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SOLUTIONS OF THE BAXTER EQUATION*

R.A. Janik

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland

e-mail: ufrjanik@jetta.if.uj.edu.pl

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Dedicated to Andrzej Bialas in honour of his 60^{th} birthday

We present a method of construction of a family of solutions of the Baxter equation arising in the Generalized Leading Logarithmic Approximation (GLLA) of the QCD pomeron. The details are given for the exchange of $N = 2$ reggeons but everything can be generalized in a straightforward way to arbitrary N. A specific choice of solutions is shown to reproduce the correct energy levels for half integral conformal weights. It is shown that the Baxter's equation must be supplemented by an additional condition on the solution.

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1. Introduction

One of the longstanding problems of QCD is the behavior of the theory in the Regge limit of high energies and fixed transferred momenta. In the Leading Logarithmic Approximation the relevant amplitudes correspond to the exchange of two reggeized gluons — the BFKL pomeron [1, 2]. Later this was generalized in the framework of GLLA to the exchange of a higher number of reggeons $N > 2$ [3, 4]. But in contrast to the BFKL case the explicit values of the intercepts are still unknown.

Recently much progress has been made with the establishment in [6, 5, 8, 9] of a very surprising link with exactly solvable lattice models. Within this framework variants of the Bethe ansatz have been tried ([7–10] and

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quasiclassical approximation [11]), all of which are based on attempts to solve the Baxter equation.

In this paper we present a method of constructing solutions to the Baxter equation, which allows for easy generalization to the still open odderon problem (the exchange of $N = 3$ reggeons). The details are explicitly worked out for the $N = 2$ case corresponding to the well known BFKL pomeron.

2. The QCD pomeron as Heisenberg XXX spin $s = 0$ chain

The Regge limit of QCD is defined as the kinematical region

$$s \gg -t \approx M^2 , \tag{1}$$

where M is the hadron mass scale, or as in the case of Deep Inelastic Scattering, as the small $x = Q^2/s$ limit. The aim is to find the Regge behavior of the amplitude $A(s,t) \sim s^{\omega_0+1}$.

It has been established that finding the Regge intercepts ω_0 is equivalent, in the GLLA and large N_C limit, to finding the energy levels of the Hamiltonian for a N-site Heisenberg XXX $s = 0$ spin chain:

$$\sum_{i=1}^{N} H(z_i, z_{i+1}) , \tag{2}$$

where N is the number of exchanged reggeons, z_i are (complex) coordinates of the i-th reggeon and the two-particle hamiltonian is given by:

$$H(z_1, z_2) = \sum_{l=0}^{\infty} \frac{2l+1}{l(l+1) - L_{12}^2} - \frac{2}{l+1} , \tag{3}$$

where

$$L_{12}^2 := -z_{12}^2 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} . \tag{4}$$

The Regge intercept is now given by

$$\omega_0 = \frac{\alpha_s N_c}{4\pi} (E + \bar{E}) , \tag{5}$$

where $\bar{E}$ is the energy level of the corresponding antiholomorphic hamiltonian.

A method of finding the energy eigenvalues based on the Functional Bethe Ansatz has been proposed in [8, 9]. One looks for holomorphic solutions to the spin $s = -1$ Baxter equation:

$$(2\lambda^N + q_2\lambda^{N-2} + \ldots + q_N) Q(\lambda) = (\lambda + i)^N Q(\lambda + i) + (\lambda - i)^N Q(\lambda - i) , \tag{6}$$

where q_i are the conserved quantities of the Heisenberg XXX model. q_2 is related to the conformal weight h by the formula:

$$q_2 = -h(h-1)\,, \tag{7}$$

where h is the conformal weight $h = \frac{1+m}{2} + i\nu$ with m integer and ν real parameters labeling the irreducible representations of $SL(2,C)$. The energy levels are given by [8, 9]

$$E = i\frac{d}{d\lambda}\log\frac{Q(\lambda-i)}{Q(\lambda+i)}_{\lfloor\lambda=0} - 2N\,. \tag{8}$$

It seems that there is no rigorous proof of this formula for the non-compact spin chain, apart from the case when $Q(\lambda)$ is a polynomial (this can occur only for integer values of h). Physically, the most interesting case is $h = {}^1\!/_2$ for which value it is expected that the largest eigenvalue occurs. This is the reason why looking for non-polynomial solutions is interesting.

In this paper we present a general method of constructing solutions to the Baxter equation (6) which can reproduce the correct eigenvalues for half-integral h. This agreement is only for a specific choice of the solutions, showing that the Baxter equation must be supplemented by some additional condition. Unfortunately, the physical understanding of this choice is apparently still lacking.

3. Solution of the Baxter equation

The difficulties associated with finding the solution to (6) for arbitrary N are the following. One approach [9, 11] is to obtain solutions for integer values of the conformal weight and then analytically continue to arbitrary h. Unfortunately, no closed form of the solutions for $N = 3$ is known to date, which makes the programme very difficult to carry out. Direct expansions in power series are plagued by convergence problems which can be overcome but then lead to non-physical solutions (*i.e.* giving infinite energy). Here we give a construction of solutions to the Baxter equation, which can be easily generalized to arbitrary N, and the energy eigenvalues can be numerically calculated from equation (8). These solutions possess no singularities and are holomorphic in the whole complex plane.

The starting point of the construction is the contour integral representation used in [8] and [9].

$$Q(\lambda) = \int\limits_C \frac{dz}{2\pi i} z^{-i\lambda-1}(z-1)^{i\lambda-1}Q(z)\,. \tag{9}$$

This is a solution to the Baxter equation if the function $Q(z)$ satisfies the ordinary differential equation:

$$\left[\left(z(1-z)\frac{d}{dz}\right)^N + z(1-z)\sum_{k=0}^{N-2} i^{N-k} q_{N-k}\left(z(1-z)\frac{d}{dz}\right)^k\right] Q(z) = 0 \tag{10}$$

and the contour C is such that one can integrate by parts. Assuming that the contour C is closed on the Riemann surface associated to the kernel $K(z,\lambda) = z^{-i\lambda-1}(z-1)^{i\lambda-1}$ this last condition amounts to the requirement that the solution $Q(z)$ of (10) at the beginning of C should coincide with the solution obtained by analytical continuation along C. The key point is, that for non-integral h, this condition cannot be met for a single contour (one can check this explicitly for $h = {}^1\!/_2$ and $N = 2$ using the known monodromy properties of the hypergeometric function [12]).

W propose a construction which involves integration along two independent contours of two different solutions of the differential equation.

$$Q(\lambda) = \int_{C_I} \frac{dz}{2\pi i} K(z,\lambda)(au_1(z) + bu_2(z)) + \int_{C_{II}} \frac{dz}{2\pi i} K(z,\lambda)(cu_1(z) + du_2(z)), \tag{11}$$

where the contours C_I and C_{II} are depicted in figure 1. u_1 and u_2 are a basis of solutions around $z = {}^1\!/_2$ such that $u_1({}^1\!/_2) = 1$, $u_1'({}^1\!/_2) = 0$ and $u_2({}^1\!/_2) = 0$, $u_2'({}^1\!/_2) = 1$.

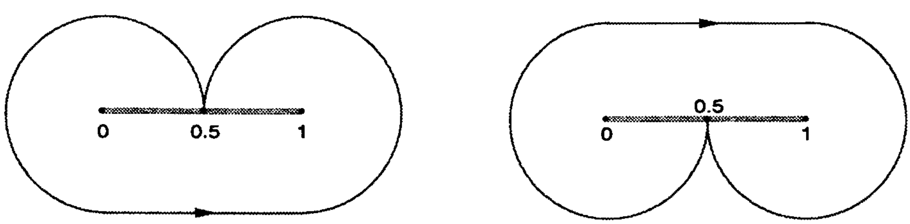

Fig. 1. The contours C_I and C_{II} in the integral representation of a solution to Baxter equation. The contours start and end at $z = {}^1\!/_2$. The shaded line is the cut for the kernel function $K(z,\lambda)$

The coefficients a, b, c and d are determined by the 'integration by parts' condition. Denoting the monodromy matrices of the contours C_I and C_{II} by M_I and M_{II} respectively, we obtain the equation:

$$M_I\begin{pmatrix} a \\ b\end{pmatrix} + M_{II}\begin{pmatrix} c \\ d\end{pmatrix} = \begin{pmatrix} a+c \\ b+d\end{pmatrix}. \tag{12}$$

Since we have two equations for four variables, a non-trivial solution always exists. One parameter is spurious, responsible only for an irrelevant change in the normalization, so we obtain a one parameter family of solutions.

A few remarks are in order. The contours do not meet any singularities, therefore the solution is holomorphic in the whole complex plane. Using the series expansion of the solutions to (10) the integrals can be carried out numerically, likewise the monodromy matrices can also be found numerically without resort to integral representation of the solutions of (10). It is this feature which makes numerical generalization to higher N possible.

Consider now explicitly the case $N = 2$. For real conformal weight h (*i.e.* $h = {}^1\!/_2, 1, {}^3\!/_2, 2, \ldots$) if $Q(\lambda)$ is a solution to the Baxter equation then so is $\overline{Q(\overline{\lambda})}$. This suggests taking the solution of the form:

$$\tilde{Q}(\lambda) = Q(\lambda) + \overline{Q(\overline{\lambda})}\,. \tag{13}$$

Now the energy is explicitly real and equal to

$$E = 2\,\mathrm{Im}\,\frac{\tilde{Q}'(i)}{\tilde{Q}(i)} - 4\,. \tag{14}$$

Furthermore we pick the solution which is maximally symmetric with respect to the symmetry $z \longleftrightarrow 1 - z$ *i.e.* $c = 1$ and $d = 0$. The numerical results for a number of conformal weights are shown in Table I, in excellent agreement with the known solutions. Note however, that the energy is different for different choices of d (the plot in figure 2 shows this dependence), making it quite remarkable that the simple symmetric choice, independent of h, gives the correct results.

TABLE I

Energy eigenvalues calculated numerically from the solutions to the Baxter equation with 40 terms in the power series expansion of $u_i(z)$.

h	E_{num}	E_{exact}	h	E_{num}	E_{exact}
0.5	5.54518	5.54518	6.5	−9.48051	−9.48051
1.5	−2.45482	−2.45482	8.5	−10.6292	−10.6292
2.5	−5.12149	−5.12149	...	...	...
4.5	−7.86435	−7.86435	16.5	−13.3999	−13.3999

A fundamental theoretical problem which remains to be solved is the physical understanding of this condition on the solution of the Baxter equation. Analytical proofs and the extension to $N = 3$ will be presented elsewhere [13].

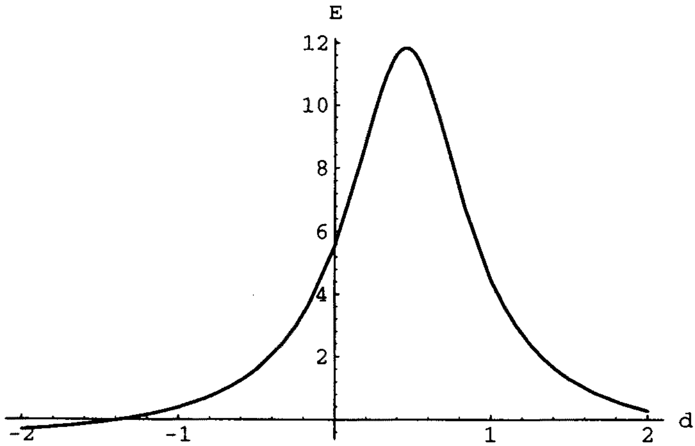

Fig. 2. Dependence of the energy calculated for different solutions of the Baxter equation for conformal weight $h = 1/2$. The correct value is obtained for the choice $d = 0$

4. Summary

In this paper we have presented a general procedure of constructing solutions to the Baxter equation arising in the theory of the QCD pomeron, for arbitrary values of the conformal weight. The correct energy eigenvalues for the $N = 2$ BFKL pomeron, are reproduced for a special choice of solution, showing that the Baxter equation must be supplemented by an additional condition.

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