UNIVERSALITY IN THE CRITICAL BEHAVIOR OF THE CORRELATION FUNCTIONS IN 2*D* SIMPLICIAL GRAVITY*

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Dedicated to Andrzej Bialas in honour of his 60th birthday

The analogue of the loop-loop correlation function in 2d gravity for the planar connected ϕ^3 diagrams is calculated. It is shown that although the discretized formulas are different the scaling limit is the same as for the loop-loop correlation function. The derivation may serve as an alternative definition of the volume-volume correlator of Euclidean quantum gravity in 2*d*.

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1. Introduction

The concept of the transfer matrix in the pure 2d gravity was introduced in [1]. The authors considered the "time" evolution of a given (closed) loop on a triangulated random lattice as a deformation of this loop by one step in the "forward" direction. The deformation consisted of removing the triangles attached to the links of the loop and could be interpreted as an evolution by the unit length in the discrete geodesic distance.

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The formulation of the problem made extensive use of the disc amplitude $F(x,g)$, which was obtained earlier in [2] in the large-N ϕ^3 matrix model. $F(x,g)$ is a generating function of Green's functions of the model:

$$F(x,g) = 1 + \sum_{k=1}^{\infty} x^k G_k(g) , \tag{1}$$

where

$$G_k(g) = \langle \mathrm{Tr}\, \phi^k \rangle$$

and the average is taken in the large-N limit of the ϕ^3 matrix model.

Green's functions defined above in the language of the planar ϕ^3 graphs contain both connected and disconnected diagrams. For loops drawn on a dual, triangulated surface this means the existence of pathologies, like two-fold links (corresponding to the disconnected bare two-point functions). These pathologies are expected to be irrelevant in the scaling limit of the theory.

In this note we show that this is indeed the case. We propose an alternative derivation of the transfer matrix and loop-loop correlators based on the connected ϕ^3 diagrams. In this formulation all pathologies mentioned above are explicitly excluded. We show that the scaling limit of the two approaches is exactly the same.

2. Connected disc amplitude

The generating function of the connected Green's functions (with one marked external line) for the ϕ^3 theory

$$\Psi(x,g) = 1 + \sum_{k=1}^{\infty} x^k \Psi(k,g) \tag{3}$$

was also considered in the paper [2]. In the planar limit this function satisfies the quadratic equation

$$g\Psi^2 - (x+g)\Psi + \frac{(s+1)(3s+1)}{8s} x + x^3 = 0 , \tag{4}$$

where the coupling constant g is related to s by

$$g^2 = 8s(1-s^2) . \tag{5}$$

In this parametrization $g = 0$ corresponds to $s = 1$. The critical point of the theory is given by

$$g_c = \frac{1}{2\, 3^{3/4}} ,$$
$$s_c = \frac{1}{3^{1/2}} . \tag{6}$$

Equation (4) can be easily solved. One gets

$$\Psi(x,g) = \frac{1}{2}(\frac{x}{g}+1) + \frac{1}{2}(1-\frac{sx}{g})\sqrt{1-\frac{4gx}{s^2}}\,. \tag{7}$$

For $g \to 0$

$$\Psi(x, g=0) = 1 + x^2\,, \tag{8}$$

where x^2 is the contribution from the bare propagator. From (7) we find the critical value x_c

$$x_c = \frac{1}{2\ 3^{1/4}}\,, \tag{9}$$

which in the scaling region $g \to g_c$ is approached from *above*.

Note that the analytic structure of $\Psi(x,g)$ is quite different from that of $F(x,g)$. However, the scaling behavior of the two functions turns out to be very similar. Introducing the parametrization

$$\begin{aligned} g &= g_c e^{-\varepsilon^2 t}\,,\\ s &= s_c(1+\frac{2\varepsilon\sqrt{t}}{\sqrt{3}})\,,\\ x &= x_c(1-\varepsilon\zeta)\,, \end{aligned} \tag{10}$$

we get the formula very similar to that obtained for $F(x,g)$ in the paper [1]:

$$\Psi = \frac{1+\sqrt{3}}{2}(1-\frac{3-\sqrt{3}}{2}\varepsilon\zeta) + \frac{1}{4}f(\zeta,\tau)\varepsilon^{3/2} + \mathcal{O}(\varepsilon^2)\,. \tag{11}$$

In (11)

$$\begin{aligned} f(\zeta,\tau) &= (2\,\zeta - \sqrt{\tau})\sqrt{\zeta+\sqrt{\tau}}\,,\\ \sqrt{\tau} &= \frac{4}{\sqrt{3}}\sqrt{t}\,. \end{aligned} \tag{12}$$

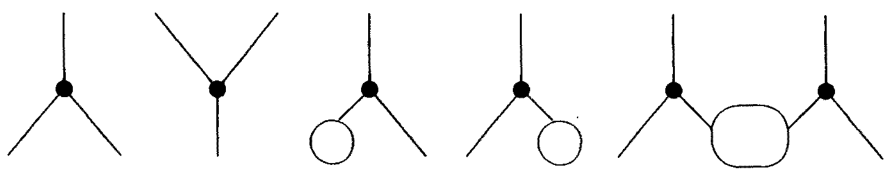
The only difference is in the value of the numerical coefficients in front of the ε and $\varepsilon^{3/2}$ terms and in the definition of τ.

3. The transfer matrix by the slicing method

Since the connected ϕ^3 diagrams rather then the triangulated surfaces will be the subject of this note, we have to modify the concept of the *slicing* to get the deformation laws. Let us consider the planar diagram with L external lines. One of them is a marked line. The deformation we shall consider will consist of eliminating all external links, together with vertices from which they emerge. After such operation we get again a connected diagram, but in general with a different length L'. Following the method proposed in [4] we consider the generating function

$$G^{(0)}_{\mu}(x,y;1)=\sum_{L,L'} x^{L}y^{L'}G^{(0)}_{\mu}(L,L';1)\,, \tag{13}$$

where $G^{(0)}_{\mu}(L,L';1)$ is the sum of all possible graphs connecting loops separated by one deformation layer with lengths L and L'. In this form the external lines are not marked. Similarly like in [4] the graphs can be obtained as a combination of five types of graphs $(a)-(e)$.



(a) (b) (c) (d) (e)

Their contributions are:

$$\begin{aligned}
(a) &= gx^2y\,,\\
(b) &= gxy^2\,,\\
(c) = (\mathrm{d}) &= gxy\frac{\Psi(x,g)-\Psi(x,g=0)}{x}\,,\\
(e) &= g^2x^2y^2\frac{\Psi(x,g)-1-x\frac{\partial\Psi(x,g)}{\partial x}|_{x=0}}{x^2}\,.
\end{aligned} \tag{14}$$

Subtraction in (c) is necessary to exclude the contribution from the bare propagator. In (e) we have to exclude from Ψ diagrams with zero and one line.

Summation of these contributions gives

$$\begin{aligned}
G^{(0)}_{\mu}(x,y;1) &= \sum_{n=1}^{\infty}\frac{1}{n}((a)+((b)+(c)+(d)+(e))^{n}\\
&= -\log(1-(a)-((b)-(c)-(d)-(e))\,.
\end{aligned} \tag{15}$$

where as in [4] the factor $1/n$ results from the cyclic symmetry.

The transfer matrix at a unit distance is closely related:

$$G_\mu(x,y;1) = y\frac{\partial}{\partial y}G_\mu^{(0)}(x,y;1)\,. \tag{16}$$

The derivative has the effect of marking one of the original external lines and providing a correct contribution from this line (compare Ref. [1]).

In the scaling limit the parameters behave as (10). For y we take

$$y = \frac{1}{x_c}(1-\varepsilon\zeta')\,. \tag{17}$$

In the small ε limit we get

$$G_\mu(x,y;1) = \frac{1}{\varepsilon}\frac{1}{\zeta+\zeta'-\varepsilon^{1/2}\alpha f(\zeta,\tau)}\,. \tag{18}$$

Again the only difference is in the finite coefficient α

$$\alpha = \frac{2\sqrt{3}-1}{11}\,. \tag{19}$$

This form of $G_\mu(x,y;1)$ leads to the continuum differential equation for the transfer matrix $G_\mu(x,y;r)$ in the scaling limit. We shall not discuss it here, but rather we rederive this equation using the *peeling* method.

4. Equation for $G_\mu(x,y;\,r)$ — the peeling method

The idea of peeling was introduced in [3]. In our context it corresponds to the following deformation of the connected diagram: we start with arbitrary external link and cut away this link together with the corresponding vertex. This operation can be represented by the diagrams presented before:

$$2\,(a)+(b)+(c)+(d)\,. \tag{20}$$

Notice the factor 2 in diagram (a), resulting from two external lines and the absence of diagram (e), which require two consecutive cuts. Notice also that a contribution of the diagram (a) is completely cancelled by subtractions of the bare propagator contribution in (c) and (d).

The peeling operation can be repeated iteratively around the connected diagram. Let us consider the function $G_\mu(L,L';r)$, where L' is the initial number of external lines (at a distance r). The peeling changes G_μ

$$G_\mu(L,L';r) \to gG_\mu(L+1,L';r) + 2g\sum_{L''=1}^{L+1}\Psi(L'',g)G_\mu(L-L''+1,L';r)\,. \tag{21}$$

This can be put in a form of a differential equation

$$\frac{1}{L}\frac{\partial}{\partial r}G_\mu(L,L';r) = gG_\mu(L+1,L';r) + 2g\sum_{L''=1}^{L+1}\Psi(L'',g)G_\mu(L-L''+1,L';r) - G_\mu(L,L';r)\,. \tag{22}$$

Multiplying by $Lx^L y^{L'}$ and performing summation over L and L' we get

$$\frac{\partial}{\partial r}G_\mu(x,y;r) = x\frac{\partial}{\partial x}\left(\left(\frac{g}{x}-1+\frac{2g}{x}(\Psi(x,g)-1)\right)G_\mu(x,y;r)\right)\,. \tag{23}$$

Recalling the form of $\Psi(x,g)$ (7) we get

$$\frac{\partial}{\partial r}G_\mu(x,y;r) = x\frac{\partial}{\partial x}\left(\frac{g}{x}f_\mu(x,g)G_\mu(x,y;r)\right)\,, \tag{24}$$

where

$$f_\mu(x,g) = \left(1-\frac{sx}{g}\right)\sqrt{1-\frac{4gx}{s^2}}\,. \tag{25}$$

The explicit solution of this equation is

$$G_\mu(x,y;r) = \frac{f_\mu(\hat{x})}{f_\mu(x)}\frac{1}{1-\hat{x}y}\,, \tag{26}$$

expressed in terms of the solution $\hat{x}(x,r)$ of the characteristic equation. We have

$$r = \int_x^{\hat{x}(x,r)}\frac{dx'}{g\,f_\mu(x')} = -\frac{1}{s\sqrt{1-\frac{4g^2}{s^3}}}\log\frac{t(x')-\sqrt{1-\frac{4g^2}{s^3}}}{t(x')+\sqrt{1-\frac{4g^2}{s^3}}}\Bigg|_x^{\hat{x}(x,r)}$$

$$t(x') = \sqrt{1-\frac{4gx}{s^2}}\,. \tag{27}$$

Formula (27) can be easily inverted. Here let us introduce notations

$$t = \sqrt{1-\frac{4gx}{s^2}}\,,$$

$$\hat{t} = \sqrt{1-\frac{4g\hat{x}}{s^2}}\,,$$

$$\delta_0 = \frac{s}{2}\sqrt{1-\frac{4g^2}{s^3}}\,. \tag{28}$$

We have

$$\hat{t} = \frac{2\delta_0}{s} \frac{\coth(\delta_0 r) + \frac{2\delta_0}{ts}}{1 + \frac{2\delta_0}{ts}\coth(\delta_0 r)} . \tag{29}$$

This formula will be very important to study the scaling behavior of the transfer matrix. To agree with conventions used in [4] we have

$$\begin{aligned} g &= g_c e^{-\Delta\mu} , \\ s &= s_c(1 + \frac{2}{\sqrt{3}}\sqrt{\Delta\mu}) , \\ \delta_0 &= \sqrt{6} g_c (\Delta\mu)^{1/4} . \end{aligned} \tag{30}$$

In the scaling region obviously $\delta_0 \to 0$. Provided x is not in the scaling region and r is not to small we get

$$\hat{t} = \frac{2\delta_0}{s}\coth(\delta_0 r) + \mathcal{O}(\delta_0^2) . \tag{31}$$

In the results presented above notice that δ_0 has *exactly* the same value as in [4] and although the formulas are different the universal large-r behavior of $G_\mu(L, L'; r)$ is the same. We show it explicitly calculating the analogue of the two-point function $G_\mu(r)$ which measures the number of links a distance r from a loop $L \to 0$. This function can be expressed in terms of the two-loop function $G_\mu(L, L'; r)$ and the one-loop function $\Psi(L', g)$. We have

$$\begin{aligned} G_\mu(r) &= \sum_{L'=1}^{\infty} G_\mu(L = 0, L') L' \Psi(L') \\ &= \oint_{C_y} \frac{dy}{2\pi i y} G_\mu(0, \frac{1}{y}; r) y \frac{\partial}{\partial y}\Psi(y) \\ &= f_\mu(\hat{x})\hat{x}\frac{\partial}{\partial \hat{x}}\Psi(\hat{x})|_{x=0} . \end{aligned} \tag{32}$$

In the derivation we made use of $f_\mu(0) = 1$. As was shown above in the scaling limit $\hat{t} \to 0$, so

$$\hat{x}\frac{\partial}{\partial \hat{x}}\Psi(\hat{x})|_{x=0} \to 2\sqrt{3} + \mathcal{O}(\delta_0^2) \tag{33}$$

is dominated by the nonuniversal part of Ψ. Expressing $f_\mu(\hat{x})$ in terms of $\hat{t}$ we have

$$f_\mu(\hat{x}) = \frac{s^3}{4g^2}\hat{t}(\hat{t}^2 - \frac{2\delta_0^2}{s}) , \tag{34}$$

and in the scaling limit we get

$$G_\mu(r) = 36\ \delta_0^3 \frac{\cosh(\delta_0 r)}{\sinh^3(\delta_0 r)}(1 + \mathcal{O}(\delta_0))\,. \tag{35}$$

Up to the numerical constant in front this is precisely the form obtained in [4].

5. Discussion

The calculation presented above is a nice demonstration of the universal character of the scaling limit in 2d simplicial gravity. Results were to large degree expected to agree with results of [1, 3, 4], however the emergence of the same asymptotics from apparently different discretized forms is a strong confirmation of these expectations.

The calculation is in many points even simpler due to the simpler analytic structure of the connected disc amplitude $\Psi(x, g)$. The scaling part of this function has almost immediately the right form.

The description in terms of ϕ^3 graphs was used in the whole paper. It is obvious that the same could be achieved using triangles as in [1, 3, 4]. The contribution both in the slicing and in the peeling methods are however different than in these references, to take into account the fact that the disc's boundary has no singularities.

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