

where the averages are calculated as in (6). The order p here need not be an integer; in fact, it can even be less than 1, but may or may not be less than 0, depending on whether there are events that have $F = 0$. For $p > 1$, C_p reflects the large F behavior of $P(F)$, which is sensitive to the spikes in phase space. For $p < 1$, C_p probes the low F behavior of $P(F)$, which is influenced mainly by bins with low multiplicities, including empty bins. Thus knowing C_p for $0 < p < 2$, say, reveals a great deal about the properties of $P(F)$, all of which are not probed by intermittency.

Collecting all the complicated properties of a complex system contributes only to a messy assemblage of facts. It is only when there is some simple, universal feature to be found in the assemblage that the phenomenological analysis becomes worthwhile. If there is self-similarity in the dynamics of particle production, we should search for power-law dependence on M. Eq. (3) exhibits one such behavior. Having generalized $\langle F \rangle$ to C_p, it is natural for us to suggest the search for the scaling behavior of C_p

$$C_p \propto M^{\psi(p)} \propto \delta^{-\psi(p)}.\tag{8}$$

While the behavior in (3) has been referred to as intermittency, we shall refer to the behavior in (8) as erraticity. Since C_p are the moments of $P(F)$, they describe the deviation of F_e from the mean $\langle F \rangle$. Consequently, C_p is sensitive to the erratic fluctuations of F_e from event to event. Those fluctuations depend on the bin size because F_e itself is a description of the spatial pattern that varies according to resolution. Thus if those fluctuations scale with bin size, then the erraticity exponent $\psi(p)$ is an economical way of characterizing an aspect of the self-similar dynamics that has some order in its erratic fluctuations.

Of particular interest is an index μ defined by

$$\mu = \left.\frac{d}{dp}\psi(p)\right|_{p=1}.\tag{9}$$

It was shown in [17, 18] that μ is related to the entropy in event space, and has been used to study chaotic behavior in branching processes in QCD.

A dynamical system that has erratic fluctuations may or may not exhibit chaotic behavior in the technical sense of chaoticity in nonlinear dynamics. The generalization of the notion of chaos in classical trajectories to quantum systems where the degrees of freedom can increase with time is still under investigation. Whatever the outcome, the notion of erraticity is independent of it, and the results of erraticity analysis describe some features that are important in their own right.

For a multifractal system one usually determines the multifractal spectrum $f(\alpha)$. It is related to a scaling exponent $\tau(q)$ by a Legendre transform [19]. In multiparticle production $\tau(q)$ appears in the scaling law of the G moments [7]

$$G_q(\delta) \propto \delta^{\tau(q)} , \tag{10}$$

where q is a continuous variable. The exponent α is defined by

$$\alpha_q = \frac{d\tau(q)}{dq} , \tag{11}$$

and the transform is

$$f(\alpha) = q\alpha - \tau(q) .$$

It can be shown that for a multifractal set $f(\alpha)$ is always $\leq \alpha$, and that the information dimension D_1 corresponds to $D_1 = \alpha_1 = f(\alpha_1)$ at $q = 1$.

Since C_p is not the same as G_q (their scaling laws (8) and (10) having opposite behaviors in δ), our measure of erraticity does not have multifractal properties. Nevertheless, we still can define a spectrum $e(\alpha)$ by the Legendre transform

$$e(\alpha) = p\alpha - \psi(p) , \tag{13}$$

$$\alpha_p = \frac{d\psi(p)}{dp} . \tag{14}$$

The function $e(\alpha)$ exhibits certain properties of erraticity more directly than $\psi(p)$. For example, we have $\alpha_1 = \mu$, which is the only point where $e(\alpha) = \alpha$. For all other values of α, one has $e(\alpha) > \alpha$. Since, by definition, $C_p = 1$ at both $p = 0$ and 1, we have $\psi(0) = \psi(1) = 0$. Thus for any $P(F)$ that becomes wider at smaller δ, as is always the case in particle production, it follows that $\psi(p) > 0$ for $p > 1$, $\psi(p) < 0$ for $0 < p < 1$, and $\psi(p) > 0$ for $p < 0$, if C_p exists. The resultant behavior of $e(\alpha)$ is therefore that $e(\alpha) = 0$ at $p = 0$ where $\alpha_0 < 0$, and $e(\alpha) > 0$ everywhere else, where α is calculable. An example of this behavior will be given in the following section.

The values of α_0 and α_1 bear specific relationships to certain averages over $P(F)$. Since (7) implies

$$\frac{d}{dp}C_p = \int\limits_0^\infty dF \, \ln\left(\frac{F}{\langle F \rangle}\right) \left(\frac{F}{\langle F \rangle}\right)^p P(F) , \tag{15}$$

it then follows from (8) and (14) that in the scaling region

$$\alpha_0 = \frac{1}{\ln M} \left(\langle \ln F \rangle - \ln \langle F \rangle \right) , \tag{16}$$

$$\alpha_1 = \frac{1}{\ln M} \left(\frac{\langle F \ln F \rangle}{\langle F \rangle} - \ln \langle F \rangle \right) . \tag{17}$$

If the scaling laws are $\langle F \rangle \propto M^{\varphi}$ and $\langle \ln F \rangle \propto \tilde{\varphi} \ln M$, then we have

$$\alpha_0 = \tilde{\varphi} - \varphi , \tag{18}$$

which is negative, except for unusual $P(F)$. On the other hand, (17) suggests that if we define $\mathcal{P} = F/ \langle F \rangle$, then

$$\langle \mathcal{P} \ln \mathcal{P} \rangle = \alpha_1 \ln M \tag{19}$$

in the scaling region. The connection of α_1 with entropy should therefore not be surprising.

In the foregoing we have suppressed the symbol q if F is the normalized factorial moment defined in (4). For such moments to describe the spatial pattern, all the relevant quantities in (5) to (19) should be labeled with an index q, *viz.* , F_q, $C_{p,q}$, $\psi_q(p)$, μ_q, $e_q(\alpha)$, $\alpha_{p,q}$, φ_q and $\tilde{\varphi}_q$.

3. An analytic example

To help make the discussion in the previous section more concrete and transparent, let us consider an example with analytic expressions. In real experiments or in computer simulations the event-to-event fluctuations of F may be so erratic that no simple formula can approximate $P(F)$, let alone its dependence on q and M. However, there are general trends characteristic of multiparticle production that can be built in. Furthermore, Monte Carlo simulations of jet fragmentation in pQCD give definite shapes of $P(F)$ that can serve as a very useful guide for the choice of analytic formulas. We shall rely on the results of [18] to generate specific expressions.

It should first be remarked that if the bin multiplicities in an event vary according to the Poisson distribution, then $F_q = 1$ for all q. Fluctuations from $F_q = 1$ have dynamical content, and usually $\langle F_q \rangle > 1$, for $q \geq 2$. However, from the simulations in [18] we have seen that $P(F_q)$ has its maximum at $F_q = 1$ for all M. That is a condition that we shall impose.

We adopt the gamma distribution for $P(F)$:

$$P(F) = A F^a e^{-bF} , \tag{20}$$

and require that its peak be located at $F = 1$. With the normalization (5), it becomes

$$P(F) = \frac{a^{a+1}}{\Gamma(a+1)} F^a e^{-aF} \,, \tag{21}$$

where a is the only parameter, dependent on q and M. All the F distributions determined in [18] have the shapes of (21) with large values of a. We adopt the following parametrization to introduce the q and M dependences:

$$a = 500/(q \ln M)^2 \,, \tag{22}$$

which reproduces the general trend of the simulated results in [18].
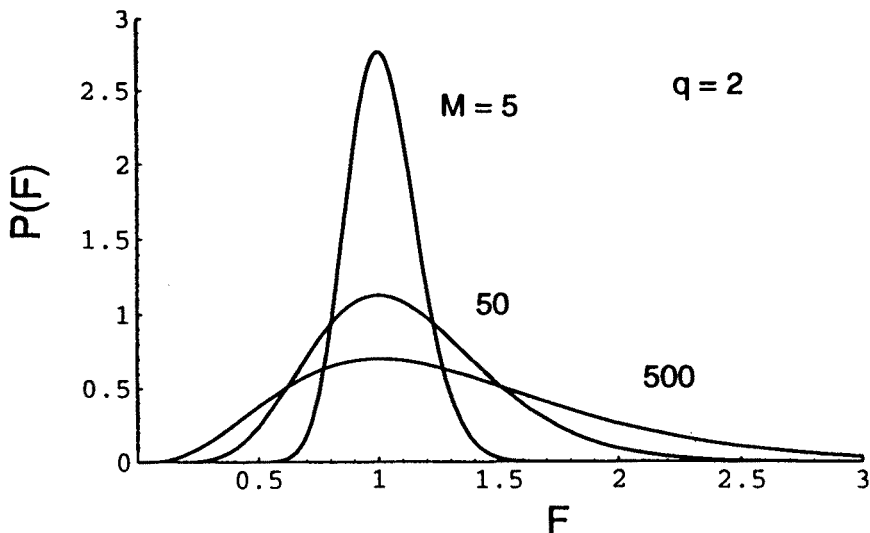
Fig. 1. Examples of the distribution $P(F)$ given by Eqs (21) and (22) for $q = 2$

In Fig. 1 are shown, as examples, the distributions $P(F)$ for $q = 2$ and $M = 5, 50, 500$. Clearly, for small bins (large M) there are large fluctuations of F from event to event. It is those fluctuations that are measured by the moments $C_{p,q}(M)$. For $p = 2$ the scaling behavior (8), as shown in Fig. 2, is satisfied at large M. Similar scaling behaviors are found for other values of p. Thus the exponents $\psi_q(p)$ can be determined by the straightline fits in the scaling region. The results are shown by the dots in Fig. 3 for $q = 2$ and 3 and for some discrete values of p. Those dots are well fitted by the formula $\psi_q(p) = \sum_{i=1}^{4} b_i p^i$, which is then used to determine $\alpha_{p,q}$. It is clear from the general behavior of $\psi_q(p)$, which is negative in the region $0 < p < 1$, that $\alpha_{0,q} < 0$ and $\alpha_{1,q} > 0$. Their numerical values are

$$\alpha_{0,2} = -0.026 \,, \quad \alpha_{0,3} = -0.047 \,,$$
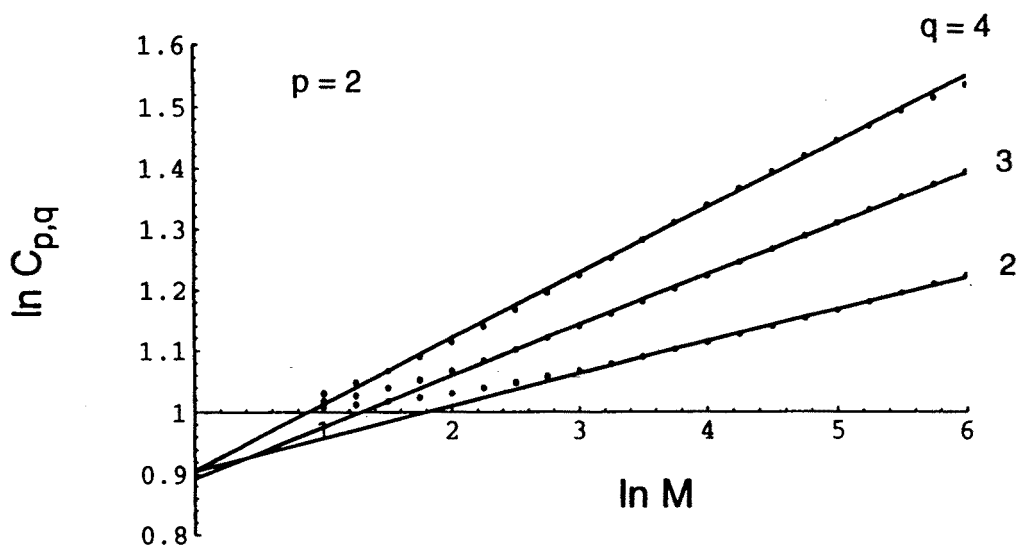$$\alpha_{1,2} = 0.024 \,, \quad \alpha_{1,3} = 0.04 \,. \tag{23}$$

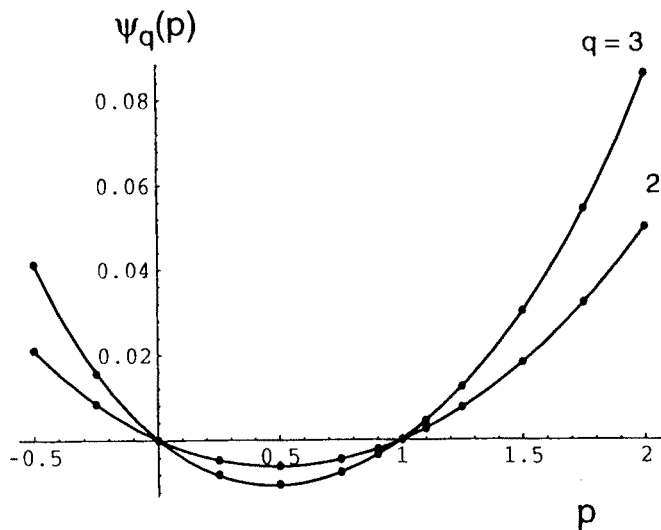Fig. 2. Scaling behaviors of $C_{p,q}(M)$ for $p = 2$ and $q = 2, 3, 4$

Fig. 3. Dependences of the erraticity exponents $\psi_q(p)$ on p for $q = 2, 3$

With analytical formulas for $\psi_q(p)$ the spectrum $e_q(\alpha)$ can be determined by use of (13). For the exponents $\psi_q(p)$ shown in Fig. 3, the corresponding $e_q(\alpha)$ are shown in Fig. 4. The straightline is for $e_q(\alpha) = \alpha$. Thus where $e_q(\alpha)$ curves touch the straightline are the values of $\alpha_{1,q}$, and where $e_q(\alpha) = 0$ give $\alpha_{0,q}$. The entropy indices are $\mu_q = \alpha_{1,q}$. They get larger at higher values of q, which is a consequence of the fact that F_q fluctuates more from event to event at higher q.
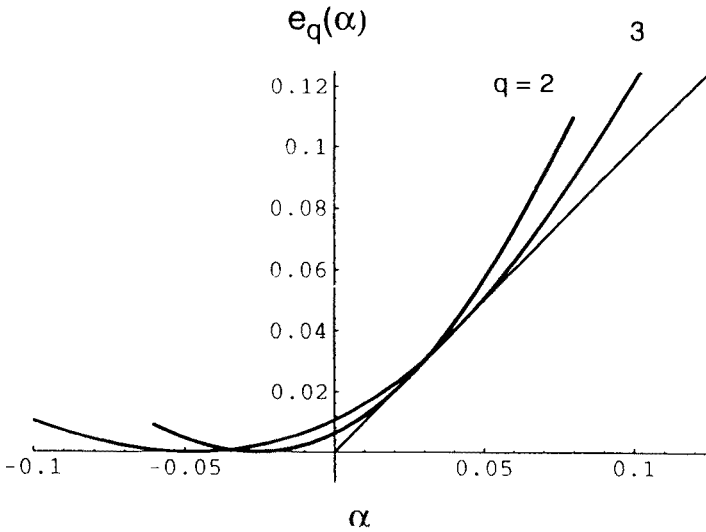
$$e_q(\alpha)$$

Fig. 4. The erraticity spectrum $e_q(\alpha)$ for $q = 2, 3$

Real data are not likely to be describable by simple formulas like (21) and (22). However, erraticity analysis can be applied to the data directly, and curves for $\psi_q(p)$ [using (7) and (8)] and $e_q(\alpha)$ [using (13) and (14)] can be determined, if scaling behavior exists. That represents the "maximum" amount of information extractable from the horizontal and vertical fluctuations of the data that exhibit properties of self-similarity. Generally speaking, positive α describes the spikes of the spatial distribution, while negative α describes the dips.

4. Conclusion

Primitive averages are performed over both spatial fluctuations and event fluctuations, which have been referred to as horizontal and vertical averages, respectively. Intermittency probes the scaling properties of one of those fluctuations, and averaging over the other. Only one kind of moments are considered, *viz.*, F_q. Erraticity probes both types of fluctuations, and therefore double moments are needed: $C_{p,q}$.

Vertical fluctuations may be due to trivial reasons, such as impact parameter variation from event to event. In heavy-ion collisions such variations should be controlled by E_T cuts. For hadronic collisions, cuts in event multiplicity may restrict event fluctuations too much and unduly suppress the erraticity to be uncovered. Those fluctuations have dynamical as well as geometrical (*i.e.*, impact-parameter related) origins and should be investigated fully. No geometrical fluctuation is present in e^+e^- annihilations, so

erraticity analysis explores the quantum fluctuations of parton branching for every fixed initial state specified by the energy.

When the dynamics of particle production is known, erraticity analysis can then describe some aspects of that dynamics, such as the (possibly) chaotic behavior of perturbative QCD [17, 18]. But the purpose of studying intermittency, and now erraticity, is to get phenomenological information from the data that can help us to learn more about the dynamics of particle production where the theory is inoperable. Specifically, it is for learning about the soft interaction. Plots of $\psi_q(p)$ or $e_q(\alpha)$ form the arena where theory and experiment should meet. Models of soft interaction that can reproduce the primitive averages may well reveal deficiencies when confronted with erraticity data.

Heavy-ion data have so far not led to interesting results in intermittency study. Perhaps too much averaging has been done. Erraticity analysis may reveal more structure.

Bose–Einstein correlation has temporarily detracted the study of intermittency. Focusing on unlike-sign charged particles in hadronic collisions and going deeper into erraticity may reveal features about the dynamics of soft interaction that may finally lead to the construction of a reliable model capable of meeting all experimental tests.

REFERENCES

[1] M. Adamus *et al.* (NA22), *Phys. Lett.* **B205**, 401 (1988).
[2] R.E. Ansorge *et al.* (UA5), *Z. Phys.* **C43**, 357 (1989).
[3] A. Białas and R. Peschanski, *Nucl. Phys.* **B273**, (1986); **308**, 867 (1988).
[4] For a review see E.A. DeWolf, I.M. Dremin, W. Kittel, *Phys. Rep.* (to be published).
[5] N. Agababyan *et al.* (NA22), *Z. Phys.* **C59**, 405 (1993).
[6] N. Neumeister *et al.* (UA1), *Phys. Lett.* **B275**, 186 (1992); *Z. Phys.* **C60**, 633 (1993).
[7] R. C. Hwa, *Phys. Rev.* **D41**, 1456 (1990); C.B. Chiu, R.C. Hwa, *Phys. Rev.* **D43**, 100 (1991).
[8] I. Derado, R. C. Hwa, G. Jancso, N. Schmitz, *Phys. Lett.* **B283**, 151 (1992).
[9] R.C. Hwa, J.C. Pan, *Phys. Rev.* **D45**, 1476 (1992).
[10] R.C. Hwa, J.C. Pan, *Phys. Rev.* **D45**, 106 (1992).
[11] R.C. Hwa, *Phys. Rev.* **D51**, 3323 (1995).
[12] J. Zhang, S. Wang, USTC-Hefei preprint (1996).

[13] P. Carruthers, *Astrophys. J.* **380**, 24 (1991); P. Lipa, P. Carruthers, H. C. Eggers, B. Buschbeck, *Phys. Lett.* **B285**, 300 (1992).

[14] M. Greiner, P. Lipa, P. Carruthers, *Phys. Rev.* **E51**, 1948 (1995); M. Greiner, J. Giesemann, P. Lipa, P. Carruthers, *Z. Phys.* **C69**, 305 (1996).

[15] A. Białas, M. Gaździcki, *Phys. Lett.* **B252**, 483 (1990).

[16] W. Ochs, *Z. Phys.* **C50**, 339 (1991).

[17] Z. Cao, R.C. Hwa, *Phys. Rev. Lett.* **75**, 1268 (1995).

[18] Z. Cao, R.C. Hwa, *Phys. Rev.* **D53**, xxx (1996); also OITS-599 (submitted to *Phys. Rev.* **D**).

[19] J. Feder, *Fractals*, Plenum Press, N.Y., 1988.