THE WEYL–WIGNER–MOYAL FORMALISM II. THE MOYAL BRACKET

J.F. PLEBAŃSKI*, M. PRZANOWSKI**

Department of Physics
Centro de Investigación y de Estudios Avanzados del IPN
Apartado Postal 14-740, México 07000, D.F., México

AND J. TOSIEK

Institute of Physics, Technical University of Łódź
Wólczańska 219, 93-005 Łódź, Poland

(Received February 5, 1996)

Relation between the Dirac problem and the Weyl–Wigner–Moyal formalism is considered. The Moyal $*_{(g)}$-product and the generalized Moyal bracket are defined and analysed. It is shown that the first heavenly equation appears to be the $\hbar \to 0$ limit of the SDYM equations for the Moyal algebra.

PACS numbers: 03.65. Ca, 04.20. Cv, 11.15. -q

1. Introduction

This paper is the second part of the work devoted to the Weyl–Wigner–Moyal formalism. In the first part [1] we have shown how some simple and natural assumptions lead to the *generalized Weyl application* W_g. Here we review those assumptions in the light of the famous *Dirac problem* (Sections 2 and 3) and then we analyse the mapping inverse to W_g *i.e.* the *generalized Weyl correspondence* W_g^{-1} (Section 4).

In Sections 3 and 4 the *generalized Stratonovich–Weyl quantizer* is defined and some of its properties are found. Section 5 is devoted to the *Moyal* $*_{(g)}$ *-product* and to the *generalized Moyal bracket* $\{\cdot,\cdot\}_M^{(g)}$. It is shown that

* E-mail: pleban@fis.cinvestav.mx

** Permanent address: Institute of Physics, Technical University of Łódź, Wólczańska 219, 93-005 Łódź, Poland.

all algebras $(\mathcal{P}, *_{(g)})$, where $\mathcal{P}$ is the set of all complex polynomials on the phase space $\Gamma_2 = R^1 \times R^1$, are isomorphic to $(\mathcal{P}, *)$ and also all Lie algebras $(\mathcal{P}, \{\cdot,\cdot\}_M^{(g)})$ are isomorphic to the *Moyal algebra on* $\mathcal{P}$, $(\mathcal{P}, \{\cdot,\cdot\}_M)$. At the end of Section 5 some comments on the problem of an extension of W_g and $*_{(g)}$ on some class of distributions are given. Finally, in Section 6 we show some interesting application of the Weyl–Wigner–Moyal formalism in self-dual gravity. Namely, with the use of this formalism we prove that the first heavenly equation can be considered to be the $\hbar \to 0$ limit of the SDYM equations in the Moyal algebra. Recently a revival of an interest in the Weyl–Wigner–Moyal formalism is observed. It has been quickly recognized that this formalism is not only a powerful tool in quantum mechanics [2-9] but also is a beautiful mathematical formalism in self-dual gravity and integrable systems [10–21]. We suppose that the Weyl–Wigner–Moyal formalism offers the most natural method of quantization in curved spaces [7, 9, 22–24].

We are going to consider this question in next parts of our work.

[*Remark*: In this paper we deal with the phase space $\Gamma_2 = R^1 \times R^1$ but the results can be easily generalized to the case of $\Gamma_{2n} = R^n \times R^n$ for any $n \geq 1$].

2. The Dirac problem

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the set of linear operators acting on $\mathcal{H}$. Moreover, let $\mathcal{P}$ denotes the set of all complex polynomials on the phase-space $\Gamma_2 = R^1 \times R^1$. For any $A, B \in \mathcal{P}$ we define the Poisson bracket to be

$$\{A, B\}_P := \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}, \tag{2.1}$$

where (q, p) stands for the coordinates on Γ_2. The symplectic form ω on Γ_2 reads

$$\omega = dq \wedge dp\,. \tag{2.2}$$

It is well known that $(\mathcal{P}, \{.,.\}_P)$ constitutes a complex Lie algebra.

Definition 2.1. [25, 26]

A *Dirac map* is a linear map $L : \mathcal{P} \to \mathcal{L}(\mathcal{H})$ such that

(i) $L(1) = \hat{1}$ (the identity operator)

(ii) $L(\{A, B\}_P) = \frac{1}{i\hbar}[L(A), L(B)]$,

for every $A, B \in \mathcal{P}$; $\hbar$ is some real constant (Planck's constant) and $[.,.]$ denotes the commutator.

Then the *Dirac problem* consists in finding a Dirac map.

We also need an associative algebra over C generated by finite linear combinations and finite powers of the operators $\hat{q}, \hat{p}, \hat{1} \in \mathcal{L}(\mathcal{H})$ satisfying the commutation relation

$$[\hat{q}, \hat{p}] = i\hbar \hat{1}\,. \tag{2.3}$$

This algebra will be denoted by $\hat{\mathcal{P}}$. $\hat{\mathcal{P}}$ is, of course, *the enveloping algebra of the Heisenberg–Weyl algebra* generated by $\hat{q}, \hat{p}$ and $\hat{1}$ [27].

According to Joseph [26] we denote by $\hat{\mathcal{P}}^m$, $m = 2, 3, \ldots$, the associative algebra over C generated by m mutually commuting independent sets of operators fulfilling (2.3).

First, we prove a theorem due to Groenewald [28] and Chernoff [29, 30].

Theorem 2.1 (Groenewald, Chernoff).

There exists no Dirac map $L : \mathcal{P} \to \mathcal{L}(\mathcal{H})$ satisfying the following conditions

$$L(q^2) = (L(q))^2 \quad \text{and} \quad L(p^2) = (L(p))^2\,. \tag{2.4}$$

Proof:

From the equalities $\{q^3, q\}_P = 0$ and $\{q^3, p\}_P = 3q^2$, and from (*ii*) it follows that $L(q^3) = \hat{q}^3 + \hat{\alpha}$, where $\hat{q} \equiv L(q)$ and $\hat{\alpha}$ is a linear operator such that $[\hat{\alpha}, \hat{q}] = 0$ and $[\hat{\alpha}, \hat{p}] = 0$, $(\hat{p} \equiv L(p))$.

Then, as $\{q^2, p^2\}_P = 4qp$, from (*ii*) and (2.4) one gets

$$L(qp) = \frac{1}{4i\hbar}[L(q^2), L(p^2)] = \frac{1}{4i\hbar}[\hat{q}^2, \hat{p}^2] = \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})\,.$$

Consider the equality $\{q^3, qp\}_P = 3q^3$.

By (*ii*) we have $\frac{1}{i\hbar}[L(q^3), L(qp)] = 3L(q^3)$ *i.e.*, using the preceding results, $\frac{1}{i\hbar}[\hat{q}^3 + \hat{\alpha}, \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})] = 3\hat{q}^3 + 3\hat{\alpha}$. But as $\hat{\alpha}$ commutes with $\hat{q}$ and $\hat{p}$ one finally obtains $\hat{\alpha} = 0$. Therefore, $L(q^3) = \hat{q}^3$.

Analogously we can show that $L(p^3) = \hat{p}^3$ and, by induction, one quickly finds the general formulas

$$L(q^m) = \hat{q}^m \quad \text{and} \quad L(p^m) = \hat{p}^m, \quad m \in \mathcal{Z}_+\,, \tag{2.5}$$

and

$$L(q^m p) = \frac{1}{2}(\hat{q}^m \hat{p} + \hat{p}\hat{q}^m) \quad \text{and} \quad L(qp^m) = \frac{1}{2}(\hat{q}\hat{p}^m + \hat{p}^m \hat{q}) \quad m \in \mathcal{Z}_+\,. \tag{2.6}$$

Consider now the relations

$$q^2p^2 = \frac{1}{9}\{q^3, p^3\}_P \quad \text{and} \quad q^2p^2 = \frac{1}{3 \cdot 6 \cdot 6}\{\{q^3, p^2\}_P, \{q^2, p^3\}_P\}_P\,. \tag{2.7}$$

Employing (*ii*) and (2.5) we get

$$L(q^2p^2) = \frac{1}{9i\hbar}[\hat{q}^3, \hat{p}^3] = \frac{1}{3}(\hat{p}^2\hat{q}^2 + \hat{p}\hat{q}^2\hat{p} + \hat{q}^2\hat{p}^2)\,,$$

$$L(q^2p^2) = \frac{1}{3\cdot 6\cdot 6(i\hbar)^3}[[\hat{q}^3, \hat{p}^2], [\hat{q}^2, \hat{p}^3]]$$

$$= \frac{1}{3}(\hat{p}^2\hat{q}^2 + \hat{p}\hat{q}^2\hat{p} + \hat{q}^2\hat{p}^2 + \hbar^2)\,.$$

Thus, as $\hbar \neq 0$ the last two equalities lead to a contradiction and the proof is complete. ■

From Theorem 2.1 one concludes that if $L : \mathcal{P} \to \mathcal{L}(\mathcal{H})$ is a Dirac map then $L(q^2) \neq \hat{q}^2$ or $L(p^2) \neq \hat{p}^2$. Therefore, the natural question is to find the general form of $L(q^m)$ and $L(p^n), n = 2, 3\ldots$

This question is solved by the lemma given in an excellent paper by Joseph [26].

Lemma 2.1 (Joseph)

Given a Dirac map $L : \mathcal{P} \to \mathcal{L}(\mathcal{H})$, then

$$(q^n) = \hat{q}^n + \sum_{k=0}^{n-2} \binom{n}{k} \hat{a}_{n-k}\hat{q}^k, n = 2, 3, \ldots, \tag{2.8}$$

$$L(p^n) = \hat{p}^n + \sum_{k=0}^{n-2} \binom{n}{k} \hat{b}_{n-k}\hat{p}^k, n = 2, 3, \ldots, \tag{2.9}$$

where $\hat{q} \equiv L(q), \hat{p} \equiv L(p)$ and the linear operators $\hat{a}_k$, $\hat{b}_k \in \mathcal{L}(\mathcal{H}), k = 2, 3, \ldots$, commute with both $\hat{q}$ and $\hat{p}$.

Proof: (By induction)

For $n = 2$ we put

$$L(q^2) = \hat{q}^2 + \hat{\gamma}_2,\ \hat{\gamma}_2 \in \mathcal{L}(\mathcal{H})\,. \tag{2.10}$$

Then the equalities $\{q^2, q\}_P = 0$ and $\{q^2, p\}_P = 2q$, and the condition (*ii*) yield

$$[\hat{\gamma}_2, \hat{q}] = 0 \text{ and } [\hat{\gamma}_2, \hat{p}] = 0\,. \tag{2.11}$$

Comparing (2.10) and (2.11) with (2.8) one concludes that for $n = 2$ the equality (2.8) holds. Analogously we show that also (2.9) holds for $n = 2$.

In the next step we assume that (2.8) is satisfied for some $n \geq 2$. Put

$$L(q^{n+1}) = \hat{q}^{n+1} + \hat{\gamma}_{n+1}, \quad \hat{\gamma}_{n+1} \in \mathcal{L}(\mathcal{H}). \tag{2.12}$$

From the relation $\{q^{n+1}, q\}_P = 0$ and (*ii*) it follows that

$$[\hat{\gamma}_{n+1}, \hat{q}] = 0. \tag{2.13}$$

Then, from (*ii*), (2.8) and (2.12) one gets

$$\frac{1}{i\hbar}[\hat{\gamma}_{n+1}, \hat{p}] = (n+1)\sum_{k=0}^{n-2} \binom{n}{k} \hat{a}_{n-k}\hat{q}^k. \tag{2.14}$$

Eq. (2.14) can be written in another form, namely

$$[\hat{\gamma}_{n+1} - \sum_{k=0}^{n-2} \binom{n+1}{k+1} \hat{a}_{n-k}\hat{q}^{k+1}, \hat{p}] = 0. \tag{2.15}$$

From (2.12), (2.13) and (2.15) we get

$$L(q^{n+1}) = \hat{q}^{n+1} + \sum_{k=0}^{n-2} \binom{n+1}{k+1} \hat{a}_{n-k}\hat{q}^{k+1} + \hat{\beta}, \tag{2.16}$$

where

$$\hat{\beta} := \hat{\gamma}_{n+1} - \sum_{k=0}^{n-2} \binom{n+1}{k+1} \hat{a}_{n-k}\hat{q}^{k+1}$$

commutes with $\hat{q}$ and $\hat{p}$.

Finally, it is easy to show that (2.16) can be rewritten in the following form

$$L(q^{n+1}) = \hat{q}^{n+1} + \sum_{k=0}^{(n+1)-2} \binom{n+1}{k} \hat{a}_{n+1-k}\hat{q}^k,$$

$$\hat{a}_{n+1} := \hat{\beta}. \tag{2.17}$$

This is exactly (2.8) with $n \to n+1$.

Similar considerations can be done for $L(p^{n+1})$, $n \geq 2$.

Thus the proof of the lemma is completed. ■

It is well known that Physics imposes some restricition on the Dirac map.

Thus, in fact, from the physical point of view one is rectricted to the *proper Dirac map* which is defined as follows:

Denifiton 2.2.

A Dirac map $L : \mathcal{P} \to \mathcal{L}(\mathcal{H})$ is called the *proper Dirac map* if for each real polynomial $A \in \mathcal{P}$ the operator $L(A)$ is self-adjoint.

Now we are going to prove the Chernoff-Joseph theorem [26, 29, 30] which shows that the original Dirac quantization program [25] cannot be realized.

Let $L^2(R^n; \mathcal{H}_d)$ denotes the Hilbert space of L^2 functions from R^n to a d-dimensional ($d < \infty$) Hilbert space $\mathcal{H}_d$.

Then the following theorem holds

Theorem 2.2 (Chernoff, Joseph)

There exists no proper Dirac map $L : \mathcal{P} \to \mathcal{L}(L^2(R^1; \mathcal{H}_d))$ such that $\hat{q} \equiv L(q) =$ the multiplication by q, and $\hat{p} \equiv L(p) = -i\hbar\frac{\partial}{\partial q}$.

Proof:

From Lemma 2.1 and the condition (*ii*) one quickly infers that

$$L(q^2) = \hat{q}^2 + \hat{a}_0, \quad L(p^2) = \hat{p}^2 + \hat{b}_0\,,$$
$$L(qp) = \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + \hat{c}_0\,, \tag{2.18}$$

where $\hat{a}_0, \hat{b}_0$ and $\hat{c}_0$ are self-adjoint operators acting on $L^2(R^1; \mathcal{H}_d)$, commuting with both $\hat{q}$ and $\hat{p}$, and satisfying the following commutation relations

$$[\hat{a}_0, \hat{b}_0] = 4i\hbar\hat{c}_0, \quad [\hat{a}_0, \hat{c}_0] = 2i\hbar\hat{a}_0, \quad [\hat{b}_0, \hat{c}_0] = -2i\hbar\hat{b}_0\,. \tag{2.19}$$

As $\hat{a}_0, \hat{b}_0$ and $\hat{c}_0$ commute with $\hat{q}$ and $\hat{p}$ they can be written in the form

$$\hat{a}_0 = \hat{1}_q \otimes \hat{A}_0, \quad \hat{b}_0 = \hat{1}_q \otimes \hat{B}_0 \quad \text{and} \quad \hat{c}_0 = \hat{1}_q \otimes \hat{C}_0\,, \tag{2.20}$$

where $\hat{1}_q$ stands for the indentity operator on $L^2(R^1)$ and $\hat{A}_0, \hat{B}_0$ and $\hat{C}_0$ are some hermitian $d \times d$ constant matrices satisfying, *mutatis mutandi*, the commutation relations (2.19). Substituting

$$\hat{A}'_0 := \frac{1}{4i\hbar}(\hat{A}_0 - \hat{B}_0), \quad \hat{B}'_0 := \frac{1}{4i\hbar}(\hat{A}_0 + \hat{B}_0), \quad \hat{C}'_0 := \frac{1}{2i\hbar}\hat{C}_0\,, \tag{2.21}$$

one gets, by (2.19), the commutation relations for A'_0, B'_0 and C'_0 to be

$$[\hat{A}'_0, \hat{B}'_0] = \hat{C}'_0, \quad [\hat{B}'_0, \hat{C}'_0] = A'_0, \quad [\hat{C}'_0, \hat{A}'_0] = -\hat{B}'_0\,. \tag{2.22}$$

Therefore the $d \times d$ anti-hermitian matrices $\hat{A}'_0, \hat{B}'_0$ and $\hat{C}'_0$ generate the finite-dimensional representation of the Lie algebra $sl(2;\mathcal{R})$ into the Lie algebra of $d \times d$ anti-hermitian matrices. As $d < \infty$ this representation must be trivial. Hence, $\hat{A}_0 = 0, \hat{B}_0 = 0$ and $\hat{C}_0 = 0$, and consequently

$$\hat{a}_0 = 0 \quad \hat{b}_0 = 0 \quad \text{and} \quad \hat{c}_0 = 0\,. \tag{2.23}$$

But then, comparing (2.23) and (2.18), we conclude that

$$L(q^2) = \hat{q}^2 \quad \text{and} \quad L(p^2) = \hat{p}^2\,. \tag{2.24}$$

Therefore by Theorem 2.1 we arrive at the contradiction. This completes the proof. ■

(It is evident that the similar theorem can be proved for the case of $L^2(R^n;\mathcal{H}_d)$ with any $n \geq 1$).

Now, to avoid the difficulties with the (proper) Dirac map we are going to weaken the conditon (*ii*). As it has been shown in [1] this procedure leads in a natural way to the *generalized Weyl application.*

In the next section we recall some results of [1] in the light of the considerations of the present section, and also several new results will be given.

(It is well known that another solution of the Dirac problem is proposed by the *geometric quantization* [31, 32]. However, in the present paper we don't touch this formalism).

3. The generalized Weyl application

From the proof of the Groenewald–Chernoff theorem one quickly concludes that the difficulties with the condition (*ii*) begin when the *third order* polynomials are considered. Therefore, first we can modify (*ii*) in such a manner that this condition is satisfied for $\{q, A\}_P$ and $\{p, A\}_P$ for every $A \in \mathcal{P}$.

As it has been shown in [1] (see also references cited in [1]) this modification can be really done.

Thus one gets the following theorem:

Theorem 3.1

There exists a linear map $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ satisfying the conditions:

(i') $W_g(1) = \hat{1}$

$$W_g(p^m q^n) = \sum_{s=0}^{min(m,n)} g(m,n,s)\hbar^s \hat{p}^{m-s}\hat{q}^{n-s}$$

$$m, n \in \mathcal{N}\,,\, m+n \neq 0\,,\, g(m,n,s) \in \mathcal{C}\,,\, g(m,n,0) = 1\,.$$

(i'i') $W_g(\{q,A\}_P) = \frac{1}{i\hbar}[\hat{q}, W_g(A)]$, $W_g(\{p,A\}_P) = \frac{1}{i\hbar}[\hat{p}, W_g(A)]$, for every $A \in \mathcal{P}$.

(i'i'i') For every *real polynomial* $A \in \mathcal{P}$ the operator $W_g(A)$ is self-adjoint.

Proof: See [1]. ■

(Recall that $\hat{\mathcal{P}}$ denotes the enveloping algebra of the Heisenberg - Weyl algebra generated by $\hat{q}, \hat{p}$ and $\hat{1}$; $\hat{q} \equiv W_g(q)$ and $\hat{p} \equiv W_g(p)$).

In [1] we have shown that any linear map $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ satisfying the conditions *(i')* and *(i'i')* is definded by the sequence of complex numbers $\{g(s,s,s)\}_{s\in\mathcal{N}}$, $g(0,0,0) = 1$; then the numbers $g(m,n,s)$, $m \geq s$ and $n \geq s$, are defined by $g(s,s,s)$ as follows

$$g(m,n,s) = \binom{m}{s}\binom{n}{s} g(s,s,s)\,. \tag{3.1}$$

Finally, the condition *(i'i'i')* imposes the restrictions on $g(s,s,s)$

$$\mathrm{Im}[g(m,m,m)] = \frac{1}{2}\sum_{s=0}^{m-1} i^{m-s-1}(m-s)!\binom{m}{s}^2 \bar{g}(s,s,s)\,, \tag{3.2}$$

where the bar stands for the complex conjugation.

Definition 3.1

A linear map $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ satisfying the conditions *(i')*, *(i'i')* and *(i'i'i')* is called a *generalized Weyl application.*

We now intend to express the generalized Weyl applicaton W_g in the integral form. This form appears to be crucial when the extension of W_g on non-polynomial functions is considered.

We prove the following:

Theorem 3.2.

Let $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ be a generalized Weyl application and let $A \in \mathcal{P}$ be any polynomial.

Then

$$W_g(A) = \frac{1}{(2\pi)^2}\int_{R^2} \tilde{A}(\lambda,\mu) f(\hbar\lambda\mu)\exp(i\lambda\hat{p})\exp(i\mu\hat{q})d\lambda d\mu\,, \tag{3.3}$$

where $\tilde{A} = \tilde{A}(\lambda,\mu)$ is the Fourier transform of $A = A(p,q)$

$$\tilde{A} = \tilde{A}(\lambda,\mu) = \int_{R^2} A(p,q)\exp[-i(\lambda p + \mu q)]dpdq\,, \tag{3.4}$$

and $f = f(\hbar\lambda\mu)$ denotes the *formal series*

$$f = f(\hbar\lambda\mu) := \sum_{s=0}^{\infty}(-1)^s \frac{(\hbar\lambda\mu)^s}{(s!)^2} g(s,s,s)\,. \tag{3.5}$$

Proof:

It is sufficient to prove the theorem for the monomials $p^m q^n$, $m, n \in \mathcal{N}$.
Denote $A_{m,n} := p^m q^n$. Then the Fourier transform of $A_{m,n}$, according to (3.4) reads

$$\begin{aligned}\tilde{A}_{m,n} = \tilde{A}_{m,n}(\lambda,\mu) &= \int_{R^2} p^m q^n \exp[-i(\lambda p + \mu q)]dpdq \\ &= (2\pi)^2 i^{m+n}\delta^{(m)}(p)\delta^{(n)}(q)\,. \end{aligned}\tag{3.6}$$

Inserting (3.6) into (3.3) one gets

$$W_g(A_{m,n}) = (-i)^{m+n}\left\{\frac{\partial^{m+n}}{\partial\lambda^m\partial\mu^n}[f(\hbar\lambda\mu)\exp(i\lambda\hat{p})\exp(i\mu\hat{q})]\right\}_{\lambda=0,\mu=0}. \tag{3.7}$$

Straightforward computation shows that (3.7) is equivalent to (i') with (3.1). Thus the proof is complete. ■

(Compare the formula (3.3) with Refs. [27, 33–35]). Using the Baker–Campbell–Hausdorff formula one can write (3.3) in other equivalent forms. Namely,

$$\begin{aligned} W_g(A) &= \frac{1}{(2\pi)^2}\int_{R^2} \tilde{A}(\lambda,\mu)\alpha(\hbar\lambda\mu)\ \exp\ [i(\lambda\hat{p} + \mu\hat{q})]d\lambda d\mu\,, \\ \alpha &= \alpha(\hbar\lambda\mu) := f(\hbar\lambda\mu)\ \exp\ (\frac{i}{2}\hbar\lambda\mu) \end{aligned}\tag{3.8}$$

or

$$\begin{aligned} W_g(A) &= \frac{1}{(2\pi)^2}\int_{R^2} \tilde{A}(\lambda,\mu)\beta(\hbar\lambda\mu)\exp(i\mu\hat{q})\exp(i\lambda\hat{p})d\lambda d\mu\,, \\ \beta &= \beta(\hbar\lambda\mu) := f(\hbar\lambda\mu)\exp(i\hbar\lambda\mu)\,. \end{aligned}\tag{3.9}$$

Note that α and β are also considered to be formal series.

In particular, the formula (3.8) is very often used in literature [2, 3, 27, 29, 33–35] and it justifies the name of the *generalized Weyl application* for W_g. Indeed if

$$\alpha = 1 \tag{3.10}$$

one gets the original Weyl application [36].

Definition 3.2

A generalized Weyl application $W_g := \mathcal{P} \to \hat{\mathcal{P}}$ for which $\alpha = 1$ is called the *Weyl application* and is denoted by W. Then one can easily show that the operator $W_g(A)$ is self-adjoint for every *real* $A \in \mathcal{P}$ if and only if the formal series $\alpha = \alpha(\hbar\lambda\mu)$ is *real*. Moreover, from the formulas (3.5) and (3.8) it follows that $g(0,0,0) = 1$ if and only if $\alpha(0) = 1$.

Gathering all that we arrive at the following important.

Theorem 3.3

Let

$$\alpha = \alpha(\hbar\lambda\mu) = \sum_{k=0}^{\infty} \alpha_k \cdot (\hbar\lambda\mu)^k, \; \alpha_k \in \mathcal{R} \tag{3.11}$$

be a real formal series such that

$$\alpha_0 = 1\,. \tag{3.12}$$

Then the linear map $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ defined by

$$\mathcal{P} \ni A = A(p,q) \mapsto W_g(A) \in \hat{\mathcal{P}}\,,$$

$$W_g(A) = \frac{1}{(2\pi)^2} \int_{R^2} \tilde{A}(\lambda,\mu)\alpha(\hbar\lambda\mu)\exp[i(\lambda\hat{p} + \mu\hat{q})]d\lambda d\mu\,, \tag{3.13}$$

is a generalized Weyl application.

Conversly, each generalized Weyl application is defined in such a manner. ■

From (3.5), (3.8), (3.11) and (3.12) one easily finds the relations between the coefficients α_k and $g(s,s,s)$ to be

$$g(s,s,s) = s! \sum_{k=0}^{s} (-1)^k \left(\frac{i}{2}\right)^{s-k} \binom{s}{k} k! \cdot \alpha_k\,. \tag{3.14}$$

In particular, for the Weyl application which is defined by (3.10) one gets

$$g(s,s,s) = \left(\frac{i}{2}\right)^s s!, \quad s \in \mathcal{N}. \tag{3.15}$$

(Compare with [1]).

The natural question arises if the condition *(i'i')* in Theorem 3.1 can be generalized to read

(i"i") $W_g(\{B,A\}_P) = \frac{1}{i\hbar}[W_g(B), W_g(A)]$ for every $B \in \mathcal{P}$ of the order ≤ 2 and for every $A \in \mathcal{P}$.

The answer to this question is the following theorem:

Theorem 3.4.

A generalized Weyl application $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ satisfies the condition (i"i") iff it is the Weyl application.

Proof:

As $\{q^2, p^m q^n\}_P = 2mp^{m-1}q^{n+1}$, $m,n \in \mathcal{N}$, we have by (ii) and (3.1)

$$W_g(\{q^2, p^m q^n\}_P) = 2m \sum_{s=0}^{min(m-1,n+1)} \binom{m-1}{s}\binom{n+1}{s} g(s,s,s)\hbar^s \hat{p}^{m-1-s}\hat{q}^{n+1-s}. \tag{3.16}$$

Then

$$\begin{aligned}\frac{1}{i\hbar}[\hat{q}^2, W_g(p^m q^n)] = & 2\sum_{s=0}^{min(m,n)} (m-s) \\ & \times \binom{m}{s}\binom{n}{s} g(s,s,s)\hbar^s \hat{p}^{m-1-s}\hat{q}^{n+1-s} \\ & + i\hbar \sum_{s=0}^{min(m,n)} (m-s)(m-s-1) \\ & \times \binom{m}{s}\binom{n}{s} g(s,s,s)\hbar^s \hat{p}^{m-2-s}\hat{q}^{n-s}.\end{aligned} \tag{3.17}$$

Comparing (3.16) and (3.17) one quickly finds that $W_g(\{q^2, p^m q^n\}_P) = \frac{1}{i\hbar}[\hat{q}^2, W_g(p^m q^n)]$ for every $m,n \in \mathcal{N}$ iff the condition (3.15) holds *i.e.*, iff the map W_g is the Weyl application W.

Then, analogously we prove that

$$W_g(\{p^2, p^m, q^n\}_P) = \frac{1}{i\hbar}[\hat{p}^2, W_g(p^m q^n)]$$

for every $m, n \in \mathcal{N}$ iff W_g is the Weyl application W.

Finally, as for the Weyl application

$$W(pq) = \tfrac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p})\ , \text{ one gets } W(\{pq, p^m q^n\}_P) = \frac{1}{i\hbar}[W(pq), W(p^m q^n)]$$

for every $m, n \in \mathcal{N}$.

Concluding, by linearity of $W_g, W_g(\{B, A\}_P) = \frac{1}{i\hbar}[W_g(B), W_g(A)]$ for every $B \in \mathcal{P}$ of the order 2 and for every $A \in \mathcal{P}$ iff W_g is the Weyl application W. Thus the theorem holds. ■

We end this section rewriting the generalized Weyl application in the form which has been used by some authors [5, 7, 9, 37-39] and appears to be crucial when the quantization for any symplectic manifold is considered. From (3.3) and (3.4) one gets

$$\begin{aligned} W_g(A) = \int_{R^2} A(p,q) \Bigg\{ \frac{1}{(2\pi)^2} \int_{R^2} & f(\hbar\lambda\mu)\ \exp[-i(\lambda p + \mu q)] \\ & \times \exp(i\lambda\hat{p}) \exp(i\mu\hat{q}) d\lambda d\mu \Bigg\} dpdq \\ = \int_{R^2} A(p,q) \hat{\Phi}_g(p,q) \frac{dpdq}{2\pi\hbar}\,, & \end{aligned} \tag{3.18}$$

where $\hat{\Phi}_g = \hat{\Phi}_g(p,q)$ is the formal series of the operator-valued distributions defined by

$$\begin{aligned} \hat{\Phi}_g = \hat{\Phi}_g(p,q) :=& \frac{2\pi\hbar}{(2\pi)^2} \int_{R^2} f(\hbar\lambda\mu) \\ & \times \exp[-i(\lambda p + \mu q)] \exp(i\lambda\hat{p}) \exp(i\mu\hat{q}) d\lambda d\mu \\ =& f\Big(-\hbar\frac{\partial^2}{\partial p \partial q}\Big) \frac{2\pi\hbar}{(2\pi)^2} \int_{R^2} \exp[i\lambda(\hat{p} - p)] \exp[i\mu(\hat{q} - q)] d\lambda d\mu \\ =& 2\pi\hbar f\Big(-\hbar\frac{\partial^2}{\partial p \partial q}\Big) \delta(\hat{p} - p)\delta(\hat{q} - q) \\ =& 2\pi\hbar \sum_{s=0}^{\infty} \frac{g(s,s,s)}{(s!)^2}\ \hbar^s\ \frac{\partial^s \delta(\hat{p} - p)}{\partial p^s}\ \frac{\partial^s \delta(\hat{q} - q)}{\partial q^s}\,. \end{aligned} \tag{3.19}$$

Using (3.8) we can write $\hat{\Phi}_g$ in the following form

$$\hat{\Phi}_g = \hat{\Phi}_g(p,q) = 2\pi\hbar\alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)\exp\Big(+\frac{i\hbar}{2}\frac{\partial^2}{\partial p\partial q}\Big)\delta(\hat{p}-p)\delta(\hat{q}-q)\,. \quad (3.20)$$

The operator $\hat{\Phi}_g$ is self-adjoint

$$\hat{\Phi}_g^+ = \hat{\Phi}_g \quad (3.21)$$

in a sense that the operator $W_g(A)$ is self-adjoint for every real $A \in \mathcal{P}$.

Definition 3.3.

The object $\hat{\Phi}_g = \hat{\Phi}_g(p,q)$ is called the *generalized Stratonovich–Weyl quantizer*, or the *generalized Grossmann–Royer operator.*

$\hat{\Phi}_g$ defined by (3.19) or, equivalently, by (3.20) generalizes in an obvious way the *Stratonovich–Weyl quantizer* or the *Grossman–Royer operator* [5, 7, 9, 37–39], which we denote here by $\hat{\Phi}$.

From (3.20) with (3.12) we obtain

$$\mathrm{Tr}[\hat{\Phi}_g(p,q)] = 1\,. \quad (3.22)$$

(*Remark:* In Section 4 we rewrite $\hat{\Phi}_g$ in a compact form (4.14) from which both (3.21) and (3.22) follow immediately).

Finally note that if $f = f(\hbar\lambda\mu)$ or, equivalently, $\alpha = \alpha(\hbar\lambda\mu)$ appear to be (suitable) analytic functions then one can extend W_g on a wide class $S'_\alpha(R^2)$ of distributions on the phase-space $\Gamma_2 = R^1 \times R^1$.

4. The generalized Weyl correspondence

The aim of this section is to find the map inverse to the generalized Weyl application. From now on we assume that the Hilbert space $\mathcal{H} = L^2(R^1)$.

First we rewrite the unitary operator $\exp[i(\lambda\hat{p}+\mu\hat{q})]$ in some useful form [2]. By the Baker–Campbell–Hausdorff formula one gets (see (3.3) and (3.8)).

$$\exp[i(\lambda\hat{p}+\mu\hat{q})] = \exp\Big(-\frac{i}{2}\hbar\lambda\mu\Big)\exp(i\lambda\hat{p})\exp(i\mu\hat{q})\,. \quad (4.1)$$

Then

$$\exp[i(\lambda\hat{p}+\mu\hat{q})] = \int_{-\infty}^{+\infty} \exp\,[i(\lambda\hat{p}+\mu\hat{q})]|q\rangle dq\langle q|\,. \quad (4.2)$$

Inserting (4.1) into (4.2) and employing the relation

$$\exp\ (i\lambda\hat{p})|q\rangle = |q - \hbar\lambda\rangle \tag{4.3}$$

one obtains the final result

$$\begin{aligned}\exp[i(\lambda\hat{p} + \mu\hat{q})] &= \int_{-\infty}^{+\infty} \exp\left[i\mu\Big(q - \frac{\hbar\lambda}{2}\Big)\right] |q - \hbar\lambda\rangle dq\langle q| \\ &= \int_{-\infty}^{+\infty} \exp(i\mu q')\Big|q' - \frac{\hbar\lambda}{2}\Big\rangle dq'\Big\langle q' + \frac{\hbar\lambda}{2}\Big| .\end{aligned} \tag{4.4}$$

Now, let $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ be a generalized Weyl application (3.13) defined by some real formal series of the form (3.11) with (3.12). Consider the matrix element $\langle q - \frac{\xi}{2}|W_g(A)|q + \frac{\xi}{2}\rangle$, $A \in \mathcal{P}$. From (3.13), employing also (3.4) and (4.4), by straightforward calculations one finds

$$\begin{aligned}\Big\langle q - \frac{\xi}{2}|W_g(A)|q + \frac{\xi}{2}\Big\rangle &= \frac{1}{2\pi\hbar}\alpha\Big(- i\xi\frac{\partial}{\partial q}\Big) \\ &\times \int_{-\infty}^{+\infty} A(p', q) \exp\Big(- \frac{i\xi p'}{\hbar}\Big) dp' .\end{aligned} \tag{4.5}$$

Hence, multiplying both sides of (4.5) by $\exp\ (\frac{i\xi p}{\hbar})$ and integrating over $d\xi$ we quickly find

$$\int_{-\infty}^{+\infty} \Big\langle q - \frac{\xi}{2}|W_g(A)|q + \frac{\xi}{2}\Big\rangle \exp\Big(\frac{i\xi p}{\hbar}\Big) d\xi = \alpha\Big(- \hbar\frac{\partial^2}{\partial p \partial q}\Big) A(p, q) . \tag{4.6}$$

Then, as $\alpha = \alpha(\hbar\lambda\mu)$ is a real formal series (3.11) satisfying (3.12) there exists a real formal series $\alpha^{-1} = \alpha^{-1}(\hbar\lambda\mu)$ *inverse* to α *i.e.*,

$$\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1 . \tag{4.7}$$

Therefore, finally one gets from (4.6)

$$\alpha^{-1}\Big(- \hbar\frac{\partial^2}{\partial p\partial q}\Big) \int_{-\infty}^{+\infty} \Big\langle q - \frac{\xi}{2}|W_g(A)|q + \frac{\xi}{2}\Big\rangle \exp\Big(\frac{i\xi p}{\hbar}\Big) d\xi = A(p, q) \tag{4.8}$$

for an arbitrary polynomial $A = A(p, q) \in \mathcal{P}$.

The formula (4.8) suggests us to consider the following linear map

$$W_g^{-1} : \hat{\mathcal{P}} \to \mathcal{P}$$

$$\hat{\mathcal{P}} \ni \hat{A} \mapsto W_g^{-1}(\hat{A}) := \alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p \partial q}\Big) \int_{-\infty}^{+\infty} \Big\langle q-\frac{\xi}{2}|\hat{A}|q+\frac{\xi}{2}\Big\rangle \exp\Big(\frac{i\xi p}{\hbar}\Big) d\xi \in \mathcal{P}. \tag{4.9}$$

By (4.8) the mapping W_g^{-1} is *onto* $\mathcal{P}$ and moreover, it is an easy matter to show that for any operators $\hat{A}, \hat{B} \in \hat{\mathcal{P}}$, $\hat{A} \neq \hat{B}$, one has $W_g^{-1}(\hat{A}) \neq W_g^{-1}(\hat{B})$. Consequently, the maps $W_g^{-1} : \hat{\mathcal{P}} \to \mathcal{P}$ and $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ are mutually inverse *i.e.*

$$W_g(W_g^{-1}(\hat{A})) = \hat{A} \quad \text{and} \quad W_g^{-1}(W_g(A)) = A \tag{4.10}$$

for arbitrary $\hat{A} \in \hat{\mathcal{P}}$ and $A \in \mathcal{P}$. Thus we get

Theorem 4.1.

For each generalized Weyl application $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ there exists the inverse mapping $W_g^{-1} : \hat{\mathcal{P}} \to \mathcal{P}$ defined by (4.9). ■

Definition 4.1.

The map $W_g^{-1} : \hat{\mathcal{P}} \to \mathcal{P}$ inverse to the generalized Weyl application $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ is called the *generalized Weyl correspondence* . Analogously, the *Weyl correspondence* $W^{-1} : \hat{\mathcal{P}} \to \mathcal{P}$ is the inverse mapping to the Weyl application $W : \mathcal{P} \to \hat{\mathcal{P}}$.

We now give some useful forms of the generalized Weyl correspondence. $W_g^{-1} : \hat{\mathcal{P}} \to \mathcal{P}$ First, employing the relation analogous to (4.3)

$$\exp(i\mu\hat{q})|p\rangle = |p + \hbar\mu\rangle$$

we get (compare with (4.4))

$$\exp[i(\lambda\hat{p} + \mu\hat{q})] = \int_{-\infty}^{+\infty} \exp(i\lambda p')\Big|p' + \frac{\hbar\mu}{2}\Big\rangle dp' \Big\langle p' - \frac{\hbar\mu}{2}\Big| . \tag{4.11}$$

Then, similar considerations to the ones which lead to (4.9) give

$$W_g^{-1}(\hat{A}) = \alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p \partial q}\Big) \int_{-\infty}^{+\infty} \Big\langle p - \frac{\eta}{2}\Big|\hat{A}\Big|p + \frac{\eta}{2}\Big\rangle \exp\Big(-\frac{i\eta q}{\hbar}\Big) d\eta, \tag{4.12}$$

Another form of W_g^{-1} can be quickly found to be

$$W_g^{-1}(\hat{A}) =\alpha^{-1}(-\hbar\frac{\partial^2}{\partial p\partial q}) \int\limits_{R^4} \exp(-\frac{i}{2}\frac{\lambda'\mu'}{\hbar}) \times \exp\left[\frac{i}{\hbar}(\lambda'(p-p')+\mu'(q-q'))\right]\frac{\langle q'|\hat{A}|p'\rangle}{\langle q'|p'\rangle}\frac{dp'dq'}{2\pi\hbar}\frac{d\lambda'd\mu'}{2\pi\hbar}. \tag{4.13}$$

Finally, we write the generalized Weyl correspondence in terms of the generalized Stratonovich–Weyl quantizer.

To this end we rewrite this object in a simple and useful form. Namely, from (3.19) with (3.8), (4.2) and (4.4) one gets

$$\hat{\Phi}_g(p,q) = \alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)\int\limits_{-\infty}^{+\infty}\exp\Big(\frac{i\xi p}{\hbar}\Big)\Big|q+\frac{\xi}{2}\Big\rangle d\xi\Big\langle q-\frac{\xi}{2}\Big|. \tag{4.14}$$

Then, for any $\hat{A}\in\hat{\mathcal{P}}$ we have

$$\begin{aligned}\mathrm{Tr}[\hat{\Phi}_g(p,q)\hat{A}] =&\alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)\int\limits_{R^2}\exp\Big(\frac{i\xi p}{\hbar}\Big)\langle q'\Big|q+\frac{\xi}{2}\Big\rangle \\ &\times\Big\langle q-\frac{\xi}{2}\Big|\hat{A}|q'\rangle d\xi dq' \\ =&\alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)\int\limits_{-\infty}^{+\infty}\Big\langle q-\frac{\xi}{2}\Big|\hat{A}\Big|q+\frac{\xi}{2}\Big\rangle\exp\Big(\frac{i\xi p}{\hbar}\Big)d\xi.\end{aligned} \tag{4.15}$$

Comparing (4.15) with (4.9) one obtains

$$W_g^{-1}(\hat{A}) = \alpha^{-2}\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)\mathrm{Tr}[\hat{\Phi}_g(p,q)\hat{A}]. \tag{4.16}$$

(Compare with Refs. [5, 7, 9]).

It is evident that the formula (4.13) can be equivalently written as follows

$$W_g^{-1}(\hat{A}) = \int\limits_{R^4}\alpha^{-1}\Big(\frac{\lambda'\mu'}{\hbar}\Big)\exp\Big(-\frac{i}{2}\frac{\lambda'\mu'}{\hbar}\Big) \times\exp\left[\frac{i}{\hbar}(\lambda'(p-p')+\mu'(q-q'))\right]\frac{\langle q'|\hat{A}|p'\rangle}{\langle q'|p'\rangle}\frac{dp'dq'}{2\pi\hbar}\frac{d\lambda'd\mu'}{2\pi\hbar}. \tag{4.17}$$

(Analogously one can also rewrite (4.9) and (4.12).)

Consequently, if α is an (appropriate) analytic function such that $\frac{1}{\alpha}$ exists almost everywhere and posesses suitable integrable properties then W_g can be extended on a wide class of distributions $S^1_\alpha(R^2)$, giving a *one to one* mapping $W_g : S'_\alpha(R^2) \to \hat{S}'_\alpha$, where $\hat{S}'_\alpha \subset \mathcal{L}(\mathcal{H})$.

It is of some interest to have a general formula for $\mathrm{Tr}[\hat{\Phi}_g(p,q)\hat{\Phi}_g(p',q')]$. Employing (4.14) one can easily find that

$$\mathrm{Tr}[\hat{\Phi}_g(p,q)\hat{\Phi}_g(p',q')] = 2\pi\hbar\alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big) \times \alpha\Big(-\hbar\frac{\partial^2}{\partial p'\partial q'}\Big)\delta(q-q')\delta(p-p')\,. \tag{4.18}$$

For $\alpha = 1$ we get the result of Refs. [5,7,8]. [To obtain (4.18) from (3.20) one uses an equality

$$\exp\left(\frac{i\hbar}{2}\frac{\partial^2}{\partial p\partial q}\right)\exp\left(\frac{i\hbar}{2}\frac{\partial^2}{\partial p'\partial q'}\right)\exp\left[\frac{i}{\hbar}(p-p')(q'-q)\right] = 2\pi\hbar\delta(p-p')\delta(q-q')]\,. \tag{4.19}$$

Finally, we make an important remark that *the formulas of sections 3 and 4 can be rewritten in terms of the Weyl application W and the Weyl correspondence* W^{-1}.

Indeed, consider the mapping $g : \mathcal{P} \to \mathcal{P}$ defined by

$$\mathcal{P} \ni A = A(p,q) \mapsto g(A) := \alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)A(p,q) \in \mathcal{P}\,. \tag{4.20}$$

In what follows we will use the notation

$$A_g \equiv g(A)\,,\ B_g \equiv g(B)\,,\ \cdots\ etc.\ A\,, B\,, \cdots\,, \in \mathcal{P}\,. \tag{4.21}$$

Then one quickly finds the fundamental relations

$$W_g = W \circ g \quad \text{and} \quad W_g^{-1} = g^{-1} \circ W^{-1}\,, \tag{4.22}$$

or, in other words

$$W_g(A) = W\Big(\alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)A\Big) \quad \text{and} \quad W_g^{-1}(\hat{A}) = \alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)W^{-1}(\hat{A}) \tag{4.23}$$

for any $A \in \mathcal{P}$ and $\hat{A} \in \hat{\mathcal{P}}$.

We have also evidently

$$\hat{\Phi}_g(p,q) = \alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)\hat{\Phi}(p,q)\,. \tag{4.24}$$

Now we are prepared to consider the Moyal $*_{(g)}$-product and the generalized Moyal bracket.

5. The Moyal $*_{(g)}$-product and the generalized Moyal bracket

In the previous sections we have found and analyzed the $\mathcal{C}$-linear isomorphisms $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ and $W_g^{-1} : \hat{\mathcal{P}} \to \mathcal{P}$ which, evidently, are not the *algebra* isomorphisms between the algebras $(\mathcal{P},\cdot)$ and $(\hat{\mathcal{P}},\cdot)$, where the dot "·" denotes the usual product of polynomials in $\mathcal{P}$ or of operators in $\hat{\mathcal{P}}$ respectively.

Here we intend to define a new multiplication in $\mathcal{P}$, which will be denoted by $*_{(g)}$, so that W_g and W_g^{-1} define the *algebra* isomorphisms between the algebras $(\mathcal{P}, *_{(g)})$ and $(\hat{\mathcal{P}},\cdot)$.

To this end consider any two operators $\hat{A}, \hat{B} \in \hat{\mathcal{P}}$. Then from (4.16) with (3.18) one gets (as usually we omit the dot "·")

$$W_g^{-1}(\hat{A}\hat{B}) = \alpha^{-2}\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)\ \mathrm{Tr}\,[\hat{\Phi}_g(p,q)\hat{A}\hat{B}] = \frac{1}{(2\pi\hbar)^2}\alpha^{-2}\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)$$
$$\times\int_{R^4} W_g^{-1}(\hat{A})(p',q')\mathrm{Tr}[\hat{\Phi}_g(p,q)\hat{\Phi}_g(p',q')\hat{\Phi}_g(p'',q'')]$$
$$\times W_g^{-1}(\hat{B})(p'',q'')dp'dq'dp''dq''\,. \tag{5.1}$$

(Compare (5.1) with [5, 7, 9]).

Then, using (4.14), we can quickly find the following formula

$$\mathrm{Tr}[\hat{\Phi}_g(p,q)\hat{\Phi}_g(p',q')\hat{\Phi}_g(p'',q'')] = 2^2\alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)$$
$$\times\,\alpha\Big(-\hbar\frac{\partial^2}{\partial p'\partial q'}\Big)\alpha\Big(-\hbar\frac{\partial^2}{\partial p''\partial q''}\Big)$$
$$\times\exp\Big\{\frac{2i}{\hbar}[(q-q')(p-p'')-(q-q'')(p-p')]\Big\}\,. \tag{5.2}$$

Substituting (5.2) into (5.1) one gets

$$W_g^{-1}(\hat{A}\hat{B})(p,q) = \frac{2^2}{(2\pi\hbar)^2}\alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)\int_{R^4} W_g^{-1}(\hat{A})(p+\lambda',q+\mu')$$

$$\times \left\{\alpha\Big(-\hbar\frac{\partial^2}{\partial\lambda'\partial\mu'}\Big)\alpha\Big(-\hbar\frac{\partial^2}{\partial\lambda''\partial\mu''}\Big)\exp\left[\frac{2i}{\hbar}(\mu'\lambda''-\mu''\lambda')\right]\right\}$$
$$\times W_g^{-1}(\hat{B})(p+\lambda'',q+\mu'')d\lambda'd\mu'd\lambda''d\mu''\,. \tag{5.3}$$

Employing (4.23) we can rewrite (5.3) in an another, equivalent, form. Indeed

$$W_g^{-1}(\hat{A}\hat{B})(p,q) = \alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)W^{-1}(\hat{A}\hat{B})(p,q) = \frac{2^2}{(2\pi\hbar)^2}\alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)$$
$$\times\int_{R^4}\left\{\alpha\Big(-\hbar\frac{\partial^2}{\partial\lambda'\partial\mu'}\Big)W_g^{-1}(\hat{A})(p+\lambda',q+\mu')\right\}\exp\left[\frac{2i}{\hbar}(\mu'\lambda''-\mu''\lambda')\right]$$
$$\times\left\{\alpha\Big(-\hbar\frac{\partial^2}{\partial\lambda''\partial\mu''}\Big)W_g^{-1}(\hat{B})(p+\lambda'',q+\mu'')\right\}d\lambda'd\mu'd\lambda''d\mu''\,. \tag{5.4}$$

The equivalent formulas (5.3) and (5.4) lead to the following new multiplication in $\mathcal{P}$. Namely we define the mapping $*_{(g)}:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ to be (see also (4.20) and (4.21))

$$*_{(g)}:(A,\,B)\mapsto A*_{(g)}B=(A*_{(g)}B)(p,q):=\frac{2^2}{(2\pi\hbar)^2}\alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)$$
$$\times\int_{R^4}A_g(p+\lambda',q+\mu')\exp\left[\frac{2i}{\hbar}(\mu'\lambda''-\mu''\lambda')\right]$$
$$\times B_g(p+\lambda'',q+\mu'')d\lambda'd\mu'd\lambda''d\mu''\,, \tag{5.5}$$

for any $A=A(p,q)$ and $B=B(p,q)$ from $\mathcal{P}$.

Definiton 5.1

The mapping $*_{(g)}:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ defined by (5.5) is called the *Moyal* $*_{(g)}$*-product* on $\mathcal{P}$.

Thus we have

$$W_g^{-1}(\hat{A}\hat{B})=W_g^{-1}(\hat{A})*_{(g)}W_g^{-1}(\hat{B})\,. \tag{5.6}$$

Consequently, as $(\hat{A}\hat{B})\hat{C}=\hat{A}(\hat{B}\hat{C})$ the Moyal $*_{(g)}$-product is associative.

Finally, one arrives at the associative but noncommutative algebra $(\mathcal{P},*_{(g)})$, and the following theorem holds

Theorem 5.1

The generalized Weyl application $W_g : \mathcal{P} \to \hat{\mathcal{P}}$ is an algebra isomorphism of $(\mathcal{P}, *_{(g)})$ onto $(\mathcal{P}, \cdot)$. ■

In the case of $\alpha = 1$ we write "$*$" instead of "$*_{(g)}$" and we say about the *Moyal $*$-product* in $\mathcal{P}$ [2, 3, 4, 5, 7, 9, 40, 42]. Now we intend to rewrite the Moyal $*_{(g)}$-product in a compact and transparent form. To this end we express $A_g(p+\lambda', q+\mu')$ and $B_g(p+\lambda'', q+\mu'')$ as follows

$$\begin{aligned}
A_g(p+\lambda', q+\mu') &= \sum_{n_1,n_2} \frac{1}{n_1!n_2!}\Big[\frac{\partial^{n_1}}{\partial p^{n_1}}\frac{\partial^{n_2}}{\partial q^{n_2}} A_g(p,q)\Big](\lambda')^{n_1}(\mu')^{n_2} \\
&\equiv \sum_{n_1,n_2} \frac{1}{n_1!n_2!} A_g(p,q)\frac{\overleftarrow{\partial}^{\,n_1}}{\partial p^{n_1}}\frac{\overleftarrow{\partial}^{\,n_2}}{\partial q^{n_2}}(\lambda')^{n_1}(\mu')^{n_2}, \\
B_g(p+\lambda'', q+\mu'') &= \sum_{n_3,n_4} \frac{1}{n_3!n_4!}(\lambda'')^{n_3}(\mu'')^{n_4}\frac{\partial^{n_3}}{\partial p^{n_3}}\frac{\partial^{n_4}}{\partial q^{n_4}} B_g(p,q) \\
&\equiv \sum_{n_3,n_4} \frac{1}{n_3!n_4!}(\lambda'')^{n_3}(\mu'')^{n_4}\frac{\overrightarrow{\partial}^{\,n_3}}{\partial p^{n_3}}\frac{\overrightarrow{\partial}^{\,n_4}}{\partial q^{n_4}}\, B_g(p,q).
\end{aligned} \tag{5.7}$$

Inserting (5.7) into (5.5) one obtains

$$\begin{aligned}
A *_{(g)} B =& (A *_{(g)} B)(p,q) \\
=& \alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p \partial q}\Big)\Big\{\sum_{n_1,n_2,n_3,n_4} A_g(p,q)\frac{\overleftarrow{\partial}^{\,n_1}}{\partial p^{n_1}}\frac{\overleftarrow{\partial}^{\,n_2}}{\partial q^{n_2}} \\
&\times \sigma(n_1,n_2,n_3,n_4)\frac{\vec{\partial}^{n_3}}{\partial p^{n_3}}\frac{\vec{\partial}^{n_4}}{\partial q^{n_4}}\, B_g(p,q)\Big\},
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
\sigma(n_1,n_2,n_3,n_4) :=& \frac{2^2}{(2\pi\hbar)^2}\,\frac{1}{n_1!n_2!n_3!n_4!}\int_{R^4} (\lambda')^{n_1}(\mu')^{n_2} \\
&\times \exp\Big[\frac{2i}{\hbar}(\mu'\lambda'' - \mu''\lambda')\Big](\lambda'')^{n_3}(\mu'')^{n_4} d\lambda' d\mu' d\lambda'' d\mu'' \\
=& \frac{2^2}{(2\pi\hbar)^2}\frac{1}{n_1!n_2!n_3!n_4!}\Big(-\frac{\hbar}{2i}\Big)^{n_1}\Big(\frac{\hbar}{2i}\Big)^{n_3}
\end{aligned}$$

$$\times \int_{R^4} (\mu')^{n_2}(\mu'')^{n_4}\frac{\partial^{n_1}}{\partial\mu''^{n_1}}\frac{\partial^{n_3}}{\partial\mu'^{n_3}}\exp\left[\frac{2i}{\hbar}(\mu'\lambda''-\mu''\lambda')\right]d\lambda'd\mu'd\lambda''d\mu''$$

$$=\frac{1}{n_1!n_2!n_3!n_4!}\left(-\frac{\hbar}{2i}\right)^{n_1}\left(\frac{\hbar}{2i}\right)^{n_3}\int_{R^2}\delta^{(n_3)}(\mu')\delta^{(n_1)}(\mu'')(\mu')^{n_2}$$

$$\times (\mu'')^{n_4}d\mu'd\mu'' = \frac{1}{n_1!n_2!}\left(-\frac{i\hbar}{2}\right)^{n_1}\left(\frac{i\hbar}{2}\right)^{n_2}\delta^{n_1}_{n_4}\delta^{n_2}_{n_3}\,. \tag{5.9}$$

Finally, substituting (5.9) into (5.8) we get

$$A *_{(g)} B = (A *_{(g)} B)(p,q) = \alpha^{-1}\left(-\hbar\frac{\partial^2}{\partial p\partial q}\right)\Bigg\{\sum_{n_1,n_2}\frac{1}{n_1!n_2!}$$

$$\left(-\frac{i\hbar}{2}\right)^{n_1}\left(\frac{i\hbar}{2}\right)^{n_2}A_g(p,q)\frac{\overleftarrow{\partial}^{\,n_1}}{\partial p^{n_1}}\frac{\overleftarrow{\partial}^{\,n_2}}{\partial q^{n_2}}\frac{\overrightarrow{\partial}^{\,n_2}}{\partial p^{n_2}}\frac{\overrightarrow{\partial}^{\,n_1}}{\partial q^{n_1}}B_g(p,q)\Bigg\}$$

$$= \alpha^{-1}\left(-\hbar\frac{\partial^2}{\partial p\partial q}\right)\Bigg\{A_g(p,q)\left[\sum_{n_2=0}^{\infty}\frac{1}{n_2!}\left(\frac{i\hbar}{2}\right)^{n_2}\frac{\overleftarrow{\partial}^{\,n_2}}{\partial q^{n_2}}\frac{\overrightarrow{\partial}^{\,n_2}}{\partial p^{n_2}}\right]$$

$$\times\left[\sum_{n_1=0}^{\infty}\frac{1}{n_1!}\left(-\frac{i\hbar}{2}\right)^{n_1}\frac{\overleftarrow{\partial}^{\,n_1}}{\partial p^{n_1}}\frac{\overrightarrow{\partial}^{\,n_1}}{\partial q^{n_1}}\right]B_g(p,q)\Bigg\}$$

$$= \alpha^{-1}\left(-\hbar\frac{\partial^2}{\partial p\partial q}\right)\left[A_g(p,q)\ \exp\left(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\right)B_g(p,q)\right], \tag{5.10}$$

where

$$\overleftrightarrow{\mathcal{P}} := \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\,. \tag{5.11}$$

In particular, for the Moyal $*$-product the formula (5.10) leads

$$A * B = A\exp\left(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\right)B\,. \tag{5.12}$$

thus

$$A *_{(g)} B(p,q) = \alpha^{-1}\left(-\hbar\frac{\partial^2}{\partial p\partial q}\right)(A_g * B_g)(p,q)\,. \tag{5.13}$$

From (5.12), (5.13) and (3.12) one finds

$$\lim_{\hbar\to 0}(A *_{(g)} B) = AB \tag{5.14}$$

for every $A, B, \in \mathcal{P}$ such that $\frac{\partial A}{\partial \hbar} = 0$ and $\frac{\partial B}{\partial \hbar} = 0$.

Therefore, the noncommutative, associative algebra $(\mathcal{P}, *_{(g)})$ appears to be a deformation of the algebra $(\mathcal{P}, \cdot)$.

Consider now the $\mathcal{C}$-linear isomorphism $g : \mathcal{P} \rightarrow \mathcal{P}$ defined by (4.20). Then, by (5.13) we have

$$g(A *_{(g)} B) = [g(A)] * [g(B)]\,. \tag{5.15}$$

Consequently, the following theorem holds.

Theorem 5.2

The mapping $g : \mathcal{P} \rightarrow \mathcal{P}$ defined by (4.20) is an algebra isomorphism of $(\mathcal{P}, *_{(g)})$ onto $(\mathcal{P}, *)$. ■

This means that *all algebras* $(\mathcal{P}, *_{(g)})$ *are mutually isomorphic and they all are isomorphic to the algebra* $(\mathcal{P}, *)$. From (5.10) with (5.11) one inmediately infers that the following relation holds

$$(A *_{(g)} B)(p,q) = \alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p \partial q}\Big)\left[B_g(p,q)\exp\Big(-\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\Big)A_g(p,q)\right]. \tag{5.16}$$

Definition 5.2

The *generalized Moyal bracket* is a mapping $\{\cdot,\cdot\}_M^{(g)} : \mathcal{P}\times\mathcal{P} \rightarrow \mathcal{P}$ defined by

$$\{A,B\}_M^{(g)} := \frac{1}{i\hbar}(A *_{(g)} B - B *_{(g)} A) \tag{5.17}$$

for any $A, B \in \mathcal{P}$.

Employing (5.16) we get

$$\{A,B\}_M^{(g)} = \alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p \partial q}\Big)\left[A_g(p,q)\frac{2}{\hbar}\sin\Big(\frac{\hbar}{2}\,\overleftrightarrow{\mathcal{P}}\Big)B_g(p,q)\right]. \tag{5.18}$$

The generalized Moyal bracket for $\alpha = 1$ will be denoted by $\{\cdot,\cdot\}_M$ and it is called the *Moyal bracket*. By (5.6) one has

$$W_g(\{A,B\}_M^{(g)}) = \frac{1}{i\hbar}[W_g(A), W_g(B)]\,. \tag{5.19}$$

This formula corresponds to the Dirac quantization rule (*ii*). It is evident that the pair $(\mathcal{P}, \{\cdot,\cdot\}_M^{(g)})$ constitutes the complex Lie algebra . Moreover, as

$$\lim_{\hbar\rightarrow 0}(\{A,B\}_M^{(g)}) = \{A,B\}_P \tag{5.20}$$

for every $A, B, \in \mathcal{P}$ such that $\frac{\partial A}{\partial \hbar} = 0$ and $\frac{\partial B}{\partial \hbar} = 0$ each Lie algebra $(\mathcal{P}, \{\cdot,\cdot\}_M^{(g)})$ appears to be a deformation of the Lie algebra $(\mathcal{P}, \{\cdot,\cdot\}_P)$ [35].

Then it is easy to see that by Theorem 5.2 the *mapping* $g : \mathcal{P} \to \mathcal{P}$ *defined by (4.20) is a Lie algeba isomorphism of* $(\mathcal{P}, \{\cdot,\cdot\}_M^{(g)})$ *onto* $(\mathcal{P}, \{\cdot,\cdot\}_M)$.

Definition 5.3

The Lie algebra $(\mathcal{P}, \{\cdot,\cdot\}_M)$ is called the *Moyal algebra* on $\mathcal{P}$.

Thus we can say that all Lie algebras $(\mathcal{P}, \{\cdot,\cdot\}_M^{(g)})$ are isomorphic to the Moyal algebra on $\mathcal{P}$.

From (5.18) with $\alpha = 1$ one gets

$$\{A, B\}_M = A\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\overleftrightarrow{\mathcal{P}}\right)B\,, \tag{5.21}$$

or, explicitly

$$\{A, B\}_M = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!}\left(\frac{\hbar}{2}\right)^{2s} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} \times (\partial_q^{2s+1-j}\partial_p^j A)(\partial_q^j \partial_p^{2s+1-j} B)\,, \tag{5.22}$$

where $\partial_q \equiv \frac{\partial}{\partial q}$ and $\partial_p \equiv \frac{\partial}{\partial p}$.

We end the present section with some remarks on the problem of extension of W_g and $*_{(g)}$ on non-polynomial functions or distributions. In the case of $\alpha = 1$ this problem has been considered by Bondia and Várilly in their distinguished paper [4]. The results obtained can be summarized as follows (see also [7, 9]). The Weyl application W ((3.13) with $\alpha = 1$) can be extended on the space of tempered distributions $S'(R^2)$.

Consequently, if $\mathcal{H} = L^2(R^1)$, then W is a linear continuous mapping from $S'(R^2)$ onto $\mathcal{L}(S(R^1), S'(R^1))$, where $\mathcal{L}(S(R^1), S'(R^1))$ denotes the space of all linear continuous mappings from the Schwartz space $S(R^1)$ into the space of tempered distributions on R^1, $S'(R^1)$.

Much more subtle is the problem of the extension of the Moyal $*$-product. As it has been shown in [4] this product (defined analogously to (5.5) with $\alpha = 1$) is well defined on the space

$$\mathcal{M}(R^2) = \mathcal{M}_L(R^2) \cap \mathcal{M}_R(R^2)\,, \tag{5.23}$$

where

$$\mathcal{M}_L(R^2) := \{T \in S'(R^2) : T * F \in S(R^2),\ \text{for all} F \in S(R^2)\}\,,$$

$$\mathcal{M}_R(R^2) := \{V \in S'(R^2) : F * V \in S(R^2), \text{ for all} F \in S(R^2)\} . \tag{5.24}$$

Here $T * F$ and $F * V$ are defined by

$$\langle T * F, G\rangle := \langle T, F * G\rangle \text{ and } \langle F * V, G\rangle := \langle V, G * F\rangle \tag{5.25}$$

for every $G \in S(R^2)$.

We have

$$S(R^2) \subset L^2(R^2) \subset \mathcal{M}(R^2) \subset S'(R^2) , \tag{5.26}$$

and also

$$\mathcal{E}'(R^2) \subset \mathcal{M}(R^2); \mathcal{P} \subset \mathcal{M}(R^2) , \tag{5.27}$$

where $\mathcal{E}'(R^2)$ denotes the space of distributions of compact support on R^2. Moreover, $\mathcal{M}(R^2)$ is invariant under the Fourier transformation.

It is evident that, in contrary to the case of polynomials, the Moyal $*$-product cannot be equivalently defined as

$$F * G = F \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}\right) G \tag{5.28}$$

for every $F, G \in \mathcal{M}(R^2)$. However, it has been shown in [4] that the formula (5.28) really holds for every $F, G \in \tilde{\mathcal{E}}'(R^2)$, where $\tilde{\mathcal{E}}'(R^2)$ stands for the Fourier image of $\mathcal{E}'(R^2)$.

What concerns the general case when $\alpha = \alpha(\hbar\lambda\mu) \neq 1$, the problem is much more involved and it will be analyzed in a separate paper. Here we only observe that this general problem can be brought to the case $\alpha = 1$ if the *Fourier transform* $\tilde{F} = \tilde{F}(\lambda, \mu)$ *is modified by* $\tilde{F}(\lambda, \mu)\alpha(\hbar\lambda\mu)$ (see (3.13)).

Finally, we make an important remark which appears to be a crucial point when some applications of the Weyl–Wigner–Moyal formalism in self-dual gravity are considered (see the next section and also Refs. [10–19]). Namely, let $F = F(p, q)$ and $G = G(p, q)$ be any $C^\infty(R^2)$ formal series in $\hbar$.

If $\alpha = \alpha(\hbar\lambda\mu)$ is also considered to be a formal series in $\hbar$, then according to (5.13) one can define

$$\begin{aligned} F *_{(g)} G = (F *_{(g)} G)(p, q) &:= \alpha^{-1}\Big(-\hbar\frac{\partial^2}{\partial p \partial q}\Big) \\ &\times \left\{ \left[\alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)F(p, q)\right] \exp\left(\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\right) \left[\alpha\Big(-\hbar\frac{\partial^2}{\partial p\partial q}\Big)G(p, q)\right]\right\} . \end{aligned} \tag{5.29}$$

$F *_{(g)} G$ is the formal series in $\hbar$.

Thus denoting the linear space of the $C^\infty(R^2)$ formal series in $\hbar$ by $\Pi(\hbar; C^\infty(R^2))$, one gets the noncommutative associative algebras $(\Pi(\hbar; C^\infty(R^2)), *_{(g)})$ which are mutually isomorphic and isomorphic to the algebra $(\Pi(\hbar; C^\infty(R^2)), *)$.

Then one defines the Lie algebras $(\Pi(\hbar; C^\infty(R^2)), \{\cdot,\cdot\}_M^{(g)})$ which are also mutually isomorphic and isomorphic to $(\Pi(\hbar; C^\infty(R^2)), \{\cdot,\cdot\}_M)$.

Definition 5.4

The algebra $(\Pi(\hbar; C^\infty(R^2)), \{\cdot,\cdot\}_M)$ is called the *Moyal algebra*.

According to the Gelfand–Dorfman–Fletcher theorem [42], all 2-index infinite Lie algebras correspond to the Moyal algebra in some basis (On the Moyal algebra see also [12, 14]).

6. The Weyl–Wigner–Moyal formalism and the first heavenly equation

In this section we present some example of application of the Weyl–Wigner–Moyal fromalism. The example, rather unexpectedly, does not concern quantum mechanics but general relativity, or more precisely, the self-dual gravity. We deal with a 4-dimensional real differential manifold X endowed with a self-dual vacuum metric ds^2 of the signature $(++--)$. It is well known that in this case the space-time (X, ds^2) appears to be hyper-kählerian and the metric ds^2 can be locally brought to the form

$$ds^2 = 2(\Omega_{,xq} dx \otimes_s dq + \Omega_{,xp} dx \otimes_s dp + \Omega_{,yq} dy \otimes_s dq + \Omega_{,yp} dy \otimes_s dp)\,, \quad (6.1)$$

where "$\otimes_s$" denotes the symmetrized tensor product *i.e.*, $dx \otimes_s dq := \frac{1}{2}(dx \otimes dq + dq \otimes dx) \ldots$, etc., and $\Omega = \Omega(x, y, p, q)$ is a real function satisfying the *first heavenly equation* [43].

$$\Omega_{,xq}\Omega_{,yp} - \Omega_{,xp}\Omega_{,yq} = 1\,. \quad (6.2)$$

Here we use the obvious notation. $\Omega_{,x} \equiv \partial_x \Omega$, $\Omega_{,xq} \equiv \partial_x \partial_q \Omega, \ldots$ etc. It is convenient to rewrite Eq. (6.2) in the form

$$\{\Omega_{,x}, \Omega_{,y}\}_P = 1\,, \quad (6.3)$$

where $\{\cdot,\cdot\}_P$ stands, for the Poisson bracket with respect to (q, p). Then the *Moyal deformation of the first heavenly equation* reads [10, 11, 17, 18].

$$\{\Omega_{,x}, \Omega_{,y}\}_M = 1\,. \quad (6.4)$$

Employing the Weyl application to (6.4) and using also (5.19) one gets

$$[\hat{\Omega}_{,x}, \hat{\Omega}_{,y}] = i\hbar\hat{1}\,, \tag{6.5}$$

where

$$\hat{\Omega} = \hat{\Omega}(x,y) := W(\Omega(x,y,p,q))\,. \tag{6.6}$$

But Eq. (6.5) really means that there exists an unitary operator-valued function

$$\hat{U} = \hat{U}(x,y),\ \hat{U}\hat{U}^{+} = \hat{U}^{+}\hat{U} = \hat{1} \tag{6.7}$$

such that

$$\hat{\Omega}_{,x} = \hat{U}\hat{q}\hat{U}^{+} \text{ and } \hat{\Omega}_{,y} = \hat{U}\hat{p}\hat{U}^{+}\,. \tag{6.8}$$

Then, from (5.6) we obtain

$$\begin{gathered}\Omega_{,x} = U * q * \bar{U}, \Omega_{,y} = U * p * \bar{U}\,,\\ U * \bar{U} = \bar{U} * U = 1\,,\end{gathered} \tag{6.9}$$

where we have used also the relation which follows inmediately from (4.8), *i.e.*

$$W^{-1}(\hat{U}^{+}) = \overline{W^{-1}(\hat{U})}\,. \tag{6.10}$$

The function $U = U(x,y,p,q)$ in (6.9) is defined by $U = U(x,y,p,q) := W^{-1}(\hat{U}(x,y))$.

Then one quickly finds

$$\begin{aligned}q * \bar{U} &= i\hbar\{q,\bar{U}\}_M + \bar{U} * q = i\hbar\partial_p\bar{U} + \bar{U} * q,\\ p * \bar{U} &= i\hbar\{p,\bar{U}\}_M + \bar{U} * p = -i\hbar\partial_q\bar{U} + \bar{U} * p\,.\end{aligned} \tag{6.11}$$

Finally, substituting (6.11) into (6.9) and employing the relation $\Omega_{,xy} - \Omega_{,yx} = 0$ we have

$$\partial_x(U * \partial_q\bar{U}) + \partial_y(U * \partial_p\bar{U}) = 0, U * \bar{U} = \bar{U} * U = 1\,. \tag{6.12}$$

Eqs. (6.12) can be rewritten in equivalent form as follows. From the relation $\hat{\Omega}_{,xy} - \hat{\Omega}_{,yx} = 0$ and (6.8) one quickly gets

$$[\hat{q}, \hat{U}^{+}\partial_y\hat{U}] - [\hat{p}, \hat{U}^{+}\partial_x\hat{U}] = 0\,. \tag{6.13}$$

Then by the Weyl correspondence we obtain

$$\begin{aligned}&\partial_q(\bar{U} * \partial_x U) + \partial_p(\bar{U} * \partial_y U) = 0\,,\\ &U * \bar{U} = \bar{U} * U = 1\,.\end{aligned} \tag{6.14}$$

Eqs. (6.12) or, equivalently, (6.14) are equivalent to the Moyal deformation of the first heavenly equation (6.4).

To give an interpretation of the equations obtained we consider the self-dual Yang–Mills (SDYM) equations on the flat 4-dimensional manifold R^4 of the metric ds'^2 of the signature $(++--)$

$$ds'^2 = 2(dx \otimes_s d\tilde{x} + dy \otimes_s d\tilde{y}) . \tag{6.15}$$

Let G be some Lie group and $\mathcal{G}$ its Lie algebra. Then the G-SDYM equations read [44]

$$\partial_x A_y - \partial_y A_x + [A_x, A_y] = 0 , \tag{6.16a}$$
$$\partial_{\tilde{x}} A_{\tilde{y}} - \partial_{\tilde{y}} A_{\tilde{x}} + [A_{\tilde{x}}, A_{\tilde{y}}] = 0 , \tag{6.16b}$$
$$\partial_x A_{\tilde{x}} - \partial_{\tilde{x}} A_x + \partial_y A_{\tilde{y}} - \partial_{\tilde{y}} A_y + [A_x, A_{\tilde{x}}] + [A_y, A_{\tilde{y}}] = 0 , \tag{6.16c}$$

where $A_i = A_i(x, y, \tilde{x}, \tilde{y}), i \in \{x, y, \tilde{x}, \tilde{y}\}$, are $\mathcal{G}$-valued functions on R^4. From (6.16 a) and (6.16 b) it follows that there exist the G-valued functions on $R^4, K = K(x, y, \tilde{x}, \tilde{y})$ and $\tilde{K} = \tilde{K}(x, y, \tilde{x}, \tilde{y})$, such that

$$A_j = K^{-1}\partial_j K \text{ for } j \in \{x, y\}, \text{ and } A_l = \tilde{K}^{-1}\partial_l \tilde{K} \text{ for } l \in \{\tilde{x}, \tilde{y}\} . \tag{6.17}$$

Define

$$J := K \cdot \tilde{K}^{-1} \in G \otimes C^\infty(R^4) . \tag{6.18}$$

Substituting (6.17) into (6.16c) and using the definiton (6.18) one finally concludes that the G-SDYM equations (6.16a), (6.16b), (6.16c) can be brought to the following form [45]

$$\partial_{\tilde{x}}(J^{-1}\partial_x J) + \partial_{\tilde{y}}(J^{-1}\partial_y J) = 0 \tag{6.19}$$

or, equivalently

$$\partial_x(J\partial_{\tilde{x}} J^{-1}) + \partial_y(J\partial_{\tilde{y}} J^{-1}) = 0 . \tag{6.20}$$

Comparing (6.19) and (6.20) with (6.14) and (6.12), respectively, we conclude that the *Moyal deformation of the first heavenly equation can be interpreted to be the U_*- SDYM equations*, where U_* is the Lie group defined by

$$U_* := \{U \in \Pi(\hbar; C^\infty(R^4)) : U * \bar{U} = \bar{U} * U = 1\} , \tag{6.21}$$

(where, as before, the Moyal $*$-product is defined according to (5.12) with (5.11).

It is evident that by the Weyl application we find that U_* appears to be isomorphic to the group $\hat{U}$ of unitary operators acting on the Hilbert space

$\mathcal{H}$. Then, the first heavenly equation can be considered to be the $\hbar \rightarrow 0$ limit of its Moyal deformation.

The relation between the first heavenly equation and the SDYM equations which has been presented in this section seems to be new with respect to the previous analysis [16–18, 46, 47, 48], as in our interpretation *no symmetry reduction is needed.* The main question is if this interpretation enables us to apply all the mathematical machinery of the SDYM equations to the first heavenly equation. This is obviously a very hard problem and it lies behind the scope of the present paper.

We are indebted to H. García-Compeán for useful discussions and to Marek Kalinowski for pointing out the recent paper by Gadella [9]. One of us (M.P.) is grateful to the staff of Departmento de Física at CINVESTAV, México, D.F., México, for a warm hospitality. The work is partially supported by CINVESTAV and CONACYT, México.

REFERENCES

[1] J. Tosiek, M. Przanowski, Weyl-Wigner-Moyal Formalism. I. Operator Ordering, submitted to *Acta Phys. Pol.* **B**.

[2] W.I. Tatarskij, *Usp. Fiz. Nauk* **139**, 587 (1983).

[3] M. Hillery, R.F. O'Connell, M.O. Scully, E.P. Wigner, *Phys. Rep.* **106**, 121 (1984).

[4] J.M. Gracia-Bondía, J. C. Várilly, *J. Math Phys.* **29**, 869 (1988).

[5] J.F. Cariñena, J.M. Gracia-Bondía, J.C. Várilly, *J. Phys. A; Math. Gen.* **23**, 901 (1990).

[6] Go. Torres-Vega, J. H. Frederick, *Phys. Rev. Lett.* **67**, 2601 (1991); *J. Chem. Phys.* **98**, 3103 (1993).

[7] J.C. Várilly, *Teoría de Grupos en Cuantización*, Publicaciones del Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del I.P.N., México D.F., México, 1992.

[8] A. Unterberger, Quantization: Some Problems, Tools and Applications, preprint 1995.

[9] M. Gadella, *Fortschr. Phys.* **43**, 229 (1995).

[10] I.A.B. Strachan, *Phys. Lett.* **B283**, 63 (1992).

[11] K. Takasaki, *J. Geom. Phys.***14**, 111 (1994); *J. Geom. Phys.* **14**, 332 (1994).

[12] D.B. Fairlie P. Fletcher, C.K. Zachos, *J. Math. Phys.***31**, 1088 (1990).

[13] I.A.B. Strachan, *J. Phys. A* **20**, 1967 (1995).

[14] I.A.B. Strachan, The Geometry of Multidimensional Integrable Systems, preprint 1995.

[15] I.A.B. Strachan, The Dispersive Self-Dual Einstein Equations and the Toda Lattice, preprint 1995.

[16] J.F. Plebański, M. Przanowski, B. Rajca, J. Tosiek, *Acta Phys,. Pol.***B26**, 889 (1995).

[17] J.F. Plebański, M. Przanowski, *The Universal Covering of Heavenly Equations via Weyl-Wigner-Moyal Formalism*, to appear.
[18] J.F. Plebański, M. Przanowski, *Phys. Lett.* **A212**, 22 (1996).
[19] J.F. Plebański, M. Przanowski, H. García-Compeán, *From Principal Chiral Model to Self-Dual Gravity*, accepted for publication in *Mod. Phys. Lett.* **A**.
[20] B.A. Kupershmidt, *Lett. Math. Phys.* **20**, 19 (1990).
[21] J. Histarinta, *Quantum Integrability and Classical Integrability*, in: *Non-Linear Equations in Classical and Quantum Field Theory*, ed. N. Sanchez, Springer-Verlag, Berlin, Heidelberg 1985. p.p. 37-44.
[22] F.A. Berezin, *Commun. Math. Phys.* **40**, 153 (1975).
[23] P. Kasperkovitz, M. Peev. *Ann. Phys.* **230**, 21 (1994).
[24] Y. Ohnuki, S. Kitakado, *Mod. Phys. Lett.* **A7**, 2477 (1992).
[25] P.A.M. Dirac, *The Principles of Quantum Mechanics*, Oxford University Press, 1958.
[26] A. Joseph, *Commun. Math. Phys.***17**, 210 (1970).
[27] K.B. Wolf, *The Heisenberg-Weyl Ring in Quantum Mechanics*, in: *Group Theory and Its Applications*, ed. E. Loebl, Academic Press, New York 1975, vol. III. p.p 189-247.
[28] H.J. Groenewald, *Physica* **12**, 405 (1946).
[29] P.R. Chernoff, *Hadronic J.* **4**, 879 (1981).
[30] R. Abraham, J. E. Marsden, *Foundations of Mechanics*, Second Edition, The Benjamin/Cummings Publishing Company, Inc., Advanced Book Program Reading, Massachusetts, 1978.
[31] L. Van Hove, *Mem. de l'Acad. Roy de Belgique (Class des Sci)*, v. **XXVI**, 61 (1951).
[32] N. Woodhouse, *Geometric Quantization*, Oxford University Press, New York 1980.
[33] L. Cohen, *J. Math. Phys.* **7**, 781 (1966).
[34] G.S. Agarwal, E. Wolf, *Phys. Rev.* **D2**, 2161 (1970); *Phys. Rev.* **D2**, 2206 (1970).
[35] F. Bayen, M.Flato, C. Fransdal, A. Lichnerowicz, D. Sternheimer, *Ann. Phys.* NY **111**, 61 (1978); *Ann. Phys.* NY, **111**, 111 (1978).
[36] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, New York 1931.
[37] A. Grossmann, *Commun. Math. Phys.* **48**, 191 (1976).
[38] A. Royer, *Phys. Rev.* **A15**, 449 (1977).
[39] R.L. Stratonovich, *Sov. Phys. JETP* **31**, 1012 (1956).
[40] J.E. Moyal, *Proc. Cambridge Phil. Soc.* **45**, 99 (1949).
[41] J. F. Plebański, Poisson Brackets and Commutators, preprint of Nicolaus Copernicus University, Toruń (1969) (in Polish).
[42] P. Fletcher, *The Moyal Bracket*, in: *Quantum Groups*, Proceedings of the Argonne Workshop, ed. T. Curtright, D.Fairlie and C. Zachos, World Scientific, Singapore 1991, p.p. 143-157.
[43] J.F. Plebański, *J. Math. Phys.* **16**, 2395 (1975).
[44] C.N. Yang, *Phys. Rev. Lett.* **38**, 1377 (1977).

[45] N. Sanchez, *Exact Solutions in Gauge Theory, General Relativity, and Their Supersymmetric Extensions*, in: *Geometric Aspects of the Einstein Equations and Integrable Systems*, ed. R. Martini, Springer-Verlag, Berlin, Heidelberg 1985, p.p. 1-76.
[46] L.J. Mason, *Twistor Newsletter* **30**, 14 (1990).
[47] J. F. Plebański, M. Przanowski, H. García-Compeán, *Acta Phys. Pol.* **B25**, 1079 (1994).
[48] Q. Han Park, *Int. J. Mod. Phys.* **A7**, 1415 (1992).