MESON SPECTROSCOPY AND SEPARABLE POTENTIALS*

L. Leśniak

Department of Theoretical Physics
Henryk Niewodniczański Institute of Nuclear Physics
Radzikowskiego 152, 31-342 Kraków, Poland

(Received May 16, 1996)

Dedicated to Andrzej Białas in honour of his 60th birthday

Coupled channel formalism for unequal masses of interacting mesons has been developed using the separable interactions. The scattering amplitudes and the corresponding channel integrals have been calculated. A practical method to determine the potential parameters from the positions of resonances in the complex energy plane has been formulated. The two-channel formalism has been extended to three interacting channels. A useful parametrization of the 3×3 S-matrix in terms of phase shifts and inelasticity parameters has been written.

PACS numbers: 11.55. Bg, 11.55. Ds, 11.80. Gw, 12.39. Pn

1. Introduction

Separable potentials can be successfully used to solve many questions encountered in meson spectroscopy. Treatment of the intermeson interactions in a framework of the coupled channel formalism has already been applied to the description of the scalar meson properties [1]. Spectroscopy of scalar mesons is, however, full of open problems [2]. A nonet of scalar mesons is not well defined in contrast to much better known nonets of pseudoscalar, vector or tensor mesons. Since many scalar mesons f_0 decay to pairs of pseudoscalar mesons [3] we have to understand the interactions of

* This work has been partially supported by the State Committee for Scientific Research (grant No 2 P03B 231 08) and by Maria Skłodowska-Curie Fund II (grant No PAA/NSF-94-158).

these mesons. Despite of many years of effort our knowledge of pion-pion, kaon-antikaon, not speaking about $\eta - \eta$ or $\rho - \rho$ interactions is very limited. If the interaction in the $K\overline{K}$ system is attractive then a kind of the $K\overline{K}$ molecule can be formed [4, 5]. In this case the intermeson physics can be similar to the well known nuclear physics. We know for example that the proton-neutron interaction and a formation of the deuteron can be well described using the separable potentials (see for example [6]). The pion-nucleon interactions have also been studied using separable potentials [7].

The separable interactions have a relatively simple analytical form which can be very helpful in solving the coupled-channel problems [8]. Unitarity constraints, which are extremely important in understanding the underlying meson dynamics, can be imposed in a consistent way. A fact that hadrons, including mesons which we wish to study, have an internal structure leads to nonlocal effects which are taken into account in the formalism of separable interactions. Scattering amplitudes calculated in the coupled-channel framework with the separable potentials have the desired factorization properties if the interaction energy is close to the energies corresponding to the quasibound or resonant states. Therefore one can easily incorporate some phenomenological information about the masses and widths of the Breit–Wigner resonances. One should remember, however, that the parameters of the scalar resonances should be obtained and confirmed by experimental analyses of many different channels. Only then we can address very interesting questions about an internal structure of scalar mesons [1] and a possible existence of scalar glueballs [9].

In [1] we have studied properties of the $\pi\pi$ and $K\overline{K}$ interactions in the $I^G(J^{PC}) = 0^+(0^{++})$ states in the energy range from the $\pi\pi$ threshold up to 1.4 GeV. Three resonances have been found in this energy region: $f_0(500)$, $f_0(975)$ and $f_0(1400)$. The $f_0(500)$ state can be associated with often postulated very broad scalar resonance under the $K\overline{K}$ threshold (sometimes called σ or ε meson). The σ meson plays an important role in understanding the nucleon- nucleon interactions and in the Bonn meson exchange model its mass has been fixed at 550 MeV [10]. Existence of a broad σ meson has been recently confirmed by Törnqvist and Roos [11]. The $f_0(975)$ state can be interpreted as a $K\overline{K}$ bound state or a $K\overline{K}$ molecule. Nature of scalar resonances near 1500 MeV is being extensively studied [12, 13]. The scalar-isoscalar resonances have the QCD vacuum quantum numbers and can be mixed with the scalar glueballs which are predicted by the lattice calculations (see Ref. [14]).

Parameters of the separable potentials can be fixed by comparison of the calculated physical quantities like the total and differential cross-sections, phase shifts, inelasticity parameters, masses, widths and branching ratios of different resonances with the corresponding quantities obtained in the experimental analyses [1]. One can also study different threshold parameters like the scattering lengths and the effective ranges [15]. In those studies the analytical structure of the scattering amplitudes and a possibility of the analytical continuation of the physical amplitudes to the complex energy plane played an important role. Some simplifications of the complex amplitudes were possible since the interacting particles in the initial or final states had equal masses (systems of two pions or two kaons). We have been able to perform analytically the calculations of the S-matrix elements (see appendices of [1]).

In Section 2 we study a case when two mesons have unequal masses. The formulae which are derived can be applied to study the production and decay properties of the isovector scalar meson $a_0(980)$. Its mass is very close to the mass of the $f_0(975)$ isoscalar meson. The $a_0(980)$ can decay not only into a pair of kaons but also into a system of π and η mesons. In the latter case two mesons have very different masses and the corresponding amplitudes have a more complicated structure. Integrals appearing in the scattering amplitudes are evaluated in Section 3. Section 4 deals with a problem of an extraction of the separable potential parameters from a very limited set of data. In Section 5 we discuss the amplitudes corresponding to a system of three interacting channels. This is a characteristic situation at higher energies when a number of coupled channels increases. Separable potentials offer a useful tool to treat those complex interactions preserving unitarity of the full S-matrix.

2. Coupled-channel formalism for unequal masses of interacting particles

Let us describe two-channel s-wave interactions. We assume that in the first channel (label 1) masses of two interacting mesons are different and in the second channel (label 2) they are equal. In view of a future application of this approach to the description of the $a_0(980)$ meson we can denote by m_π and m_η the pion and η masses in the first channel and by m_K the kaon mass in the second channel. For simplicity we use rank-one separable potentials in the momentum space

$$\langle \boldsymbol{p} \mid V_{ij} \mid \boldsymbol{q} \rangle = \lambda_{ij}\, g_i(p) g_j(q)\ , \qquad (i,j = 1,2)\,. \tag{1}$$

In this equation $\lambda_{i,j}$ are the coupling constants and g_i are the form factors which are functions of the relative momenta q (in the initial channel) or p (in the final channel). V_{ii} is the elastic interaction in the channel i and $V_{1,2}$ or $V_{2,1}$ are the transition potentials satisfying the following symmetry relation:

$$\langle \boldsymbol{p} \mid V_{12} \mid \boldsymbol{q} \rangle = \langle \boldsymbol{q} \mid V_{21} \mid \boldsymbol{p} \rangle \tag{2}$$

in addition to $\lambda_{12} = \lambda_{21}$.

The scattering amplitude T satisfies the Lippmann–Schwinger equation:

$$\langle \boldsymbol{p} \mid T \mid \boldsymbol{q} \rangle = \langle \boldsymbol{p} \mid V \mid \boldsymbol{q} \rangle + \int \frac{d^3 s}{(2\pi)^3} \langle \boldsymbol{p} \mid V \mid \boldsymbol{s} \rangle \langle \boldsymbol{s} \mid G \mid \boldsymbol{s} \rangle \langle \boldsymbol{s} \mid T \mid \boldsymbol{q} \rangle \,, \tag{3}$$

where V, G, T are 2×2 matrices, V is the interaction matrix defined in (1) and G is the diagonal matrix of propagators written in the center of mass system:

$$\langle \boldsymbol{s} \mid G_{ij} \mid \boldsymbol{s} \rangle = G_i(s)\, \delta_{ij} \qquad (i,j = 1,2)\,, \tag{4}$$

In Eq. (4)

$$G_1^{-1}(s) = E - E_\pi(s) - E_\eta + i\varepsilon \,, \qquad \varepsilon \to 0(+)\,, \tag{5}$$

s is the relative momentum, E is the total energy,

$$E_\pi(s) = \sqrt{s^2 + m_\pi^2}\,, \tag{6}$$

and

$$E_\eta(s) = \sqrt{s^2 + m_\eta^2}\,. \tag{7}$$

The average pion mass m_π=137.27 MeV and the η mass is m_η=547.45 MeV. Similarly we write:

$$G_2^{-1}(s) = E - 2E_K(s) + i\varepsilon \,, \tag{8}$$

where $E_K(s) = \sqrt{s^2 + m_K^2}$ is the relativistic kaon energy , m_K=495.69 MeV is the average of the charged and neutral kaon masses.

The form factors have the Yamaguchi form [16]:

$$g_i(p) = \sqrt{\frac{2\pi}{m_i}} \frac{1}{p^2 + \beta_i^2}\,, \tag{9}$$

where m_i are the reduced masses: $m_1 = m_\pi\, m_\eta/(m_\pi + m_\eta)$, $m_2 = m_K/2$ and β_i are the range parameters. The potential matrix has altogether five parameters which can be fixed by comparison of the scattering amplitudes T_{ij} with experiment. We postulate the following form of the scattering matrix elements:

$$\langle \boldsymbol{p} \mid T_{ij} \mid \boldsymbol{q} \rangle = g_i(p)\, t_{ij}\, g_j(q)\,, \tag{10}$$

where t_{ij} are the energy dependent reduced amplitudes. In (10) T_{11} and T_{22} are the $\pi\eta$ and $K\overline{K}$ elastic scattering amplitudes and T_{21} and T_{12} are the transition $K\overline{K} \to \pi\eta$ and $\pi\eta \to K\overline{K}$ amplitudes. The system (3) of the coupled integral equations satisfied by the T_{ij} elements can be replaced by a set of algebraic equations for the reduced amplitudes t_{ij} written in the 2×2 matrix form:

$$t = \lambda + \lambda I t \,. \tag{11}$$

Here λ is the symmetric 2×2 matrix of the coupling constants

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} , \tag{12}$$

and I is the diagonal 2×2 matrix consisting of the integrals

$$I_{ii} = \int \frac{d^3 s}{(2\pi)^3} g_i(s) G_i(s) g_i(s) \,. \tag{13}$$

The matrix equation (11) is easily solved:

$$t = (1 - \lambda I)^{-1} \lambda \,. \tag{14}$$

Substituting the elements of (14) to (10) we obtain a formal solution of the scattering problem.

If the energy E is higher than the $K\overline{K}$ threshold mass we can write the on-shell scattering matrix elements $T_{ij}(k_1, k_2)$ in terms of the S-matrix elements

$$S_{ij} = \delta_{ij} - \frac{i}{2\pi} \sqrt{k_i \alpha_i k_j \alpha_j}\, T_{ij}(k_i, k_j) \,, \tag{15}$$

where the $\pi\pi$ channel and $K\overline{K}$ channel momenta k_1 and k_2 are defined by the energy conservation condition:

$$E = E_\pi + E_\eta = 2E_K \,, \tag{16}$$

$$\alpha_1 = \frac{E_\pi E_\eta}{E_\pi + E_\eta} \tag{17}$$

and

$$\alpha_2 = \frac{1}{2} E_K \,. \tag{18}$$

In (16) the pion on-shell energy $E_\pi = \sqrt{k_1^2 + m_\pi^2}$, the η meson energy $E_\eta = \sqrt{k_1^2 + m_\eta^2}$ and the kaon energy $E_K = \sqrt{k_2^2 + m_K^2}$. The inverse relations for the center-of-mass momenta are:

$$k_1 = \frac{1}{2E} \sqrt{[E^2 - (m_\eta + m_\pi)^2][E^2 - (m_\eta - m_\pi)^2]} \,, \tag{19}$$

and

$$k_2 = \sqrt{\frac{E^2}{4} - m_K^2}\,. \tag{20}$$

All the functions t_{ij} in (11) are inversely proportional to the Jost function

$$D(E) = det(1 - \lambda I)\,. \tag{21}$$

Explicit form of the Jost function, which can be written as a function of two momenta k_1 and k_2, is

$$D(k_1, k_2) = D_1(k_1)D_2(k_2) - F(k_1, k_2)\,, \tag{22}$$

where

$$D_i(k_i) = 1 - \Lambda_{ii} J_{ii}(k_i), \qquad (i, j = 1, 2)\,, \tag{23}$$

and

$$F(k_1, k_2) = \Lambda_{12}^2 J_{11}(k_1) J_{22}(k_2)\,. \tag{24}$$

The function $F(k_1, k_2)$ describes the interchannel coupling. In (23), (24), we use the dimensionless coupling constants defined as

$$\Lambda_{ij} = \frac{\lambda_{ij}}{2(\beta_i\beta_j)^{3/2}}\,, \tag{25}$$

and the redefined integrals

$$J_{ii} = 2\beta_i^3\, I_{ii}\,. \tag{26}$$

All the S-matrix elements can be expressed by the Jost function ratios as follows:

$$S_{11} = \frac{D(-k_1, k_2)}{D(k_1, k_2)}\,, \tag{27}$$

$$S_{22} = \frac{D(k_1, -k_2)}{D(k_1, k_2)}\,, \tag{28}$$

$$S_{12}^2 = S_{11}S_{22} - \frac{D(-k_1, -k_2)}{D(k_1, k_2)}\,. \tag{29}$$

Above the $K\overline{K}$ threshold the S-matrix can be parametrized in terms of the inelasticity parameter η and the phase shifts $\delta_{\pi\eta}$ and $\delta_{K\overline{K}}$:

$$S = \begin{pmatrix} \eta e^{2i\delta_{\pi\pi}} & i\sqrt{1-\eta^2}\, e^{i(\delta_{\pi\eta} + \delta_{K\overline{K}})} \\ i\sqrt{1-\eta^2}\, e^{i(\delta_{\pi\eta} + \delta_{K\overline{K}})} & \eta e^{2i\delta_{K\overline{K}}} \end{pmatrix}\,. \tag{30}$$

Some formulae presented in this chapter can be generalized or derived from the appropriate formulae in [1] but they have been written in this chapter for a completeness and a further use.

3. Evaluation of channel integrals

Calculation of the channel integrals J_{11} and J_{22} defined in (13) and (26) is an important step in solving the meson scattering problem. Let us start from the expression for J_{11}:

$$J_{11} = \frac{2\beta^3}{\pi m_1} \int_0^\infty ds \frac{s^2}{(s^2+{\beta_1}^2)^2} \frac{1}{E + i\varepsilon - \sqrt{s^2+{m_\pi}^2} - \sqrt{s^2+{m_\eta}^2}} . \tag{31}$$

After a simple algebra this integral can be rewritten in the following equivalent form:

$$J_{11} = \frac{{\beta_1}^3}{E\pi m_1}(c_1+c_2+c_3+c_4) , \tag{32}$$

where

$$c_1 = \int_0^\infty ds \frac{s^2(E_\pi E_\eta + k_1^2 - s^2)}{(s^2+{\beta_1}^2)^2(k_1^2+i\varepsilon - s^2)} , \tag{33}$$

$$c_2 = E_\eta Y(k_1, m_\pi, \beta_1) , \tag{34}$$

$$c_3 = E_\pi Y(k_1, m_\eta, \beta_1) , \tag{35}$$

$$c_4 = \int ds \frac{s^2\sqrt{s^2+{m_\pi}^2}\sqrt{s^2+{m_\eta}^2}}{(s^2+{\beta_1}^2)^2(k_1^2+i\varepsilon - s^2)} , \tag{36}$$

and

$$Y(k,m,\beta) = \int_0^\infty ds \frac{s^2\sqrt{s^2+m^2}}{(s^2+\beta^2)^2(k^2+i\varepsilon-s^2)} . \tag{37}$$

The integral c_1 is elementary:

$$c_1 = \frac{\pi}{4\beta_1}\left[1 - \frac{E_\pi E_\eta}{\beta_1^2}\frac{1}{(1-ik_1/\beta_1)^2}\right] . \tag{38}$$

The integral $Y(k,m,\beta))$ can be done analytically as explained in the Appendix A of [1]. It reads:

$$Y(k,m,\beta) = -\frac{1}{\beta^2}\frac{1}{(1+a^2)^2}\left[\frac{1}{2}(1+a^2)(1-b^2d) + a^2r^2(d-H)\right] , \tag{39}$$

where $a = k/\beta$, $b = m/\beta$, $r = \sqrt{k^2 + m^2}/k$, $H = F(r^2)$ and $d = F(1 - b^2)$. The integral $F(x^2)$ is defined by

$$F(x^2) = \int_0^1 \frac{dy}{1 - y^2 x^2 + i\varepsilon} . \tag{40}$$

If x^2 is real then

$$F(x^2) = \begin{cases} \frac{1}{2x} \ln\left(\frac{x+1}{x-1}\right) - \frac{i\pi}{2x} & \text{if } x^2 > 1 , \\ \frac{1}{2x} \ln\left(\frac{1+x}{1-x}\right) & \text{if } 0 < x^2 < 1 , \\ 1 & \text{if } x^2 = 0 , \\ \arctan(|x|)/|x| & \text{if } x^2 < 0 . \end{cases} \tag{41}$$

The integral c_4 cannot be expressed by elementary functions. Using (36) it can be calculated numerically for real values of k_1 or in the complex ${k_1}^2$ plane for positive values of $\mathrm{Im}\, {k_1}^2$. For negative values of $\mathrm{Im}\, {k_1}^2$ it must be, however, analytically continued from an upper half-plane of ${k_1}^2$. In order to calculate it properly in the whole ${k_1}^2$ plane we first use the following integral representation

$$\frac{1}{\sqrt{s^2 + {m_\pi}^2}\sqrt{s^2 + {m_\eta}^2}} = \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \frac{1}{s^2 + x m_\pi^2 + (1-x) m_\eta^2} \tag{42}$$

obtaining the c_4 in the form:

$$c_4 = \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \int_0^\infty \frac{ds\, s^2 (s^2 + m_\pi^2)(s^2 + m_\eta^2)}{(s^2 + \beta_1^2)^2 (k_1^2 + i\varepsilon - s^2)[s^2 + x m_\pi^2 + (1-x) m_\eta^2]} . \tag{43}$$

The integral over s can be done and finally we get

$$c_4 = \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{x(1-x)}} [(\frac{A}{2\beta_1^3} + \frac{B}{\beta_1}) \frac{1}{Z - \beta_1^2} - \frac{A}{\beta_1} \frac{1}{(Z - \beta_1^2)^2} - \frac{1}{\sqrt{Z}}$$
$$+ \frac{A}{\sqrt{Z}(Z - \beta_1^2)^2} - \frac{B}{\sqrt{Z}(Z - \beta_1^2)} - \frac{C}{\sqrt{Z}(k_1^2 + Z)} + \frac{iC}{k_1} \frac{1}{k_1^2 + Z}] , \tag{44}$$

where

$$Z = xm_\pi^2 + (1-x)m_\eta^2\,, \tag{45}$$

$$A = -\frac{\beta_1^2}{k_1^2+\beta_1^2}(\beta_1^2 - m_\pi^2)(\beta_1^2 - m_\eta^2)\,, \tag{46}$$

$$B = -\frac{\beta_1^2(m_\pi^2 + m_\eta^2 - 2\beta_1^2)}{k_1^2+\beta_1^2} + \frac{k_1^2(\beta_1^2 - m_\pi^2)(\beta_1^2 - m_\eta^2)}{(k_1^2+\beta_1^2)^2}\,, \tag{47}$$

$$C = -\frac{k_1^2(k_1^2 + m_\pi^2)(k_1^2 + m_\eta^2)}{(k_1^2+\beta_1^2)^2}\,. \tag{48}$$

If $m_\pi^2 \le \beta_1^2 \le m_\eta^2$ then to avoid zeroes in the integrand denominators for $Z = \beta_1^2$ at $x = (m_\eta^2 - \beta_1^2)/(m_\eta^2 - m_\pi^2)$ the value of the integrand function in the parenthesis [...] of Eq. (44) should be replaced by

$$\frac{3A}{8\beta_1^5} + \frac{1}{2\beta_1^3}(-k_1^2 + 2\beta_1^2 - m_\pi^2 - m_\eta^2 - C) - \frac{1}{\beta_1}(1 + \frac{C}{k_1^2+\beta_1^2}) + \frac{iC}{k_1(k_1^2+\beta_1^2)}\,. \tag{49}$$

The formula (44) completes the sum of integrals needed to evaluate J_{11} in (32).

The integral J_{22} for the $K\overline{K}$ channel has been calculated in [1] and can be expressed as

$$J_{22} = -\frac{E}{4m_K(1 - ik_2/\beta_2)^2} + \frac{2\beta_2^3}{\pi m_K} Y(k_2, m_K, \beta_2)\,. \tag{50}$$

4. Determination of potential parameters

Determination of the separable potential parameters is fairly easy if we have to our disposal a set of many scattering data describing the meson-meson interaction amplitudes. But even in a case when we know only positions and widths of resonances appearing in different reaction channels we are able to fix some of the unknown potential parameters. Let us discuss a special case when two resonances at the complex energies E^r and E^R exist. Then the S-matrix has poles at these energies and the corresponding Jost function vanishes (Eqs. (27)–(29),(22)). So two complex equations:

$$D(k_1^r, k_2^r) = 0\,, \tag{51}$$

and

$$D(k_1^R, k_2^R) = 0 \tag{52}$$

have to be satisfied for the pairs of complex momenta k_1^r, k_2^r and k_1^R, k_2^R related to E^r and E^R. Substituting (23) and (24) into (22) we get the following two equations equivalent to (51):

$$1 - \Lambda_{11}\,\mathrm{Re}\,J_{11}(k_1^r) - \Lambda_{22}\,\mathrm{Re}\,J_{22}(k_2^r) + (\Lambda_{11}\Lambda_{22} - \Lambda_{12}^2)\,\mathrm{Re}[J_{11}(k_1^r)J_{22}(k_2^r)] = 0\,, \quad (53)$$

$$-\Lambda_{11}\,\mathrm{Im}\,J_{11}(k_1^r) - \Lambda_{22}\,\mathrm{Im}\,J_{22}(k_2^r) + (\Lambda_{11}\Lambda_{22} - \Lambda_{12}^2)\,\mathrm{Im}[J_{11}(k_1^r)J_{22}(k_2^r)] = 0\,. \quad (54)$$

Similar two equations can be written changing the set of variables k_1^r, k_2^r into the set k_1^R, k_2^R. In this way we obtain a system of four equations which can be used to fix four parameters. Thus solving this system of algebraic equations we can eliminate three coupling constants:

$$\Lambda_{11} = \frac{a\,\mathrm{Im}\,J_{22}(k_2^R) + c\,\mathrm{Im}[J_{11}(k_1^R)\,\mathrm{Im}\,J_{22}(k_2^R)]}{(a-d)\,\mathrm{Im}[J_{11}(k_1^R)\,\mathrm{Im}\,J_{22}(k_2^R)] - \mathrm{Im}\,J_{11}(k_1^R) - b\,\mathrm{Im}\,J_{22}(k_2^R)}\,, \quad (55)$$

$$\Lambda_{22} = a + b\,\Lambda_{11}\,, \quad (56)$$

$$\Lambda_{12}^2 = b\,\Lambda_{11}^2 + d\,\Lambda_{11} + c\,. \quad (57)$$

In (55), (56) and (57) we define the following constants:

$$a = [\mathrm{Re}\,J_{22}(k_2^r) - r\,\mathrm{Im}\,J_{22}(k_2^r)]^{-1}\,, \quad (58)$$

$$r = \frac{\mathrm{Re}[J_{11}(k_1^r)J_{22}(k_2^r)]}{\mathrm{Im}[J_{11}(k_1^r)J_{22}(k_2^r)]}\,, \quad (59)$$

$$b = -a\,[\mathrm{Re}\,J_{11}(k_1^r) - r\,\mathrm{Im}\,J_{11}(k_1^r)]\,, \quad (60)$$

$$c = -a\,\frac{\mathrm{Im}\,J_{22}(k_2^r)}{\mathrm{Im}[J_{11}(k_1^r)J_{22}(k_2^r)]}\,, \quad (61)$$

$$d = a - \frac{b\,\mathrm{Im}\,J_{22}(k_2^r) + \mathrm{Im}\,J_{11}(k_1^r)}{\mathrm{Im}[J_{11}(k_1^r)J_{22}(k_2^r)]}\,. \quad (62)$$

The fourth constant β_1 or β_2 can be obtained from the equation:

$$1 - \Lambda_{11}\,\mathrm{Re}\,J_{11}(k_1^R) - \Lambda_{22}\,\mathrm{Re}\,J_{22}(k_2^R) + (\Lambda_{11}\Lambda_{22} - \Lambda_{12}^2)\,\mathrm{Re}[J_{11}(k_1^R)J_{22}(k_2^R)] = 0\,, \quad (63)$$

in which the integrals J_{11} and J_{22} depend also on the range parameters β_1 and β_2, respectively (see Section 3). The last equation has to be solved numerically.

In a possible application of the method outline above to the description of the scalar $I = 1$ mesons we should mention the Crystal Barrel paper

([17]) in which a resonance of a mass $M = 1450$ MeV and a width 270 MeV has been observed. This may be a partner of $a_0(980)$, so in principle one can try to fix four potential parameters in the $K\overline{K}$ and $\pi\eta$ channels.

5. Three channels

If the total energy grows up more and more channels are open and we have to extend the model to include new channels. So after a discussion of the two-channel model let us pass to a formalism of three coupled channels. It is not always the case that we have two-body interaction in each of the three channels but from a practical point of view it happens quite often that mesons are grouped into resonances decaying into pairs of particles. Examples of such channels are $\eta\eta$, $\sigma\sigma$ or $\rho\rho$. If the energy is higher than a sum of the meson masses in the third channel then we are dealing with a system of three dynamically open channels. There are, however, effects which can be observed even at energies below the third channel threshold. We have discussed such a case in [18] introducing a third closed channel in addition to the $\pi\pi$ and $K\overline{K}$ channels and investigating the properties of the $f_0(975)$ meson. In [19] an attempt has been made to treat S-wave $\pi\pi$, $K\overline{K}$ and $\rho\rho(\omega\omega)$ interactions using a generalization of a simple nonrelativistic two-channel model developed in [20].

In this chapter we first extend the t-matrix formulae obtained in Section 2 for the two-channel case. The generalization is very easy if the separable interactions are rank-one potentials as in (1). Then the λ matrix is the symmetric 3×3 matrix of coupling constants (compare (12)) and we introduce the additional channel integral I_{33} as in (13). In this integral we define the third propagator $G_3(s)$ which has a structure similar to (5) or (8) depending on the masses of particles in the third channel. Below we give a formula for the corresponding Jost function (compare (22),(23)):

$$D(k_1, k_2, k_3) = D_1(k_1)D_2(k_2)D_3(k_3) - F(k_1, k_2, k_3)\,, \tag{64}$$

where

$$F(k_1, k_2, k_3) = \Lambda_{12}^2\, J_{11}(k_1)J_{22}(k_2) + \Lambda_{13}^2\, J_{11}(k_1)J_{33}(k_3)$$

$$+ \Lambda_{23}^2\, J_{22}(k_2)J_{33}(k_3) + 2\Lambda_{12}\Lambda_{13}\Lambda_{23}\, J_{11}(k_1)J_{22}(k_2)J_{33}(k_3)\,. \tag{65}$$

This formula can be further extended to a case of more complicated interactions like the rank-two potentials discussed in [1].

We have already seen in Section 2 that the S-matrix elements can be expressed in terms of the Jost functions (compare (27)–(29)). Similar formulae are valid in the three channel case. For example:

$$S_{11} = \frac{D(-k_1, k_2, k_3)}{D(k_1, k_2, k_3)}, \tag{66}$$

$$S_{33} = \frac{D(k_1, k_2, -k_3)}{D(k_1, k_2, k_3)}, \tag{67}$$

$$S_{13}^2 = S_{13}S_{33} - \frac{D(-k_1, k_2, -k_3)}{D(k_1, k_2, k_3)}. \tag{68}$$

Other S-matrix elements can be obtained from the above formulae by a proper permutation of indices. A generalization of the convenient 2×2 S-matrix parametrization (30) in terms of the phase shifts and the inelasticity coefficient is not obvious. Remembering, however, that the S-matrix must satisfy the unitarity condition $S S^{\dagger} = 1$, we can parametrize the diagonal elements in terms of three phase shifts δ_i ($i = 1, 2, 3$) and three inelasticity coefficients η_i:

$$S_{ii} = \eta_i \, e^{2i\delta_i}. \tag{69}$$

The total number of independent parameters is six. The inelasticity parameters have to satisfy the following boundary conditions:

$$0 \leq \eta_i \leq 1, \tag{70}$$

and

$$| 1 - \eta_j - \eta_k | \leq \eta_i \leq 1 - | \eta_j - \eta_k |; \quad i, j, k = 1, 2, 3; \; i \neq j \neq k. \tag{71}$$

The nondiagonal elements have more complicated form than those elements in (30):

$$S_{ij} = i\gamma_{ij} \, e^{i\phi_{ij}} \; ; \quad i \neq j = 1, 2, 3, \tag{72}$$

$$\gamma_{ij}^2 = \tfrac{1}{2}(1 + \eta_k^2 - \eta_i^2 - \eta_j^2) \; ; \quad k \neq i, k \neq j, i \neq j, \tag{73}$$

$$\phi_{ij} = \delta_i + \delta_j + \alpha_{ij}, \tag{74}$$

$$\alpha_{ij} = \arcsin \sqrt{\frac{u_{ij}^2 - w_{ij}^2}{v_{ij}^2 - w_{ij}^2}}, \tag{75}$$

$$u_{ij} = \frac{\gamma_{ik}\gamma_{jk}}{\gamma_{ij}}, \quad \gamma_{ij} \neq 0, \; k \neq i, k \neq j, \tag{76}$$

$$w_{ij} = \eta_i - \eta_j, \tag{77}$$

$$v_{ij} = \eta_i + \eta_j. \tag{78}$$

The S-matrix is symmetric: $S_{ij} = S_{ji}$.

Let us notice that in the case where one of the three channels (for example $i = 1$) is not coupled to another channel (for example $j = 2$) then not only the corresponding matrix element $S_{12} = 0$ but also $S_{13} = 0$ *or* $S_{23} = 0$. In the former case the channel 1 is completely decoupled from the channels 2 and 3 or if $S_{23} = 0$ then the channel 2 is decoupled from 1 and 3. This is a result of the unitarity of the full S-matrix.

It is a pleasure to dedicate this article to Andrzej Białas — my teacher of mathematical methods of physics. I am grateful to J.-P. Dedonder for very helpful discussions and a fruitful collaboration within the French–Polish Convention.

REFERENCES

[1] R. Kamiński, L. Leśniak, J.-P. Maillet, *Phys. Rev.* **D50**, 3145 (1994).
[2] N.A. Törnqvist, *Z. Phys.* **C68**, 647 (1995).
[3] Particle Data Group, Phys. Rev. **D50**, part I (1994).
[4] J. Weinstein, N. Isgur, *Phys. Rev.* **D27**, 588 (1983).
[5] L. Leśniak, in Proc. of the III Int. Conf. on Hadron Spectroscopy: HADRON '89, eds F. Binon, J.-M. Frère and J.-P. Peigneux, Editions Frontières, 1989, p. 269.
[6] J.M. Eisenberg, W. Greiner, *Microscopic Theory of the Nucleus*, North-Holland Publishing Company, Amsterdam 1976, p. 66.
[7] J.T. Londergan, K.W. McVoy, E.J. Moniz, *Ann. Phys.* **86**, 147 (1974).
[8] F. Cannata, J.-P. Dedonder, L. Leśniak, *Z. Phys.* **A334**, 457 (1989).
[9] C. Amsler, F.E. Close, *Phys. Rev.* **D53**, 295 (1996).
[10] R. Machleidt, K. Holinde, Ch. Elster, *Phys. Rep.* **149**, 1 (1987).
[11] N.A. Törnqvist, M. Roos, *Phys. Rev. Lett.* **76**, 1575 (1996).
[12] D.V. Bugg, in Proceedings of the International Europhysics Conference on High Energy Physics, Marseille 1993, eds J. Carr and M. Perrottet, Editions Frontières,1994, p. 717.
[13] C. Amsler, in Proceedings of XXVII International Conference on High Energy Physics, Glasgow 1994, eds P. Bussey and I. G. Knowles, Institute of Physics Publishing Ltd, Bristol 1995, p. 199.
[14] G. Bali *et al.*, *Phys. Lett.* **B309**, 378 (1993).
[15] R. Kamiński, L. Leśniak, *Phys. Rev.* **C51**, 2264 (1995).
[16] Y. Yamaguchi, Y. Yamaguchi, *Phys. Rev.* **95**, 1635 (1954).
[17] C. Amsler *et al.*, Crystal Barrel Collaboration *et al.*, *Phys. Lett.* **B333**, 277 (1994).
[18] F. Cannata, J.-P. Dedonder, L. Leśniak, *Z. Phys.* **A343**, 451 (1992).
[19] W.M. Kloet, B. Loiseau, *Z. Phys.* **A353**, 227 (1995).
[20] F. Cannata, J.-P. Dedonder, L. Leśniak, *Phys. Lett.* **B207**, 115 (1988).