DYNAMICS OF OVERLAPPING VORTICES IN COMPLEX SCALAR FIELDS*

J. DZIARMAGA**

Jagellonian University, Institute of Physics
Reymonta 4, 30-059 Kraków, Poland

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We investigate dynamics of overlapping vortices in the nonlinear Schrödinger equation, the nonlinear heat equation and in the equation with an intermediate Schrödinger-diffusion dynamics. Because of formal similarity on a perturbative level we discuss also the nonlinear wave equation (Goldstone model). Special solutions are found like vortex helices, double-helices and braids, breather states and vortex mouths. A pair of vortices in the Goldstone model scatters by the right angle in the head-on collision. It is found that in a dissipative system there is a characteristic length scale above which vortices can be entangled but below which the entanglement is unstable.

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1. Introduction

The dynamics of the superfluid helium condensate at zero temperature [1] was proposed to be described by the nonlinear Schrödinger equation [2]. Since then many approaches have been developed to study the dynamics of vortices being topological excitations of the NLSE. Lund [3] derives equations of motion for 3-dimensional vortex filaments with the help of the effective action method. The effective model takes the form of an action at a distance theory. As anticipated [3], such a method can not describe

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** address from October 1, 1995: Departmentof Mathematical Sciences, University of Durham, South Road, Durham, DH1 3LE, United Kingdom; e-mail address: J.P.Dziarmaga@durham.ac.uk

many details of the dynamics. Lee [4] rederives Lund's equations but in addition takes into account sound radiation from a moving vortex filament. His method originates from earlier similar developments in relativistic field theories [5].

The dynamics of well separated vortices on a plane has been studied by Neu [6]. The basic assumption of his method is that the modulus of the scalar field around a given vortex is undisturbed by other distant vortices. Interactions are mediated only by the phase of the complex scalar field. Such a simplification is not reliable for vortices with overlapping cores.

In this paper we are going to consider the dynamics of vortices when their cores strongly overlap. The method can be outlined as follows. We take as a background configuration a planar vortex solution with winding number, say, 2. Then we consider small fluctuations around this background. The double vortex can be viewed as a superposition of two unit vortices. As the double vortex has higher energy then two single vortices, it is likely to split into vortices with winding number 1. This splitting is described in our perturbative calculation by a mode with Z_2 symmetry. Its energy is negative as it should be for a splitting mode. One can also consider axially symmetric modes which do not split the double vortex but instead they change its width. They are usually called breathers or pulsons. Another possibility is a mode with Z_3 symmetry. The double vortex might uniformly split into 3 unit vortices leaving one antivortex at the center. The perturbative calculation shows this way of splitting does not take place for the double vortex. However for a vortex with winding number 3 there is a Z_4-symmetric mode which describes a decay of the triple vortex into 4 unit vortices with one antivortex left at the origin. We generalize these considerations allowing the modes to vary along the line of the background vortex. In this setting the double vortex can be split into parallel unit vortices and later on these vortices can be twisted to form a twisted pair or a double helix. It turns out that there is a critical wave-length at which such a double-helix is static.

The unperturbed configuration is a straight-linear vortex. Of course in reality such a configuration can not extend to infinity. It must either terminate on container walls or form a closed vortex ring. This two cases mean different boundary conditions which have to be imposed on the excitations we find. If the vortex segment terminates on container walls, the traveling waves excitations have to be combined to form standing waves. Periodic boundary conditions have to be imposed, if a straight vortex segment is intended to be an approximation to a large and smooth vortex ring. Such boundary conditions allow the traveling waves to move in opposite directions but they quantize the spectrum of their wave-lengths. Some of our perturbative solutions diverge for large time or large z. In the intervals of t or z where they are large the perturbative method is no longer valid. Our perturbative solution have to be replaced there by a solution obtained within one of the models for long range interactions of vortices. The two solutions have to be sewn together in an intermediate interval.

Our analysis is carried out not only for the nonlinear Schrödinger equation (NLSE) but also for a diffusive model we call after Neu [6] the nonlinear heat equation (NLHE). We discuss an intermediate Schrödinger-diffusive dynamics case and that of the nonlinear wave equation (NLWE) also known as the Goldstone model. The physical reason to consider diffusive and hamiltonian dynamics in one paper is that systems like superfluid helium change their dynamical properties according to external conditions such as temperature. At zero temperature the dynamics is hamiltonian but as we move towards the phase transition it gradually becomes more diffusive. Thus the different equations can describe one physical system but in different regimes. There are also important mathematical reasons for the unified treatment. First of all there is a set of static solutions, like static double helices, which are common to all the considered equations. The eigenvalue problem for time-dependent modes turns out to be formally the same for the NLWE and the NLHE. Solving this problem we kill two birds with one stone. The other reason is that there is a direct correspondence between the modes in the NLHE case and those in the NLSE. Although the at first sight tempting analytical continuation in the complex time plane fails, a large class of perturbative NLHE's solutions can be mapped into corresponding perturbative solutions of the NLSE. The analytical continuation does not apply to the modes themselves but it does apply to their eigenvalues. Some NLHE's modes do not map to the NLSE. However they still can be mapped to the intermediate diffusive equation with some admixture of the Schrödinger dynamics.

2. Field equations and vortex solutions

We will consider three types of nonlinear field equations. The first two are the nonlinear heat equation (NLHE)

$$\psi_{,t} = \Delta\psi + (1- \mid \psi \mid^2)\psi \equiv -\frac{\delta F}{\delta\psi^\star} \tag{1}$$

and the nonlinear Schrödinger equation (NLSE)

$$-i\psi_{,t} = \Delta\psi + (1- \mid \psi \mid^2)\psi \equiv -\frac{\delta H}{\delta\psi^\star}\,. \tag{2}$$

where $\Delta = \nabla^2$ is a Laplacian. The free energy and the Hamiltonian are

$$F = H = \int d^3x\, [\nabla\psi^\star\nabla\psi + \tfrac{1}{2}(1- \mid \psi \mid)^2]\,. \tag{3}$$

The third equation is the nonlinear wave equation (NLWE)

$$\psi_{,tt} - \Delta\psi - (1- \mid \psi \mid^2)\psi \equiv \frac{\delta L}{\delta\psi^\star} = 0\,, \tag{4}$$

with the Lagrangian

$$L = \int d^3x\, [\partial_\mu\psi^\star\partial^\mu\psi - \tfrac{1}{2}(1- \mid \psi \mid^2)^2]\,. \tag{5}$$

For time-independent fields all the three models reduce to the same static equation

$$\Delta\psi + (1- \mid \psi \mid^2)\psi = 0\,. \tag{6}$$

It is well known that such an equation admits topological vortices. A rotationally symmetric vortex solution can be obtained in the form

$$\tilde{\psi}(r,\theta) = f_n(r)\mathrm{e}^{in\theta}\,, \tag{7}$$

where (r,θ) are polar coordinates. Substitution of this Ansatz to Eq. (6) yields

$$\Delta_n f_n + (1 - f_n^2)f_n = 0\,, \tag{8}$$

where

$$\Delta_n \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\,.$$

The boundary condition at infinity is $f_n(\infty) = 1$. The phase of the complex scalar field is multivalued at the origin so its modulus must vanish there, $f_n(0) = 0$. If we restrict to regular solutions the asymptotes at the origin and at infinity will turn out to be

$$\begin{aligned} f_n(r) &\approx f_0 r^n + \ldots\,, \quad r \to 0\,, \\ f_n(r) &\approx 1 - \frac{n^2}{2r^2} + \ldots\,, \quad r \to \infty\,. \end{aligned} \tag{9}$$

From now on we will restrict to positive n.

3. Fluctuations around vortex background in nonlinear heat and nonlinear wave equation

Let the background solution be a straight-linear vortex along the z-axis with winding number n and the wave function $\tilde{\psi}$, see (7)–(9). The particle density of the background will be denoted by $\rho = \tilde{\psi}^\star\tilde{\psi}$. The equation satisfied by the field fluctuation $\delta\psi$ within the NLHE is

$$-\delta\psi_{,t} = -\Delta\delta\psi + (2\rho - 1)\delta\psi + \rho\mathrm{e}^{2in\theta}\delta\psi^\star\,. \tag{10}$$

One could try decomposition of the solution into Fourier modes both in θ and z, say $e^{ip\theta+ikz}$. However such a single Fourier mode can not be a solution to Eq. (10) because of the last term, $\rho e^{2in\theta}\delta\psi^\star$, which couples Fourier modes into pairs. The general solution can be obtained as a sum (or an integral) over pairs of Fourier modes like

$$f_0 e^{-Et}[\xi(E,k,p)e^{ikz}e^{ip\theta}u_1(r)+\xi^\star(E,k,p)e^{-ikz}e^{i(2n-p)\theta}u_2(r)] \qquad (11)$$

or

$$f_0 e^{-Et}e^{-kz}[\bar{\xi}(E,k,p)e^{ip\theta}u_1(r)+\bar{\xi}^\star(E,k,p)e^{i(2n-p)\theta}u_2(r)]\,, \qquad (12)$$

where ξ's are complex amplitudes of the modes and f_0 is a constant from Eqs (9) which is introduced to provide the modes with a convenient normalization. The z-dependence factorizes out because the background solution is z-independent. p's are integers which can range from $-\infty$ to n. Modes with $p > n$ are identical to those with $p < n$ *e.g.* the $p = n+1$ modes can be identified with the $p = n-1$ modes. The eigenvalues (E,k) can in principle take arbitrary real values but only the modes with finite energy (or free energy) are physical. The functions u_1, u_2 in Eq. (11) are solutions to the eigenvalue problem

$$\begin{aligned}(E-k^2)u_1 &= -\Delta_p u_1 + (2\rho-1)u_1 + \rho u_2\,,\\ (E-k^2)u_2 &= -\Delta_{2n-p} u_2 + (2\rho-1)u_2 + \rho u_1\,,\end{aligned} \qquad (13)$$

where Δ's are defined below Eq. (8). The functions u_1, u_2 in Eq. (12) are solutions of the same eigenvalue problem as in Eqs (13) but with k^2 replaced by $-k^2$. From now on we will allow k^2 to be a positive or negative real number. $k^2 > 0$ will refer to the modes in Eq. (12). $k^2 < 0$ will refer to those in Eq. (12) so that k is to be replaced by $\mathrm{Im}(k)$ there.

We can simplify Eqs (13) by a replacement $E - k^2 = \omega$,

$$\begin{aligned}\omega u_1 &= -\Delta_p u_1 + (2\rho-1)u_1 + \rho u_2\,,\\ \omega u_2 &= -\Delta_{2n-p} u_2 + (2\rho-1)u_2 + \rho u_1\,.\end{aligned} \qquad (14)$$

With equations (14) at hand we are able to give a physical interpretation to the modes with various p. If one restricts to regular solutions one can easily find the asymptotes of the profile functions close to the origin

$$\begin{aligned}u_1(r) &\approx u_1^0 r^{|p|}\,,\\ u_2(r) &\approx u_2^0 r^{|2n-p|} + \dots\,,\end{aligned} \qquad (15)$$

where u_1^0, u_2^0 are real constants which have to be chosen so as to meet the conditions of regularity and fast convergence at infinity. Eqs (15) are a set of linear homogeneous differential equations so the constants can be multiplied by a common factor. In other words we can redefine the constants by an appropriate rescaling of the overall amplitudes ξ in Eqs (11), (12). Thus, provided that $u_1^0 \neq 0$, we can choose $u_1^0 = 1$ and call the still free $u_2^0 = a$. In this case we are left with just one coefficient a which we can vary to obtain solutions with acceptable asymptotes at infinity. For $u_1^0 = 0$ the coefficient u_2^0 can be rescaled to 1, $u_2^0 = 1$. In this case, which is a set of measure zero, there is no free parameter to remove singularities or slowly convergent asymptotes at infinity and it turns out that there indeed is no solution. Thus from now on we restrict our attention to the first case

$$\begin{aligned} u_1(r) &\approx r^{|p|} + \dots , \\ u_2(r) &\approx a r^{|2n-p|} + \dots . \end{aligned} \tag{16}$$

In the limit of very small $w = x + iy$ or close to the vortex axis, for the modes in Eq. (11), the total scalar field looks like

$$\psi = \tilde{\psi} + \delta\psi \sim w^n + \xi(E,k,p) \mathrm{e}^{-Et+ikz} w^p \tag{17}$$

for $p \geq 0$. The zeros of this polynomial coincide with the zeros of the total scalar field.

Let us consider the double vortex background, $n = 2$. For $p = 0$ we have $\psi \sim w^2 + \xi(E,k,0)\mathrm{e}^{-Et+ikz}$. At $t = 0$ and on the plane $z = 0$ we have two zeros at complex roots of $-\xi$. For $E > 0$ these zeros will with time shrink down to $w = 0$. For nonzero k the two lines of vanishing scalar field form a double helix. If there is a solution for k then there is also a solution for $-k$. We can combine the two to obtain *e.g.* $\psi \sim w^n + \xi \mathrm{e}^{-Et} \cos kz$ with ξ, say, real and positive constant. Let us fix time t but consider what happens as we vary z. Close to $z = 0$ the zeros lie on y-axis. As we increase z the zeros shrink to $w = 0$ but then reappear but rotated by the right angle. This kind of solution will be called a double-braid.

For general n, p we get $n - p$ uniformly split vortices and p vortices left at the origin. If $p < 0$ there are $-p$ antivortices at the origin. The case $n = p$ is exceptional. There is no splitting but just a change of vortex width.

Similar mode decomposition as for the NLHE (11), (12) can be performed for the NLWE,

$$f_0[\xi(E,k,p)\mathrm{e}^{iEt+ikz}u_1(r) + \xi^\star(E,k,p)\mathrm{e}^{-iEt-ikz}\mathrm{e}^{i(2n-p)\theta}u_2(r)] . \tag{18}$$

Substitution of this Ansatz to the NLWE linearized in fluctuations

$$-\frac{\partial^2}{\partial t^2}\delta\psi = -\Delta\delta\psi + (2\rho - 1)\delta\psi + \rho \mathrm{e}^{2in\theta}\delta\psi^\star . \tag{19}$$

once again leads to Eqs (14) but this time $\omega = E^2 - k^2$. The Ansatz (18) can be generalized to admit E^2 or k^2 negative. If say E^2 is negative one should replace the two time-dependent exponents in the Ansatz (18) by one exponent $e^{-\mathrm{Im}(E)t}$ standing in front of the mode. An analogous rearrangement should be done if $k^2 < 0$.

Since both the fluctuations in the NLHE and in the NLWE satisfy Eqs (14) it is the highest time to look for their solutions.

3.1. Asymptotes of the splitting modes at infinity

At infinity the asymptotic form of Eqs (14) is

$$\begin{aligned} 0 &= \Delta u_1 + (\omega - 1)u_1 - u_2 \,, \\ 0 &= \Delta u_2 - u_1 + (\omega - 1)u_2 \,. \end{aligned} \tag{20}$$

These equations can be combined to give

$$\begin{aligned} 0 &= \Delta u_+ + (\omega - 2)u_+ \,, \\ 0 &= \Delta u_- + \omega u_- \,, \end{aligned} \tag{21}$$

where $u_+ = u_1 + u_2$ and $u_- = u_1 - u_2$.

For $\omega > 2$ both u_+ and u_- approach combinations of Bessel functions. In the range $0 < \omega < 2$, u_+'s asymptote is a linear combination of the modified Bessel functions while u_- still falls down like $1/\sqrt{r}$. Finally for $\omega < 0$ both u_+ and u_- are independent combinations of the modified Bessel functions. The cases of $\omega = 0, 2$ need a separate treatment.

We can accept only localized modes with finite energy or free energy. With such a restriction we have to exclude both exponentially divergent solutions and all the solutions with Bessel function-like asymptotes. For $\omega = 2$ there are slowly falling down oscillations in the asymptote of u_-. Thus the range of ω where we can look for acceptable solutions is restricted to $\omega \leq 0$. In this range there are two divergent asymptotes at infinity. For any given ω we can use the parameter a in Eq. (16) to fine tune one of the divergent asymptotes to zero. The second may be removed by an appropriate choice of ω. A distinguished ω for which it happens is just the eigenvalue we are looking for.

3.2. Bound states for $p = 0 \ldots n - 1$

There are analytical solutions for $p = n - 1$ and $\omega = 0$. They can be constructed as $\delta\psi = \partial_x \tilde{\psi}(r, \theta)$, where $\tilde{\psi} = f(r)e^{in\theta}$ is the background

n-vortex solution. The profile functions of such zero modes are

$$u_1 = \tfrac{1}{2}[f'(r) + \frac{nf(r)}{r}]$$
$$u_2 = \tfrac{1}{2}[f'(r) - \frac{nf(r)}{r}]\,. \qquad (22)$$

The energy of these zero modes similarly as the energy of the background vortex is logarithmically divergent. They give rise to the following solution of the NLWE for small t

$$\delta\psi/f_0 = -vt[u_1(r) + e^{2i\theta}u_2(r)]\,, \qquad (23)$$

where v is a complex velocity. For $n = 1$ the zero of the scalar field coincides with the zero of the polynomial $w - vt$. The solution is just a perturbative approximation to a planar vortex moving with a constant velocity. This solution becomes more interesting if we admit $k^2 \neq 0$ to excite traveling waves with a dispersion relation $E^2 - k^2 = 0$. The line of vanishing scalar field coincides with zeros of a polynomial $w(t,z) - F_1(t+z) - F_2(t-z)$, where F's are arbitrary functions. This solution is a perturbative approximation to Vachaspatis' traveling waves [9].

For the solution with $\omega = 0$ and $n = 1, p = 0$, the dispersion relation in the NLHE case is $E - k^2 = 0$. The line of zero scalar field is deformed from the z-axis, $w(t,z) = 0$, to the helix $w(t,z) = \xi(k^2,k,0)\exp(-k^2t + ikz)$. Any initial disturbance of this form (for example a helical standing wave) will shrink down to the unperturbed straight-linear vortex.

We have found some bound states for $\omega < 0$. Let me explain the method first. The asymptotes at infinity are

$$u_- = A\frac{\exp(-\sqrt{|\,\omega\,|}\,r)}{\sqrt{r}} + B\frac{\exp(+\sqrt{|\,\omega\,|}\,r)}{\sqrt{r}}\,,$$
$$u_+ = C\frac{\exp[-\sqrt{|\,\omega\,|+2}\,r]}{\sqrt{r}} + D\frac{\exp[+\sqrt{|\,\omega\,|+2}\,r]}{\sqrt{r}}\,. \qquad (24)$$

We want to find such ω's that B and D vanish simultaneously. In Eqs (16) there is only one free parameter a. For each ω we can choose such a value of a that the coefficient D vanishes. Once D is removed the coefficient B remains in general nonzero. This procedure may be used to determine the function $B(\omega) \equiv B(\omega \mid D = 0)$. We assume the function to be continuous. If there are two ω's with opposite signs of B, there must be such a value of ω in between that $B = 0$. This is the way in which one can find a bound state solution to Eqs (14).

For $n = 1$ and $p = 0$ we have scanned a wide range of negative ω's finding the function B to be always positive. More fruitful was the search for $n = 2, p = 0$. We have found a bound state for $\omega = -0.168 \equiv -k_0^2$. The static modes which arise from this state are the same for all the considered models. The modes satisfy $k^2 = k_0^2$. The lines of zero scalar field are in general the roots of the complex polynomial $w^2 - \xi_1 e^{ik_0 z} - \xi_2 e^{-ik_0 z}$. If only $\xi_1 \neq 0$, the solution is a static double helix, see the bottom part of Fig. 1. Another characteristic solution is that with $\xi_1 = \xi_2$. This is a static double braid, see Fig. 2. The static double helices or braids exist only for the special characteristic wave-length $L_0 = 2\pi/k_0$. Perturbations with other wave-lengths are no longer static.

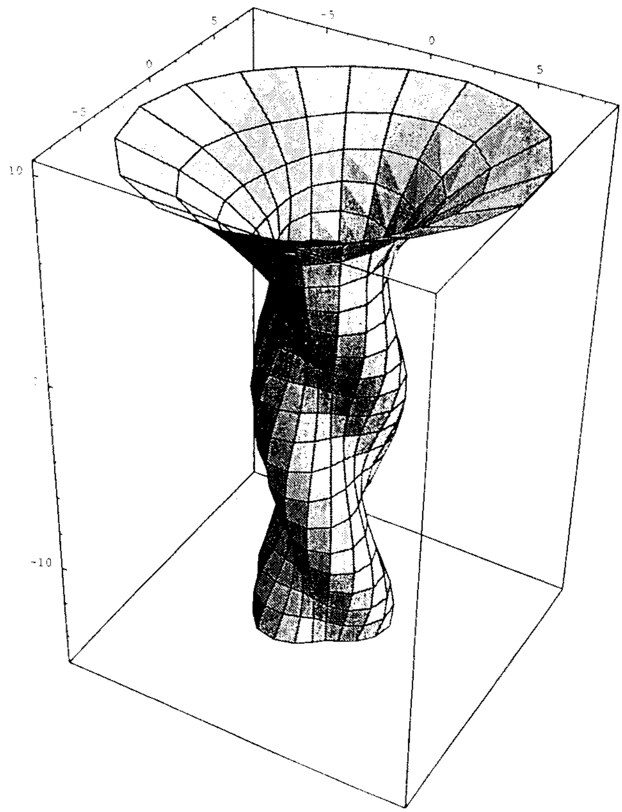

Fig. 1. Superposition of two modes on a double vortex. The bottom part is the double helix. In the top part the vortex mouth opens. On the plotted surface the scalar field's modulus is equal to $^1/_2$.

For the NLHE the dispersion relation is $E = k^2 - k_0^2$. E is negative for fluctuations with wave-lengths greater then L_0. The external radius of such

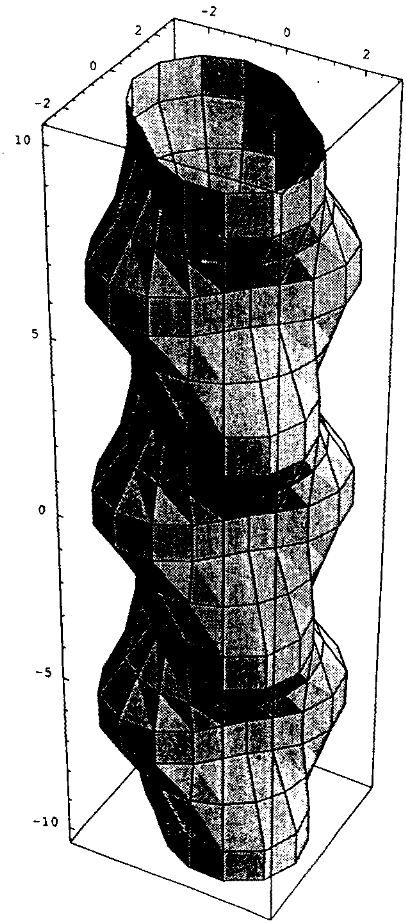

Fig. 2. The double braid. The double vortex splits into two unit vortex cosine waves of the same polarization. At each nodal point the polarization plane turns by the right angle. For n-braids the polarization would turn by the angle π/n at each nodal point.

double-helices or braids is growing with time like $\exp(^1\!/\!_2 Et)$. Because of the string tension they must stabilize at some larger radius which we are not able to reach by our perturbative analysis. For short wave-length fluctuations E is positive and such double-helices or braids shrink down like $\exp(-^1\!/\!_2 Et)$ to the unperturbed straight-linear vortex configuration. The NLWE has a different dispersion relation, namely $E^2 - k^2 = -k_0^2$. For generic values of positive k^2 and E^2 the double-helix at the bottom of Fig. 1 and the braid in Fig. 2 move up or down the z-axis with a phase velocity less then 1. Such modes should have been expected since they are just Lorenz boosted static helices or braids. More interesting are the modes with negative E^2: $-E^2 = k_0^2 - k^2$. They exist for long wave-length fluctuations as compared to the critical wave-length L_0. Such double helices can decay or expand with a passage of time. There is also a combination for which the external radius evolves like $\sqrt{\sinh(\sqrt{-E^2}\,t)}$. This can describe a double-helix (braid) shrinking to the basic solution and then expanding but rotated around the z-axis by the right angle.

In the limit of infinite wave-length we obtain a pair of parallel vortices. Vortices in the NLHE repel one another and the separation of their zeros grows like $\exp({}^1\!/\!_2 k_0 t)$. If they started right from the unperturbed solution they would need infinite time to separate but any generic initial dipole fluctuation should substantially lower this time. For a similar pair in the NLWE, the most interesting time evolution is that in which the relative position evolves like $\sqrt{\sinh(k_0 t)}$. The positions of the zeros of the scalar field are the same as the zeros of the complex polynomial $w^2 - \sinh(k_0 t)$. For negative times vortices approach one another along the y-axis, at the time $t = 0$ they coincide and for later times they split but this time along the x-axis. This is the right angle scattering in the head-on collision of two vortices. This result can be generalized to symmetric collisions of n vortices to give the π/n scattering. A physical importance of this simple right-angle scattering has been realised by Manton [7]. Its net effect for multivortex system's thermodynamics is that vortices behave as if they had finite cores — there is a net excluded area for a planar vortex liquid [7].

3.3. Bound states for $p = n$

We have seen, see Eq. (21), that the asymptotic equations for u_+ and u_- decouple. For $p = n$ the two equations decouple for any r. Eqs (14) can be rewritten as

$$\begin{aligned} \omega u_+ &= -\Delta_n u_+ + (3\rho - 1)u_+ \,, \\ \omega u_- &= -\Delta_n u_- + (\rho - 1)u_- \,. \end{aligned} \tag{25}$$

What are the bound states of these stationary Schrödinger equations? As u_- is concerned, we have not found any bound states for $n = 1$ and $n = 2$. The proof in Section 3.1 of Ref. [8] applies here. According to this argument there are no bound states for u_-.

The results for u_+ are more interesting. For n=1 there is one bound state with energy $\omega_1 = 1.806$. For n=2 the potential $(3\rho - 1)$ is broader so that the bound state energy is lower and amounts to $\omega_2 = 1.613$. It seems that the energy of the ground state is decreasing with increasing winding number. For sufficiently large n the next bound state is likely to appear. The existence of at least one bound state for any n can be proved following the lines in Section 4 of Ref. [8].

The two modes look qualitatively the same for $n = 1$ and $n = 2$. Once again let me begin with static modes. They satisfy $k^2 = -\omega_n$. The vortex exponentially broadens or tightens with z. An example is shown in the top part of Fig. 1. Such solutions may seem at first sight to be simply diverging outside the perturbative regime and thus useless. However looking at Fig. 2 one can imagine that there is a surface of the superfluid somewhere above and our mode is the way in which the vortex mouth begins. Vortices prefer to be wider at the surface then in the bulk.

Now about the time-dependent solutions. In the NLHE the dispersion relation is $E = \omega_n + k^2$. One can perturb the vortex changing its core radius but the distortion will shrink down to the unperturbed solution like $\exp -Et$. It shrinks the faster the shorter is the wave-length of the fluctuation. In the NLWE, where the dispersion relation is $E^2 - k^2 = \omega_n$, such a z-independent distortion oscillates in time. Something like a "breather" or "pulson" state forms. Making the last solution z-dependent, with $k^2 > 0$, we obtain waves traveling along the vortex with a phase velocity greater then 1.

There are also solutions of Eqs (25) just on the verge between bound states and scattering states. One of them is $u_- = f(r)$ for $\omega = 0$, where $f(r)$ is the moduli of the background solution. Another solution is $u_+ = f(\sqrt{3}r)$ for $\omega = 2$. These solutions would exist for any background, they have nothing to do with the vortex. The first solution is a zero mode due to a global gauge transformation $\tilde{\psi} \rightarrow \tilde{\psi}e^{i\theta_0}$. The second one would be $u_+ = 1$ for a uniform background $\rho = 1$. It is just an uniform massive oscillation of the scalar field's modulus around its equilibrium value equal to 1.

3.4. Bound states for $p < 0$

As discussed in Section 3 such modes describe splitting of the n-vortex into n unit vortices and in addition $\mid p \mid$ vortex-antivortex pairs. Such a decay is in principle possible because the energy of n-vortex ($\sim n^2$) is higher then the energy of n unit vortices ($\sim n$). The extra energy can be used to create vortex-antivortex pairs. The maximal number of such pairs has been estimated to be at best $\mid p \mid = n - 1$, see [11].

For $n = 1$ and $n = 2$ the result is negative. Double-vortex can be split into 2 unit vortices but there is not enough energy to create a $v - \bar{v}$ pair. Such a decay turns out to be possible for $n = 3, p = -1$. There is one antivortex left at the origin and 4 uniformly split vortices. The corresponding eigenvalue is $\omega = -0.103$. For the NLHE the mode is a diffusive splitting with creation of one $v - \bar{v}$ pair. This way of splitting has been observed in a direct numerical simulation, see the Figure 3 in [6]. z-dependence can be introduced giving rise to vortex 4-helix around a straight-linear antivortex or 4-braid with nodal points on the antivortex.

For $k^2 = -\omega$ the helices and braids are static and for larger k^2 they are damped. For the NLWE there are 4-helices and 4-braids at the critical wavelength. At other wavelengths they are traveling waves. For parallel vortices the mode describes head-on collision of 3 vortices on an antivortex center resulting in π/n scattering.

4. Fluctuations in the nonlinear Schrödinger equation regime

The discussion of the NLSE has been postponed until this section. The NLSE can be obtained from the NLHE by a formal Wick rotation $t = i\tau$

$$i\partial_\tau \psi = -\nabla^2\psi + (\psi^\star\psi - 1)\psi\,. \tag{26}$$

However it does not mean we can do the same with solutions of these equations. For example the eigenvalue problem for the fluctuations around a vortex background in the NLSE

$$f_0\left[\xi(E,k,p)\mathrm{e}^{iEt+ikz}\mathrm{e}^{ip\theta}u_1(r) + \xi^\star(E,k,p)\mathrm{e}^{-iEt-ikz}\mathrm{e}^{i(2n-p)\theta}u_2(r)\right] \tag{27}$$

takes a slightly different form

$$\begin{aligned}&(-E + k^2 + 2\rho - 1 - \triangle_p)u_1 + \rho u_2 = 0\,,\\ &(+E + k^2 + 2\rho - 1 - \triangle_{2n-p})u_2 + \rho u_1 = 0\end{aligned} \tag{28}$$

than in the NLHE. Note that the equations differ by a sign in front of E. This makes the eigenvalue problem different from that for the NLHE and NLWE, compare with Eqs (13). This is where the analytical continuation in the complex time plane fails on the perturbative level.

Let us take a closer look at asymptotic properties of the solutions to the eigenvalue problem (28). The asymptotes at the origin are the same as in the NLHE case, see Eq. (28). As E and k^2 can not be combined now into just one variable $\omega = E - k^2$, the asymptotic behavior at infinity will depend on both E and k^2 independently. In this limit the Eqs (28) become

$$\begin{aligned}&\triangle u_1 + (E - k^2 - 1)u_1 - u_2 = 0\,,\\ &\triangle u_2 - u_1 + (-E - k^2 - 1)u_2 = 0\,.\end{aligned} \tag{29}$$

These equations can be diagonalized. The eigenvalues turn out to be

$$-(1 + k^2) \pm \sqrt{1 + E^2}\,. \tag{30}$$

For a positive eigenvalue we obtain a Bessel function-like asymptote which does not converge fast enough to be acceptable. The two eigenvalues are negative for (E,k^2) belonging to the area defined by $k^2 > -1+\sqrt{1+E^2}$. In this "convergence area" there are two exponentially decaying asymptotes and two exponentially divergent ones. For any (E,k^2) one of the divergent asymptotes can be removed with an appropriate choice of the constant a in Eq. (16). The other one may happen to vanish along some lines in the (E,k^2)-plane. These distinguished lines are just the dispersion lines we are looking for.

A straightforward but laborious way of solving the problem (28) would be to scan the (E,k^2)-plane in search of bound state solutions. Let us check first what could we learn from analytical continuation of the NLHE's bound states to the NLSE.

Let us consider the bound state of the NLHE

$$\delta\bar{\psi} = e^{-Et}[e^{ikz}e^{ip\theta}u_1(r) + e^{-ikz}e^{i(2n-p)\theta}u_2(r)], \tag{31}$$

where the functions $u_1(r), u_2(r)$ satisfy Eqs (13). The Wick rotated configuration takes the form

$$\delta\bar{\psi} = e^{-iEt}[e^{ikz}e^{ip\theta}u_1(r) + e^{-ikz}e^{i(2n-p)\theta}u_2(r)]. \tag{32}$$

This configuration is a solution of the linearized NLSE but only at $t=0$. Later on the fluctuation $\delta\psi$ must deviate from the continued mode $\delta\bar{\psi}$ to $\delta\psi = \delta\bar{\psi} + \phi$. We look for the deviation in the form

$$\begin{aligned}\phi = {} & e^{ikz+ip\theta}[e^{iEt}w_1(r) + e^{-iEt}v_1(r)] \\ & + e^{-ikz+i(2n-p)\theta}[e^{-iEt}w_2(r) + e^{iEt}v_2(r)].\end{aligned} \tag{33}$$

The profile functions w's and v's must satisfy the following sets of inhomogeneous differential equations

$$\begin{aligned}(+E+k^2+2\rho-1-\Delta_p)w_1 + \rho w_2 &= -2\rho u_2, \\ (-E+k^2+2\rho-1-\Delta_{2n-p})w_2 + \rho w_1 &= 2\rho u_1\end{aligned} \tag{34}$$

and

$$\begin{aligned}(-E+k^2+2\rho-1-\Delta_p)v_1 + \rho v_2 &= 2\rho u_2, \\ (+E+k^2+2\rho-1-\Delta_{2n-p})v_2 + \rho v_1 &= -2\rho u_1.\end{aligned} \tag{35}$$

The R.H.S.'s of the above inhomogeneous equations are regular sources which vanish exponentially at infinity. Close to $r=0$ the w's and v's look like

$$\begin{aligned}w_1(r) &\approx w_1^0 r^{|p|} + \dots, \\ w_2(r) &\approx w_2^0 r^{|2n-p|} + \dots, \\ v_1(r) &\approx v_1^0 r^{|p|} + \dots, \\ v_2(r) &\approx v_2^0 r^{|2n-p|} + \dots,\end{aligned} \tag{36}$$

with w^0's and v^0's being constants which have to be chosen so as to make the solutions convergent at infinity. The equations are inhomogeneous so this time the constants can not be rescaled to remove a half of them. For each set of equations (34), (35) there are two adjustable parameters, (w_1^0, w_2^0) and (v_1^0, v_2^0) respectively. As to the asymptotic properties at infinity, it is enough to note that the differential operators in both sets (34), (35) are the same as in the Eqs (28). If the eigenvalues (E, k^2) belong to the convergence area $k^2 > -1+\sqrt{1+E^2}$ then, for each set of equations (34), (35), there are two divergent asymptotes at infinity. These two asymptotes can be removed with the two free parameters in the asymptotes close to the origin (36).

Thus it turns out that ϕ is a quickly convergent function provided that $k^2 > -1 + \sqrt{1+E^2}$. For (E, k^2) belonging to this convergence area the NLHE mode $\delta\bar{\psi}$ when analytically continued to the NLSE gives rise to the mode $\delta\psi = \delta\bar{\psi} + \phi$. This means that the parts of dispersion lines which belong to the convergence area can be analytically continued from the NLHE to the NLSE. We have confirmed this observation looking directly for the solutions of Eqs (28). Let us then discuss the most characteristic solutions of the NLSE.

For $n = 2$ and $p = 0$ the NLHE's dispersion relation is $E = k^2 - k_0^2$. The dispersion relation remains unchanged for the NLSE modes obtained by mapping NLHE's solutions. For short wave-length double helices and braids (Fig. 1, Fig. 2) the energy is positive. They can move up or down the z axis as traveling waves. For $k^2 = k_0^2$ they become static. In the range of k^2 from k_0^2 to $1/2(1 - \sqrt{1 - 4k_0^2})$ there are once again traveling waves. Now the problem arises if we can continue the dispersion line $E = k^2 - k_0^2$ outside the convergence area, in particular to the point $(E = -k_0^2,\ k^2 = 0)$ which corresponds to a pair of parallel vortices rotating anticlockwise around their common center of mass with angular velocity $1/2 k_0^2$. Outside the convergence area the long-range Bessel function-like asymptotes are unavoidable. A well defined NLHE's mode when Wick rotated to the NLSE develops a long range deviation ϕ. Thus it seems that a rotating parallel pair of vortices dissipates energy radiating sound waves. As the potential between vortices is repulsive their mutual distance should be growing with time.

For the excitations $(n = 1, p = 0)$ of a single vortex the dispersion relation is $E = k^2$. This line remains in the convergence area for all $k^2 > 0$. The traveling waves on the single vortex move up or down the z-axis. The phase velocity is falling down to zero as the wave-length tends to infinity. These excitations can be identified with the experimentally observed Kelvin modes [1].

The dispersion relation for the breather states ($p = n$) is $E = \omega_n + k^2$ with ω_n positive but smaller then 2. For $n = 1, 2$ the whole dispersion line lies outside the convergence area. It may happen that for some large enough n, $\omega_n < 1$. Then there would exist a finite part of the dispersion line belonging to the convergence area. In any other case the breather modes must radiate. For $k^2 > 0$ they travel up or down the vortex with some phase velocity dependent on the wave-length. In the z-independent case $k^2 = 0$ there is a breather state like in the NLWE — vortex width is oscillating in time around the equilibrium value. For $k^2 < 0$ there exists an exceptional point ($E = 0$, $k^2 = \omega_n$) where we have a static vortex throat common to all the considered equations.

In the $n = 3, p = -1$ case 4 vortex lines uniformly split from the origin. For k^2 large enough, the 4-helix or the 4-braid wave is traveling along the central antivortex axis. The point on the dispersion line with $k^2 = 0$ lies outside the convergence area thus the mode with parallel vortices rotating around the central antivortex is at best a radiative one.

The fluctuations are similar to those in the NLWE. The difference is that the dispersion relations for the modes in the NLSE are a nonrelativistic limit of the dispersion relations for corresponding modes in the NLWE. Some of the NLSE modes, if they exist at all, are unstable against decay by radiation of sound waves.

5. Mixed Schrödinger-diffusive dynamics

So far we have considered only the NLHE and NLSE. In general we can replace t in the NLHE (10) by $t\exp(i\gamma)$ to obtain the mixed Schrödinger-diffusion equation

$$-\psi_{,t}\cos\gamma + i\psi_{,t}\sin\gamma = -\Delta\psi - (1 - |\psi|^2)\psi\,. \tag{37}$$

γ can vary from 0 to $\pi/2$. The fluctuation modes now take the form

$$f_0 e^{-Et\cos\gamma}\Big[\xi(E,k,p)e^{-iEt\sin\gamma}e^{ikz}e^{ip\theta}u_1(r) + \xi^\star(E,k,p)e^{+iEt\sin\gamma}e^{-ikz}e^{i(2n-p)\theta}u_2(r)\Big]\,, \tag{38}$$

where this time u_1, u_2 are complex. To answer the question if the exponentially localized NLHE's modes can be mapped into modes of the intermediate equation we have first to find its convergence area.

The homogeneous equations which u_1, u_2 have to satisfy are

$$\begin{aligned} &[\Delta_p - k^2 + (1-2\rho) + E]u_1 - \rho u_2 = 0\,,\\ &[\Delta_{2n-p} - k^2 + (1-2\rho) + E\exp(-i\gamma)]u_2 - \rho u_1 = 0\,. \end{aligned} \tag{39}$$

The real part of the eigenvalues of the matrix operator (for very large r) is

$$-(1+k^2) + E\cos^2\gamma + \mathrm{Re}\sqrt{1 - E^2\sin^2\gamma e^{-2i\gamma}}\,. \tag{40}$$

The convergence area is a set of all such (E, k^2) that

$$k^2 > -1 + E\cos^2\gamma + |\,\mathrm{Re}\sqrt{1 - E^2\sin^2\gamma e^{-2i\gamma}}\,|\,, \tag{41}$$

where the real parts of both eigenvalues are negative. In this area, by similar arguments as in the previous section, a deviation ϕ from a continued NLHE's solution can be always made convergent.

The dispersion relations are once again formally the same as for the NLHE or the NLSE but as for the NLHE the sign of E determines whether a given mode is growing or decaying with time.

A pair of parallel vortices not only rotates around the common center of mass with the angular velocity $^1\!/_2 k_0^2 \sin\gamma$ but also their mutual distance is growing with time. The zeros of the scalar field move along spiral lines. There is such a $\delta\gamma > 0$ that for $\gamma < (\pi/2) - \delta\gamma$ the dispersion relation $E = k^2 - k_0^2$ for the $(n = 2, p = 0)$ mode can be continued from large positive k^2 down to the axis $k^2 = 0$ and a little bit below. Thus a pair of parallel vortices rotating one around another is radiating sound waves in the NLSE regime but a small dissipation makes them an exponentially localized solution. The external radius of the long wave-length double-helix (Fig. 1) or the braid (Fig. 2) is also growing with time until it stabilizes thanks to the string tension. If its wave-length is short, it prefers to shrink down to the coincident two-vortex configuration.

For single vortex modes $(n = 1, p = 0)$ the dispersion relation $E = k^2$ remains in the convergence area for all positive k^2. A single-helix gradually shrinks down to the straight linear vortex. Fluctuations in the vortex width, the same as in the NLSE case, remain outside the convergence area.

6. Summary

We have obtained a wide class of perturbative solutions like helices, double-helices or braids. The question arises whether such configurations really correspond to some exact solutions. We suppose the answer to be yes. Configurations like helices (Kelvin waves) or double helices were analyzed within the models for widely separated vortices (Kelvin waves are exact solutions to the NLWE [9]). The braids can not be described by such models because even if their amplitude is large vortices have to cross one another at the nodal points. In the Bogomol'nyi limit of the Abelian Higgs model some extra symmetries of the model enable an exact construction of double-helices and braids [10].

The analysis was substantially simplified by various symmetries which connect some of the considered equations. The perturbative calculations in the NLWE and the NLHE regimes lead to the same stationary eigenvalue problem. The perturbative solutions to the whole family of nonrelativistic equations interpolating between the NLHE and the NLSE were constructed from the NLHE's solutions. This construction shows that although the analytical continuation of solutions in the complex time plain fails one can still continue the dispersion relation, provided that dispersion line belongs to the convergence area. The calculations in the NLHE were substantially simpler then they would be in the corresponding problem for the NLSE. The lesson is that it is convenient to map such problems to the NLHE and later on continue the obtained NLHE's dispersion lines to the equation under consideration.

Note added. A month after the first version of this paper the preprint by Goodband and Hindmarsh [11] appeared. The papers overlap in the analysis on the NLWE.

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