RELATIVISTIC CONTRIBUTIONS TO THE ELECTROMAGNETIC BINDING AND SCATTERING OF PIONS BY ATOMS

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Dedicated to Andrzej Białas in honour of his 60^{th} birthday

A relativistic two-particle theory is used for the calculation of the lowest order correction to the electromagnetic binding and scattering of a pion by an atom. The binding energy is found to be equal to the result obtained with the Klein-Gordon equation. The relativistic correction to the differential cross section, however, differs from the corresponding Klein-Gordon result.

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1. Introduction

Many years ago the relativistic fine-structure splitting in pionic Ti and Fe atoms was measured by Delker *et al.* [1] and by Wang *et al.* [2] The observed splitting agreed with the predictions of the Klein-Gordon equation, but also with those of the relativistic Schrödinger equation

$$\left[\sqrt{p^2+m^2}-\frac{\alpha}{r}\right]\psi = E\psi . \tag{1.1}$$

In both cases the spectrum up to terms of order α^4 is given by $E_{nl} = m\varepsilon_{nl}$ [3], with

$$\varepsilon_{nl} = 1-\frac{\alpha^2}{2n^2}-\frac{\alpha^4}{2n^3}\left\{\frac{1}{l+1/2}-\frac{3}{4n}\right\}+\cdots . \tag{1.2}$$

The Klein-Gordon equation was also used by Kang and Brown [4] to calculate the differential cross section for scattering of a pion by an atom. To order α^3 they found

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(q^2+m^2)}{4q^4\sin^4\frac{1}{2}\theta}\left[1 + R^{\mathrm{KG}}(q,\theta) + \cdots\right], \tag{1.3}$$

with

$$R^{\mathrm{KG}}(q,\theta) = \frac{\pi\alpha q \sin\frac{1}{2}\theta}{\sqrt{q^2+m^2}}. \tag{1.4}$$

In the same way as for the spectrum of bound states [5], it can be shown that also for the differential cross section the relativistic Schrödinger equation gives the same result (1.4). Therefore, when it comes to the question as to which is the best one-particle equation to describe pionic atoms, the answer is not clear. In addition to the experimental difficulty of distinguishing between several theories, there is on the theoretical side the uncomfortable property of the eigenfunctions of the Klein-Gordon equation not being orthogonal. In 1926 Pauli [6] therefore wrote in a letter to Schrödinger that he did *"... not believe that the relativistic equation of 2. order with the many fathers corresponds to reality."* For a critical discussion of the work that has since been done on the particle interpretation of the Klein–Gordon equation, the article by Rizov, Sazdjian and Todorov [7] should be consulted.

In the present paper the Coulomb binding and scattering of a negatively charged pion by an atom will again be calculated, but now in the frame work of a relativistic invariant two-particle theory, which was put forward in 1975 [8] and further developed and applied in later years. See [9] for references. The next section contains a concise exposition of this theory. In Sections 3 and 4 it will then be applied to the calculation of ε_{nl} and of $d\sigma/d\Omega$ respectively. The result will be that to order α^4 the binding energy will again coincide with (1.2), obtained from the Klein–Gordon equation. The correction to the differential cross section for pion-atom scattering will, however, differ from Eq. (1.4).

2. The ingredients of the theory

For low energies the scattering of two particles with an interaction potential $V_{\alpha\beta}$ is usually described by the Lippmann–Schwinger equation [10]

$$M_{\alpha\beta}(z) = V_{\alpha\beta} - \int_\gamma \frac{V_{\alpha\gamma}M_{\gamma\beta}(z)}{\varepsilon_\gamma - z}\delta(\vec{P}_\gamma - \vec{P}_\beta) \quad \text{for} \quad \vec{P}_\alpha = \vec{P}_\beta. \tag{2.1}$$

The notation α,β,γ is used for the two momenta of the particles in the final, initial and intermediate state. The integration element is $\int_\gamma \cdots =$

$\int d\vec{k}_1 d\vec{k}_2 \cdots$, while $\vec{P}_\alpha = \vec{P}_\beta = \vec{P}_\gamma$ stand for the total kinetic momentum of the two particles, which is not changed during the interaction. The total kinetic energy of the intermediate state γ is written as ε_γ. The scattering amplitude $M_{\alpha\beta}(z)$ for $z \to \varepsilon_\beta$ is used in the standard way for the calculation of the differential cross section, whereas the energies of possible bound states are given by the poles of the partial wave components of $M_{\alpha\beta}(z)$. Many authors have taken this Lippmann–Schwinger equation as starting point for the construction of a Lorentz invariant theory. The first two of these so called "quasipotential theories", were formulated by Logunov and Tavkhelidze [11] and by Blankenbecler and Sugar [12]. They always have the form of a Lippmann–Schwinger equation, with an integration over the three-momenta of the particles, such that the total three-momentum is conserved. No matter whether the quasipotential theory is obtained by a certain reduction technique from the Bethe-Salpeter equation, or rather postulated as a phenomenological theory, there is always some freedom in the way the particle masses are allowed to go off shell and how the propagator $(\varepsilon_\gamma - z)^{-1}$ is changed. The theories of Todorov [13] and of Gross [14] are examples of this kind. In order to get an impression of the vast literature on the subject, the reader should consult the bibliography compiled by Pyykkö [15], with its 6577 references to papers on relativistic theory of atoms and molecules, published between 1916 and 1992. The theory that will be presented here is again of the phenomenological quasipotential type. It avoids, however, the arbitrariness of which particle to put on the mass shell: they are all *on* mass shell. The theory will also be more symmetric in that in intermediate states not only the total kinetic energy of the particles goes off shell, but also their total kinetic momentum. The idea is that if the mediating field carries energy, it should in a relativistic theory also carry three-momentum. Still there should be some conserved three-vector, so as to get back the Lippmann–Schwinger equation in the nonrelativistic limit. The possibility to do all this without violating any sacred principle, is related to the assumption that in intermediate states not the total momentum is conserved, but rather the centre of mass velocity, which for N particles is defined as

$$\vec{v} = \frac{\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_N}{p_1^0 + p_2^0 + \cdots + p_N^0} = \frac{\vec{P}}{P^0}, \tag{2.2}$$

with the free particle relation

$$p_i^0 = +\sqrt{|\vec{p}_i|^2 + m_i^2} \qquad \text{for all} \quad i = 1, \ldots, N \tag{2.3}$$

between the momentum components and the rest mass m_i. In a classical two-particle system it is almost obvious, at least it was to Møller [16], that this total particle velocity is constant. It must be stressed, however, that this is not a mathematical identity, but rather a physical hypothesis. For two particles it amounts to assuming *action* = *reaction* and therefore has a certain degree of plausibility. One could even think of an experimental verification of the constancy of $\vec{v}$, but a classical system with two relativistic particles, for which $\vec{v}$ is a measurable quantity, will be hard to find.[1] Perhaps the binary pulsars, a number of which have now been discovered [17, 18], could be used to test the hypothesis.

After these preliminaries the new quasipotential theory for relativistic transition amplitudes is now defined by the following generalisation of the Lippmann–Schwinger equation

$$M_{\alpha\beta}(s) = V_{\alpha\beta} - \int\limits_{\gamma} V_{\alpha\gamma} L_{\gamma}(\vec{v}, s) M_{\gamma\beta}(s) \quad \text{for} \quad \vec{v}_{\alpha} = \vec{v}_{\beta} \equiv \vec{v}\,. \tag{2.4}$$

The integration element for the intermediate state $\gamma = (p_1, \cdots, p_n)$ is

$$\int\limits_{\gamma} \cdots = \int dp_1 \ldots dp_n \prod_{j=1}^{n} \delta(p_j^2 - m_j^2)\theta(p_j^0) \cdots \tag{2.5}$$

and the velocities $\vec{v}_{\alpha}$ and $\vec{v}_{\beta}$ are defined by (2.2). The propagator on the upper rim of the unitarity cut is taken as

$$L_{\gamma}(\vec{v}, s_0 + i0) = \int\limits_{0}^{\infty} \frac{ds'}{s' - s_0 - i0} \delta_4(P_{\gamma} - \frac{s'}{s_0} P_0)\,, \tag{2.6}$$

in which the four-momenta P_{γ} and P_0 are equal to

$$P_{\gamma} = \sqrt{\frac{s_{\gamma}}{1 - |\vec{v}_{\gamma}|^2}}(1, \vec{v}_{\gamma}) \quad \text{and} \quad P_0 = \sqrt{\frac{s_0}{1 - |\vec{v}|^2}}(1, \vec{v})\,. \tag{2.7}$$

The unitarity of the S-matrix is guaranteed by the hermiticity of $V_{\alpha\beta}$ and by the equation

$$\lim_{s \to s_0 + i0} \operatorname{Im} L_{\gamma}(\vec{v}, s) = \pi \delta_4(P_{\gamma} - P_0)\,. \tag{2.8}$$

The form (2.6) of the propagator furthermore ensures the equality of the total velocities in the initial and intermediate state $\vec{v} = \vec{v}_{\gamma}$. The invariance under proper Lorentz transformations is manifest once an invariant potential

[1] Dr P. Hoyng suggested the following possibility.

$V_{\alpha\beta}$ has been chosen. Also invariance under the full Poincaré group has been proved. The situation is like in Dirac's so-called "point form" of classical relativistic mechanics [19]. The generators for rotations and boosts are not changed by the presence of interaction, while the generators for the total momentum can be written as

$$P_\mu^{\text{tot}} = P_\mu + W u_\mu , \tag{2.9}$$

in which P_μ is the total kinetic four-momentum of the particles, W is a scalar proportional to the interaction and

$$u_\mu = \frac{1}{\sqrt{1 - v^2}}(1, \vec{v}) \tag{2.10}$$

is the (conserved) four-velocity. The proof that these ten generators really satisfy the commutation relations of the Poincaré group is too lengthy to be given here. The relative simplicity of the present method as compared to other attempts to construct these generators [20], is due to the use of the velocity instead of the kinetic momentum as conserved three-vector. The one-photon exchange diagram of Figure 1 is assumed to define the interaction between two charged particles.

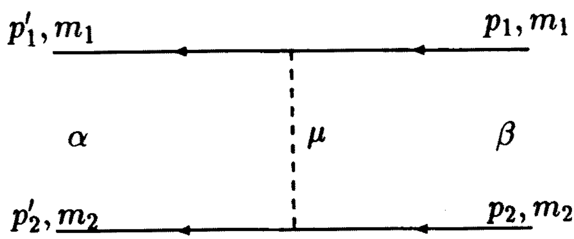

Fig. 1. One-photon exchange diagram. The mass of the exchanged particle is $\mu = 0$.

The corresponding potential is taken as

$$V_{\alpha\beta} = \frac{\alpha}{(2\pi)^3} j_1^\mu D_{\mu\nu} j_2^\nu , \tag{2.11}$$

where the current for a spin-0 or spin-1/2 particle is given by

$$j^\mu = (p' + p)^\mu \quad \text{or} \quad j^\mu_{\lambda'\lambda} = \bar{u}_{\lambda'}(p')\gamma^\mu u_\lambda(p). \tag{2.12}$$

The photon propagator $D_{\mu\nu}$ is expressed in terms of the momentum gain in the upper vertex $q_1 = p_1' - p_1$ and the momentum loss $q_2 = p_2 - p_2'$ in the lower vertex

$$D_{\mu\nu} = \frac{4\pi}{t}\left[g_{\mu\nu} - (1 - \beta)\frac{q_{1\mu} q_{2\nu}}{q_1 \cdot q_2}\right], \tag{2.13}$$

in which β is the gauge fixing parameter.

Because the particles are always on mass shell, unlike in other quasipotential theories, the currents defined in (2.12) are conserved in the sense that $j^\mu q_\mu = 0$. For this reason the term proportional to $(1-\beta)$ in (2.13) gives no contribution to the potential and the basic equation (2.4) becomes manifestly gauge invariant. The variable $\bar{t}$, which was not defined yet, replaces the Mandelstam variable t, which in the upper vertex has the value $t_1 = (p_1' - p_1)^2 = q_1^2$ and in the lower vertex $t_2 = (p_2 - p_2')^2 = q_2^2$. In the usual theories the conservation of four-momentum $p_1' + p_2' = p_1 + p_2$ implies that q_1 and q_2 are equal and therefore $t_1 = t_2 \equiv t$. In the present theory this equality is replaced by the conservation of velocity, which can be expressed as

$$\frac{p_1' + p_2'}{\sqrt{s'}} = \frac{p_1 + p_2}{\sqrt{s'}} \quad \text{with} \quad s = (p_1 + p_2)^2 \quad \text{and} \quad s' = (p_1' + p_2')^2. \qquad (2.14)$$

Defining the variable $\bar{t}$ by $\bar{t} = q_1 \cdot q_2$ it is seen that on shell, *i.e.*, when $s' = s$, the new and old variables are equal, $\bar{t} = t$. Therefore Eq. (2.11) corresponds on shell to the standard Coulomb potential in the momentum representation. The same equation defines the off shell continuation of the potential, which is not identical to the continuation used in other quasipotential theories.

It can be shown that the eigenvalue problem for the masses M_{nl} of the bound states is

$$2\pi m_1 m_2 \int_{s_+}^{\infty} \frac{V_l(s', s)\Phi_{nl}(s')}{[s' s \lambda(s')\lambda(s)]^{1/4}} ds' = (M_{nl} - \sqrt{s})\Phi_{nl}(s) , \qquad (2.15)$$

with

$$\lambda(s, m_1^2, m_2^2) = s^2 + m_1^4 + m_2^4 - 2sm_1^2 - 2sm_2^2 - 2m_1^2 m_2^2$$
$$= (s - s_+)(s - s_-) ,$$

with

$$s_+ = (m_1 + m_2)^2 \quad \text{and} \quad s_- = (m_1 - m_2)^2 , \qquad (2.16)$$

and

$$V_l(s, s') = \frac{\alpha\tau(s, s')}{8\pi^2 m_1 m_2}\delta_{l0} - \frac{\alpha[s + s' - 2(m_1^2 + m_2^2)]}{8\pi^2 m_1 m_2} Q_l(z_0). \qquad (2.17)$$

The argument of the Legendre function of the second kind is $z_0 = -\frac{t_0(s,s')}{\tau(s,s')}$ with t_0 and τ defined by writing the dependence of $\bar{t}$ on the c.m.s. azimuthal angle θ^* between $\vec{p}$ and $\vec{\mathbf{p}}'$ as $\bar{t} = t_0 + \tau\cos\theta^*$.

3. Coulomb bound states

In this section the spectrum is calculated of two charged spinless particles, which are bound by a one-photon exchange potential. For the Klein–Gordon equation this photon exchange is described by the "minimal coupling", *i.e.*, by making the replacement $p_\mu \longrightarrow p_\mu - \frac{e}{c}A_\mu$. Only the static limit will be considered, in which $m_2 \longrightarrow \infty$. In this limit the exact expression for the reduced mass $m_1\varepsilon_{nl}$ of the bound particle, can be found [3]. For small coupling it reduces to Eq. (1.2). If the exchanged particle were scalar (spin 0), the two massive particles could only be coupled through the $(mass)^2$–term of the Klein–Gordon equation [3]. This, however, leads to a spectrum of energy levels, which does not depend on the relative sign of the charge of the two particles. This may not be a great disadvantage, because scalar massless bosons do not exist anyway, but it is an indication that the Klein–Gordon equation cannot be considered as the static limit of a quasipotential theory. In contrast to the Klein–Gordon equation, the present theory does allow a coupling to the mass itself and with the correct static limit. In this limit it is found that the reduced mass $m_1\varepsilon_{nl}$ of the bound particle, is given by

$$\varepsilon_{nl} = 1 - \frac{\alpha^2}{2n^2}\left[1 - \frac{\alpha^2}{n}\left\{\frac{1}{2l+1} + \frac{1}{4n}\right\} + \cdots\right]. \tag{3.1}$$

Now consider the more realistic case of the exchange of a massless vector boson. The exact solution of the Klein–Gordon equation gives for the energy levels the expression

$$\varepsilon_{nl} = [1 + \frac{\alpha^2}{\{n - l - 1/2 + \sqrt{(l+1/2)^2 - \alpha^2}\}^2}]^{-1/2}, \tag{3.2}$$

which for small α reduces to (1.2). When in Eq. (3.2) $l + 1/2$ is replaced by $l + 1$, the fine structure formula of Sommerfeld [21] is obtained, while the replacement $l+1/2 \longrightarrow j+1/2$ gives the spectrum of the Dirac equation. No deep reason for this coincidence is known. In the framework of the present theory Eq. (2.15), must be solved. Only the static limit will be considered. Replacing m_1 by m and m_2 by M, this means that $m/M \longrightarrow 0$. In this limit the term in (2.17), that is effective for s–states only, can be neglected with respect to the other part of the potential. This shows that it is a pure recoil term, which does not exist for other than s–states. Define new variables y and y' by

$$\sqrt{s} = M + m\sqrt{1+y^2} \quad \text{and} \quad \sqrt{s'} = M + m\sqrt{1+y'^2}. \tag{3.3}$$

The eigenvalue M_{nl} of (2.15), is written as

$$M_{kl} = M + m\varepsilon_k^l \quad (k = 0, 1, 2, \cdots \text{labels the eigenvalues}) \tag{3.4}$$

and the function $F_k^l(y)$ is defined by

$$\Phi_k^l(s) = \frac{1}{\sqrt{2Mm}} \frac{(1+y^2)^{1/4}}{\sqrt{y}} F_k^l(y). \tag{3.5}$$

It can then be shown after some simple algebra, that in the static limit the eigenvalue equation (2.15) becomes

$$(\sqrt{1+y^2} - \varepsilon_k^l) F_k^l(y) = \frac{\alpha}{2\pi} \int_0^\infty \left[\left(\frac{1+y^2}{1+y'^2} \right)^{1/4} + \left(\frac{1+y'^2}{1+y^2} \right)^{1/4} \right] \times Q_l(z_0) F_k^l(y')\, dy', \tag{3.6}$$

with $z_0 = \frac{1}{2}(\frac{y}{y'} + \frac{y'}{y})$. The normalisation reads

$$\int_0^\infty F_{k'}^{l*}(y) F_k^l(y)\, dy = \delta_{k'k}. \tag{3.7}$$

It has not been possible to find the exact solution of (3.6), so that it had to be solved numerically. The results of this numerical calculation will be presented at the end of this section. For small α standard perturbation theory is applied by writing $y = \alpha x$ and $y' = \alpha x'$ and expanding in powers of α^2. Substituting

$$F_k^l(y) = G_k^l(x) + \alpha^2 D_k^l(x) + \cdots$$

and

$$\varepsilon_k^l = 1 - \frac{1}{2}\alpha^2 x_0^2 + \frac{1}{8}\beta\alpha^4 x_0^4 + \cdots, \tag{3.8}$$

gives the equations

$$\frac{1}{2}(x^2 + x_0^2) G_k^l(x) = \frac{1}{\pi} \int_0^\infty Q_l(\frac{1}{2}(\frac{x}{x'} + \frac{x'}{x})) G_k^l(x')\, dx' \tag{3.9}$$

and

$$\frac{1}{2}(x^2 + x_0^2) D_k^l(x) - \frac{1}{8}(x^4 + \beta x_0^4) G_k^l(x) = \frac{1}{\pi} \int_0^\infty Q_l\left(\frac{1}{2}\left(\frac{x}{x'} + \frac{x'}{x}\right)\right) D_k^l(x')\, dx'. \tag{3.10}$$

In order to solve these equations again new variables u and v and new functions $c_k^{l+1}(v)$ and $d_k^{l+1}(v)$ are defined by

$$x = x_0 \tan\frac{v}{2} \quad \text{and} \quad x' = x_0 \tan\frac{u}{2} \tag{3.11}$$

and

$$G_k^l(x) = \cos\frac{v}{2}\cdot c_k^{l+1}(v) \quad \text{and} \quad D_k^l(x) = \cos\frac{v}{2}\cdot d_k^{l+1}(v). \tag{3.12}$$

The equations (3.9) and (3.10) now take the following forms

$$x_0 c_k^{l+1}(v) = \frac{1}{\pi}\int_0^\pi Q_l\left(\frac{1-\cos v\cdot\cos u}{\sin v\cdot\sin u}\right) c_k^{l+1}(u)\,du \tag{3.13}$$

and

$$x_0 d_k^{l+1}(v) - \frac{1}{4}x_0^3\left(\tan^4\frac{v}{2}+\beta\right)\cos^2\frac{v}{2}\cdot c_k^{l+1}(v) = \frac{1}{\pi}\int_0^\pi Q_l\left(\frac{1-\cos v\cdot\cos u}{\sin v\cdot\sin u}\right) d_k^{l+1}(u)\,du\,. \tag{3.14}$$

The first of these equations is equivalent to the nonrelativistic Schrödinger equation. It was solved by Eriksen [22], who showed that

$$x_0 = \frac{1}{n} \quad \text{with} \quad n = k+l+1\,, \quad k = 0,1,2,\cdots \tag{3.15}$$

and that $c_k^\nu(u)$ is related to the Gegenbauer polynomial $C_k^\nu(\cos u)$ by

$$c_k^\nu(u) = 2^\nu \Gamma(\nu)\left[\frac{(k+\nu)k!}{2\pi\Gamma(k+2\nu)}\right]^{1/2}(\sin u)^\nu C_k^\nu(\cos u)\,. \tag{3.16}$$

This wave function in momentum space was first obtained by Podolski and Pauling [23]. The solution of (3.14) is not unique, because any multiple of $c_k^{l+1}(v)$ can be added to $d_k^{l+1}(v)$, to give another solution. This fact is used to choose $d_k^{l+1}(v)$ such that it is orthogonal to $c_k^{l+1}(v)$, *i.e.*,

$$\int_0^\pi d_k^{l+1}(u)c_k^{l+1}(u)\,du = 0\,. \tag{3.17}$$

Using (3.13), together with the normalisation

$$\int_0^\pi c_{k'}^{l+1}(u) c_k^{l+1}(u)\, du = \delta_{k'k}\,, \tag{3.18}$$

it then follows from (3.14) that β is given by

$$\beta = -2 \int_0^\pi \cos^2 \frac{v}{2} \cdot \tan^4 \frac{v}{2} \cdot |c_k^{l+1}(v)|^2\, dv = 3 - 4 \int_0^\pi \frac{|c_k^{l+1}(v)|^2}{1+\cos v}\, dv. \tag{3.19}$$

The second integral can be found in [24], resulting in

$$\beta = 3 - \frac{4n}{l+1/2} \quad \text{with} \quad n = k + l + 1. \tag{3.20}$$

When this β is substituted into the expression for ε_k^l of (3.8), the same formula (1.2) as derived from the Klein–Gordon equation is obtained. For larger values of the coupling constant Eq. (3.6) was solved numerically. For the integration through the logarithmic singularity of $Q_l(z_0)$ in $y' = y$ the algorithm of Wheeler was used. For a description see [25]. For the three lowest s- and p-states the energy levels are shown in figures 2 and 3 respectively.

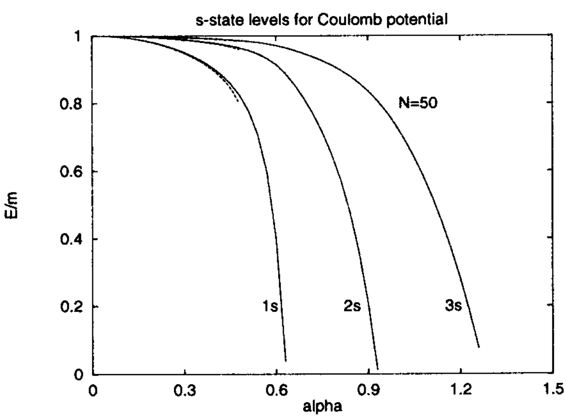

Fig. 2. The tree lowest s–state masses as functions of α. The dashed curves show the Klein–Gordon masses of (3.2).

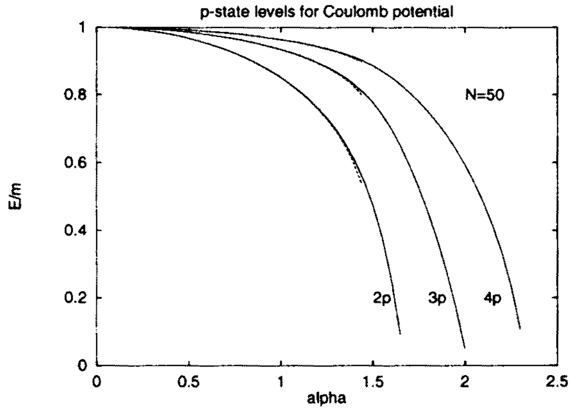

Fig. 3. The three lowest p-state masses as functions of α. The dashed curves show the Klein–Gordon masses of (3.2).

Comparison with the results of the Klein–Gordon equation (3.2) for those values of α for wich the solution is defined, *i.e.*, for $l + 1/2 > \alpha$, shows only small deviations. The almost perfect agreement between the experimentally determined energy levels of the pionic atoms of Ti($Z = 22$) and Fe($Z = 26$) [1, 2] and (1.2), can therefore not be considered as support for the Klein–Gordon equation, because it is identical to the perturbation result of the present theory as well as of the relativistic Schrödinger equation [26]. In order to discriminate between these theories the spectrum of pionic atoms with larger Z-values should be measured.

Two more differences with the Klein–Gordon equation should be pointed out. They are both related to the fact that in the present theory no negative energy states occur. For the binding of a π^--meson to a nucleus of charge Z, the Klein–Gordon equation gives a critical charge $Z_{cr} = 137(l + 1/2)$, which is the same for all states with the same angular momentum. With the new theory the Klein–Gordon curves extend into the region of coupling constants in which the Klein–Gordon equation has no solution. This results in larger values of the critical charge. These values are still not very realistic in the case of heavy pionic atoms, because the effects of the finite nuclear size should also be considered. It is clear, however, that the parameters which are always used to take this effect into account, shall have to be changed.

A second important difference with the Klein–Gordon equation lies in the fact that, while for the Klein–Gordon equation a dissolution into the negative energy continuum takes place, this so called Brown–Ravenhall disease [27] does not exist here.

4. Coulomb scattering

When trying to give a correct theory for Rutherford scattering, it is not easy to avoid all pitfalls. It is known that the exact scattering amplitude, as calculated from the Schrödinger equation, differs from the Born amplitude by a mere phase factor, so that lowest order perturbation theory already gives the exact expression for the differential cross section. However, the calculation of this phase factor, using an expansion in partial waves, leads to a divergent series. The history of the efforts to overcome or to ignore this difficulty was written by Marquez [28].

Another way to obtain the correct expression for the cross section, is to treat the Coulomb potential as the limit of a Yukawa potential with infinite range. This method was first used by Dalitz [29], who showed for the Dirac equation that divergences in the perturbation series arise from the expansion of a phase factor, with a phase which approaches infinity with the range of the potential. Therefore this infinity is harmless. For the Klein–Gordon equation the same was shown by Kang and Brown [4]. For the nonrelativistic case Kacser [30] found that the absolute value of the scattering amplitude did not acquire contributions of second and third order in the fine structure constant α. For the Dirac- and for the Klein–Gordon equation such contributions do exist.

The method of Dalitz [29] will now be applied to the new theory and it will be shown that again the divergences, which arise in perturbation theory, can be hidden in a phase factor. The next order contribution to the differential cross section will also be calculated. Coulomb scattering for larger values of α has been studied in [31] and [32].

The starting point is (2.4) for the two–particle scattering amplitude in the c.m.s. In this system the variables $\sqrt{s_\beta}$ and $\sqrt{s_\gamma}$ can be written as

$$\sqrt{s_\beta} = \sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2} \quad \text{and} \quad \sqrt{s_\gamma} = \sqrt{k^2+m_1^2} + \sqrt{k^2+m_2^2}\,. \tag{4.1}$$

With these expressions substituted into the energy denominator of (2.4) one finds

$$\frac{1}{\sqrt{s_\gamma} - \sqrt{s_\beta} - i0} = \frac{2R(k,p)}{k^2 - p^2 - i0}\,, \tag{4.2}$$

with $R(k,p)$ defined by

$$\frac{1}{2R(k,p)} = \frac{1}{\sqrt{k^2+m_1^2} + \sqrt{p^2+m_1^2}} + \frac{1}{\sqrt{k^2+m_2^2} + \sqrt{p^2+m_2^2}}\,. \tag{4.3}$$

For $k = p = 0$ this function is equal to the reduced mass, because then

$$\frac{1}{R(0,0)} = \frac{1}{m_1} + \frac{1}{m_2}\,. \tag{4.4}$$

For k much larger than each of the masses, however, this reduced mass function behaves as $R(k,p) \approx \frac{k}{4}$. Equation (2.4) can now be written as

$$M(\vec{q},\vec{p}) = V(\vec{q},\vec{p}) - \frac{1}{4}\int \frac{R(k,p)}{\sqrt{(k^2+m_1^2)(k^2+m_2^2)}} \frac{V(\vec{q},\vec{k})M(\vec{k},\vec{p})}{k^2-p^2-i0}\, d\vec{k}\,. \quad (4.5)$$

Once the scattering amplitude has been solved from this equation, the differential cross section is obtained by putting $|\vec{p}| = |\vec{q}| = q$ and by calculating

$$\frac{d\sigma}{d\Omega} = \frac{\pi^4}{s}|M(q,\theta)|^2, \quad (4.6)$$

with

$$s = (\sqrt{q^2+m_1^2} + \sqrt{q^2+m_2^2})^2 \quad \text{and} \quad \vec{p}\cdot\vec{q} = q^2\cos\theta\,. \quad (4.7)$$

The variable $\bar{t}$, which occurs in the potential of Eq. (4.5), can also be written in terms of the c.m.s. momenta. It then takes the form

$$\bar{t} = -|\vec{q}-\vec{k}|^2 - \mu^2(q,k)\,. \quad (4.8)$$

In this formula

$$\mu^2(q,k) = \{\sqrt{q^2+m_1^2} - \sqrt{k^2+m_1^2}\}\{\sqrt{q^2+m_2^2} - \sqrt{k^2+m_2^2}\} \geq 0 \quad (4.9)$$

represents an off-shell screening of the Coulomb potential. For $k = q$ it disappears, however. Also in the static limit $m_2 \longrightarrow \infty$, which will be assumed from now on, this $\mu^2(q,k)$ goes to zero for all q and k. In order to prevent the occurrence of divergent integrals, however, μ^2 will be replaced by a constant λ^2. Only at the end of the calculations, *i.e.*, in the expression for the differential cross section, will it be possible to let this cut–off mass go to zero, without the recurrence of infinities. If the potential (2.11) for the Coulomb interaction between two spinless particles, is written as

$$V(\vec{q},\vec{k}) = -\frac{m_2}{\pi^2}\bar{V}(\vec{q},\vec{k}) \quad \text{and also} \quad M(\vec{q},\vec{k}) = -\frac{m_2}{\pi^2}\bar{M}(\vec{q},\vec{k})\,, \quad (4.10)$$

the static limit of (4.5) becomes

$$\bar{M}(\vec{q},\vec{p}) = \bar{V}(\vec{q},\vec{p}) + \frac{1}{8\pi^2}\int \frac{\{\sqrt{q^2+m^2}+\sqrt{k^2+m^2}\}}{\sqrt{k^2+m^2}} \frac{\bar{V}(\vec{q},\vec{k})\bar{M}(\vec{k},\vec{p})}{k^2-q^2-i0}\, d\vec{k}\,, \quad (4.11)$$

with $m_1 = m$ and $|\vec{p}|^2 = |\vec{q}|^2$. The static limit of this potential is

$$\bar{V}(\vec{q},\vec{k}) = \alpha\frac{\sqrt{q^2+m^2}+\sqrt{k^2+m^2}}{|\vec{q}-\vec{k}|^2+\lambda^2}\,. \quad (4.12)$$

The formula for the cross section now becomes

$$\frac{d\sigma}{d\Omega} = |\bar{M}(q,\theta)|^2. \tag{4.13}$$

In the Born approximation this is equal to

$$\frac{d\sigma^{\mathrm{B}}}{d\Omega} = \frac{\alpha^2(q^2+m^2)}{4q^4\sin^4\frac{1}{2}\theta}, \tag{4.14}$$

which is equal to the lowest order result obtained from the Klein–Gordon equation [4], and which for nonrelativistic energies reduces to the Rutherford formula. The second order contribution to the scattering amplitude is equal to

$$\bar{M}_2(q,\theta) = \frac{\alpha^2}{8\pi^2}\int \frac{\{\sqrt{k^2+m^2}+\sqrt{q^2+m^2}\}^3}{\sqrt{k^2+m^2}} \times \frac{d\vec{k}}{(|\vec{k}-\vec{q}|^2+\lambda^2)\cdot(|\vec{k}-\vec{p}|^2+\lambda^2)\cdot(k^2-q^2-i0)}. \tag{4.15}$$

By separating this function into its real and imaginary part the total amplitude becomes

$$\bar{M}(q,\theta) = \bar{V}(q,\theta) + \bar{M}_2^R(q,\theta) + i\bar{M}_2^I(q,\theta) + O(\alpha^3), \tag{4.16}$$

which to this order in α can also be written as

$$\bar{M}(q,\theta) = (\bar{V}(q,\theta) + \bar{M}_2^R(q,\theta))\exp\left(i\frac{\bar{M}_2^I}{\bar{V}}\right). \tag{4.17}$$

The imaginary part of the amplitude is equal to the pole–contribution to the integral in (4.15). It can be expressed in terms of elementary functions and for small values of λ it takes the form

$$\bar{M}_2^I(q,\theta) \approx \frac{\alpha^2(q^2+m^2)}{2q^3\sin\frac{1}{2}\theta}\log\left(\frac{4q^2\sin\frac{1}{2}\theta}{\lambda^2}\right). \tag{4.18}$$

In the limit $\lambda \longrightarrow 0$ it goes to infinity, but this is harmless, because it only makes the phase of the amplitude indeterminate. The differential cross section (4.13) now becomes

$$\frac{d\sigma}{d\Omega} = \bar{V}^2(q,\theta) + 2\bar{V}(q,\theta)\bar{M}_2^R(q,\theta) + O(\alpha^4), \tag{4.19}$$

in which $\bar{M}_2^R(q,\theta)$ is given by the principal part of the integral in (4.15). After analytically performing the integration over the directions of $\vec{k}$, an integral over its magnitude remains, which cannot be expressed in terms of elementary functions. It has been shown, however, that the limit $\lambda \longrightarrow 0$ does exist, so that $\frac{d\sigma}{d\Omega}$ is finite. With proper care for the logarithmic singularity at $k = q$ in the remaining integral, this could be calculated numerically. The quantity of interest is the relative change in the cross section as compared to the Born approximation. To first order in α this is given by

$$R(q,\theta) = \frac{\frac{d\sigma}{d\Omega} - \frac{d\sigma^{\mathrm{B}}}{d\Omega}}{\frac{d\sigma^{\mathrm{B}}}{d\Omega}} = \frac{2\bar{M}_2^R(q,\theta)}{\bar{V}(q,\theta)} + O(\alpha^2). \tag{4.20}$$

For the Klein–Gordon equation the same quantity was calculated by Kang and Brown [4]. They found

$$R^{\mathrm{KG}}(q,\theta) = \frac{\pi\alpha q \sin\frac{1}{2}\theta}{\sqrt{q^2+m^2}}. \tag{4.21}$$

The ratio $R(q,\theta)/R^{\mathrm{KG}}(q,\theta)$ is plotted in figure 4 as a function of θ and for two extreme values of the velocity $v = \frac{q}{\sqrt{q^2+m^2}}$.

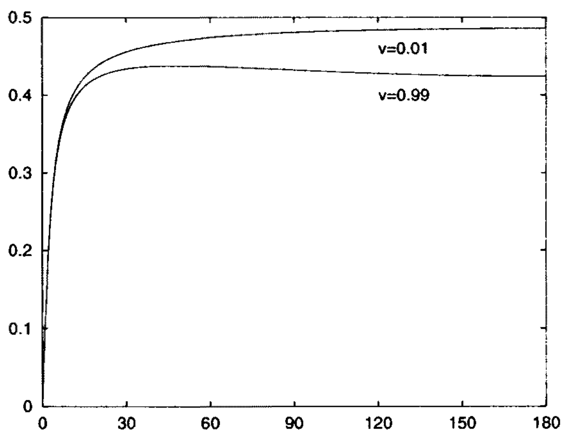

Fig. 4. The ratio $R(q,\theta)/R^{\mathrm{KG}}(q,\theta)$ for two velocities

For intermediate velocities the curves do not differ much from the two which are shown. Two features in this figure should be noticed. The first is that for small scattering angles and for all energies the relative correction to the Born approximation in the present theory is much smaller than in the

Klein–Gordon theory. This is closer to the Schrödinger theory, in which the second order term is exactly equal to zero. The second feature is that the difference between relativistic- and nonrelativistic scattering can be seen only for larger scattering angles. Also there the correction to the Born approximation is still smaller (by a factor of more than two) than in the case of the Klein–Gordon equation. Finally it should be remarked that the second order effects which have been considered here, have a different origin in the two theories. In the Klein–Gordon equation the real part of the amplitude gets no contribution from the two–photon exchange, and is completely determined by the $|\vec{A}|^2$–term in the Lagrangian. In contrast, for the present theory the $|\vec{A}|^2$–term does not exist at all and the two–photon exchange amplitude has a finite real part.

The main conclusion of this paper can be formulated by comparing it with an investigation of Rawitscher in 1964 [33]. He observes that

> ... the limit $v/c \to 0$, where v is the velocity of the incident particle, and c that of light, is not equivalent to the limit $c \to \infty$ in the case of Coulomb scattering and therefore the question arises whether the Schrödinger equation should be used at all in the Coulomb case. This is illustrated by the Coulomb phase shifts obtained in the Klein–Gordon equation ....

The conclusion of the present work is rather that, since the Klein–Gordon equation has a number of undesirable features and differs from the Schrödinger equation in the nonrelativistic limit, it should not be used for the description of Coulomb scattering.

REFERENCES

[1] L. Delker, *et al.*, *Phys. Rev. Lett.* **42**, 89 (1979).
[2] K.-C. Wang, *et al.*, *Phys. Rev.* **A22**, 1072 (1980).
[3] W. Greiner, *Relativistic Quantum Mechanics*, Springer, Berlin 1990.
[4] Ik-Ju Kang, L.M. Brown, *Phys. Rev.* **128**, 2828 (1962).
[5] See *e.g.* the appendix in [2].
[6] H. Kragh, *Am. J. Phys.* **52**, 1024 (1984).
[7] V.A. Rizov, H. Sazdjian, I.T. Todorov, *Ann. Phys. (N.Y.)* **165**, 59 (1985).
[8] E.H. de Groot, Th.W. Ruijgrok, *Nucl. Phys.* **B101**, 95 (1975).
[9] H. Hersbach, *Phys. Rev.* **C50**, 2562 (1994).
[10] B.A. Lippmann, J. Schwinger, *Phys. Rev.* **79**, 469 (1950).
[11] A.A. Logunov, A.N. Tavkhelidze, *Nuovo Cim.* **29**, 380 (1963).
[12] R. Blankenbecler, R. Sugar, *Phys. Rev.* **142**, 1051 (1966).
[13] I.T. Todorov, *Quasipotential Approach to the Two-Body Problem in Quantum Field Theory*, in: A. Zichichi,(ed.) *Properties of the Fundamental Interactions*, Vol.9, Part C, Bologna 1973, p.953.

[14] F. Gross, *Phys. Rev.* **C26**, 2203 (1984).
[15] P. Pyykkö, *Relativistic Theory of Atoms and Molecules I and II*, Springer, Berlin 1986, 1993.
[16] C. Møller, *The Theory of Relativity*, Clarendon Press, Oxford, 1952, Chapter III, S31.
[17] J.H. Taylor, *Class. Quantum Grav.* **10**, S167 (1993).
[18] N. Straumann, *Europhys. News* **24**, 191 (1993).
[19] P.A.M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949).
[20] B.D. Keister, W.N. Polyzou, *Relativistic Hamiltonian Dynamics in Nuclear and Particle Physics.* In: J. W. Negele and E. Vogt, eds., *Advances in Nuclear Physics,* Vol. 20, Plenum Press, New York 1991, p.225.
[21] A. Sommerfeld, *Ann. Phys.* **51**, 1 (1916).
[22] E. Eriksen, *Physica Norvegica*, **1**, 121 (1962).
[23] B. Podolski, L. Pauling, *Phys. Rev.* **34**, 109 (1929).
[24] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products,* Academic Press, New York 1980, eq. 7.314.1.
[25] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, *Numerical Recipes in C,* Cambridge Un. Press, Cambridge, 1992, 2nd ed., section (4.5).
[26] M.L. Goldberger, K.M. Watson, *Collision Theory* Wiley, New York 1964, p.24.
[27] G.E. Brown, D.G. Ravenhall, *Proc. Roy. Soc.* **A208**, 552 (1951).
[28] L. Marquez, *Am. J. Phys.* **40**, 1420 (1972).
[29] R.H. Dalitz, *Proc. Roy. Soc. (London)* **A206**, 509 (1951).
[30] C. Kacser, *Nuovo Cim.* **13**, 303 (1959).
[31] D.M. Fradkin, T.A. Weber, C.L. Hammer, *Ann. Phys. (N.Y.)* **27**, 338 (1964).
[32] E.H. de Groot, Th.W. Ruijgrok, *Acta Phys. Pol.* **B17**, 365 (1986).
[33] G.H. Rawitscher, *Phys. Lett.* **9**, 337 (1964).