QUANTUM CHAOS IN CONTINUOUS SPECTRUM

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The continuous analogy to the spacing is introduced: the first derivative of energy with respect to its continuous labelling index and its distribution is calculated. As an example of application the Schrödinger particle with a random effective mass is investigated. The notion of quantum chaos and quantum integrability in continuous spectrum is widely discussed.

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1. Introduction

Very complicated interactions in solid states may lead to chaotic behaviour of quantum excitations. For systems with extended states, where classical motion is unrestricted in space, the spectrum is not discrete. These two properties of solid states create the following question: how to investigate quantum chaos in an uncountable spectrum? Some problems concerning this question have been investigated in recent papers. Geisel *et al.* [1] investigate Cantor spectrum, *i.e.* an uncountable one. Silberbauer *et al.* [2] report on inter-mini-band statistics, performed for the magnetic band structure [3]. In both papers the quantum chaos is detected by the distribution of spacing, which is a suitable tool for investigation of the discrete spectrum. Therefore, the method discretizes the continuous spectrum by itself. Another approach to the statistical properties of the continuous spectrum is achieved by analysis of the quantum chaotic scattering [4, 5]. The chaotic characteristics of the system manifest themselves obviously in the properties of the S-matrix [6]. Instead of investigating the S-matrix elements one can consider the scattering system by the properties of the corresponding phase shifts [7, 8]. In this paper instead of the distribution of spacing we

introduce the distribution of the first derivative of energy with respect to the continuous parameter. As an example of application we investigate one-dimensional Schrödinger particle with randomized effective mass. Meaning of quantum chaos in the continuous spectrum is discussed in the relation to the quantum chaos in the discrete spectrum.

2. Statistical measure

In the discrete spectrum one can treat the i^{th} energy level E_i as a value of the discrete function f of its index i, namely $f : I \ni i \to E_i \in E_R$, where I is the discrete index set and E_R is the discrete energy set, and $f(i) = E_i$. Hence the i^{th} spacing s_i is the asymmetrical two point first difference of the adjacent energies E_i, E_{i+1} and it is also the first differential quotient of the function f at the point i

$$s_i = \frac{\Delta^1 f(i)}{\Delta^1 i} = \frac{E_{i+1} - E_i}{i+1-i} . \tag{1}$$

The spacing distribution includes complete information about the interaction between the energy levels for the two level system.

In the continuous spectrum limit the first differential quotient converges to the first derivative of energy with respect to the continuous parameter labelling allowed energies $s(k) = dE(k)/dk$. Therefore in the case of continuous spectrum the statistics of the first derivative of the energy is a natural tool for investigation of repulsion between the levels separated by the infinitesimal distance. This fact has a deeper explanation. The first derivative can be approximated by the three adjacent points much better than by the two points:

$$\Delta_a^1 E_i = \frac{1}{2(i+1-i)}(-3E_i + 4E_{i+1} - E_{i+2}) , \tag{2}$$

(compare [9]; we named $\Delta_a^1 E_i$ the i^{th} asymmetrical three point first finite element. Shortly we call it asymmetrical element.) and

$$\Delta_s^1 E_{i+1} = \frac{1}{2(i+1-i)}(E_{i+2} - E_i) , \tag{3}$$

(compare [9]; we called $\Delta_s^1 E_{i+1}$ the i^{th} symmetrical three point first finite element or just symmetrical element). We have shown analytically [10] that their distributions, for the GOE(3), GUE(3) and GSE(3) ensembles, as well as for the ordered sequence of the three uncorrelated adjacent energy levels, are universal and properly describe the level repulsion (GOE(3), GUE(3),

and GSE(3) mean Gaussian Orthogonal Ensemble, Gaussian Unitary Ensemble, and Gaussian Symplectic Ensemble of dimensions 3, respectively). This result shows that even in the case of discrete spectrum the idea of the first derivative works.

3. Chaotic model

Let us assume a purely chaotic one dimensional quantum system and derive the distribution of $s(k)$. The system is described by the Schrödinger equation with the random effective mass m^*:

$$-\frac{\hbar^2}{2m^*}\frac{d^2}{dx^2}\Psi = E\Psi\,. \tag{4}$$

The energy of a particle in an energy band is given by the following formula [11]

$$E(k) = \frac{\hbar^2 k^2}{2m^*}\,, \tag{5}$$

where $E(k)$ is the energy, k is the wave vector. The energy $E(k)$ is a continuous function E of wave vector k

$$E : E_B \ni k \to E(k) \in E_R\,,$$

where E_B is its continuous (non discrete) domain set, E_R is its continuous range set. In order to delete the double degeneration of the energy $E(k)$ with respect to k we restrict its domain E_B to the set of nonnegative real numbers:

$$E_B = \{k : k \geq 0\}\,.$$

The random effective mass might assume negative values for its sample points and the function $E(k)$ might not to be monotonously increasing. Therefore energy levels $E(k)$ will not be ordered increasingly and we could not introduce the continuous analogue of spacing. We solve this problem by computing the absolute value $B(k)$ of $E(k)$:

$$B : E_B \ni k \to B(k) = |E(k)| = \frac{\hbar^2 k^2}{2|\,m^*\,|} \in \boldsymbol{R}\,. \tag{6}$$

The function B is monotonously increasing and the absolute values of energy levels $B(k)$ are ordered into an ascending sequence. Now, we are ready to introduce continuous analogue $A(k)$ of the spacing. First, we compute the first derivative of the function E with respect to k:

$$E'(k) = \frac{dE(k)}{dk} = \frac{\hbar^2 k}{m^*}\,. \tag{7}$$

Then, we derive the absolute value $A(k)$ of first derivative $E'(k)$, which is also the first derivative of B:

$$A(k) = |E'(k)| = \frac{dB(k)}{dk} = \frac{\hbar^2 k}{|m^*|}. \tag{8}$$

If we assume, that the index i, which is the argument of the discrete function f, can be interpreted as a discrete wave vector, then we can treat the energy function E as a limit of a sequence of discrete functions f_n, namely

$$\lim_{n \to \infty} f_n = E. \tag{9}$$

and its domain E_B as a limit of a sequence of discrete domains I_n of functions f_n

$$\lim_{n \to \infty} I_n = E_B, \tag{10}$$

where the limits describe the transition from discrete energy domains I_n to the continuous domain E_B and n is a natural number.

Let us assume that both f_n and E are unknown and unknowable. Then $f_n(i)$ are the random variables for every $i \in I$ and for every n, namely

$$f_n(i) : \Omega_n \ni \omega_{n,i} \to f_n(i)(\omega_{n,i}) \in \boldsymbol{R},$$

where $\omega_{n,i}$ is the sample point and Ω_n is the sample space. Also $E(k)$ are the random variables for every $k \in E_B$:

$$E(k) : \Omega \ni \omega_k \to E(k)(\omega_k) \in \boldsymbol{R},$$

where ω_k is the sample point and Ω is the sample space. Thence we can introduce the random function (random process) $\bar{E}$

$$\bar{E} : E_B \times \Omega \ni (k, \bar{\omega}) \to \bar{E}(k, \bar{\omega}) = E(k)(\bar{\omega}) \in \boldsymbol{R},$$

where $\bar{\omega}$ is the sample point, and the map

$$\Omega \ni \bar{\omega} \to \bar{E}(k, \bar{\omega}) \in \boldsymbol{R}$$

is the random variable $E(k)$ for every $k \in E_B$. Let us assume that m^* is M_0-centred Gaussian distributed, with the variance equal to σ_m^2

$$m^* : M \ni m \to m^*(m) \in \boldsymbol{R}, \tag{11}$$

where M is the sample space and m is the sample point. Its probability density function f_{m^*} is given by the following formula

$$f_{m^*}(m) = \frac{1}{\sqrt{2\pi}\sigma_m} \exp\left(-\frac{(m - M_0)^2}{2\sigma_m^2}\right). \tag{12}$$

Its inverse $X = \frac{1}{m^*}$ is a random variable

$$X : X_D \ni x \to X(x) \in \boldsymbol{R}, \tag{13}$$

where X_D is the sample space and x is the sample point. From the change of the variable formula [12] one easily obtains that the inverse X has the following distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_m x^2} \exp\left(-\frac{(\frac{1}{x} - M_0)^2}{2\sigma_m^2}\right) \quad \text{for } x \neq 0\,. \tag{14}$$

The function f_X can be made continuous in the whole domain if we put

$$f_X(0) = \lim_{x \to 0} f_X(x) = 0\,. \tag{15}$$

It follows from (5), (12), (14), (15) and from the change of the variable formula [12] that the distribution of the random variable $E(k)$ is

$$f_{E(k)}(\omega_k) = \begin{cases} \frac{\hbar^2 k^2}{2\sqrt{2\pi}\sigma_m \omega_k^2} \exp\left(-\frac{(\frac{\hbar^2 k^2}{2\omega_k} - M_0)^2}{2\sigma_m^2}\right) & \text{for} \quad \omega_k \neq 0\,, \\ 0 & \text{for} \quad \omega_k = 0\,, \end{cases} \tag{16}$$

for every $k \in E_B$.

The absolute value of energy $B(k)$ is the random variable

$$B(k) : Q \ni q_k \to B(k)(q_k) \in \boldsymbol{R},$$

for every $k \in E_B$, where q_k is the sample point and Q is the sample space. We define the random function $\bar{B}$

$$\bar{B} : E_B \times Q \ni (k, \bar{q}) \to \bar{B}(k, \bar{q}) = B(k)(\bar{q}) = \mid E(k)(\bar{q}) \mid \in \boldsymbol{R},$$

where $\bar{q}$ is the sample point, and the map

$$Q \ni \bar{q} \to \bar{B}(k, \bar{q}) \in \boldsymbol{R}$$

is the random variable $B(k)$ for every $k \in E_B$. The distribution of $B(k)$ will be derived after the computation of the distribution of $A(k)$. The first derivative $E'(k)$ is also the random variable

$$E'(k) : O \ni o_k \to E'(k)(o_k) \in \boldsymbol{R},$$

for every $k \in E_B$, where o_k is the sample point and O is the sample space. Hence we define the random function $\bar{E}'$

$$\bar{E}' : E_B \times O \ni (k, \bar{o}) \to \bar{E}(k, \bar{o}) = E(k)(\bar{o}) \in \boldsymbol{R},$$

where $\bar{o}$ is the sample point, and the map

$$O \ni \bar{o} \to \bar{E}(k, \bar{o}) \in \boldsymbol{R}$$

is the random variable $E'(k)$ for every $k \in E_B$. From (7), (12), (14), (15), and from the change of the variable formula [12] one easily obtains that its distribution is

$$f_{E'(k)}(o_k) = \begin{cases} \frac{\hbar^2 k}{\sqrt{2\pi}\sigma_m o_k^2} \exp(-\frac{(\frac{\hbar^2 k}{o_k} - M_0)^2}{2\sigma_m^2}) & \text{for} \quad o_k \neq 0\,, \\ 0 & \text{for} \quad o_k = 0\,, \end{cases} \tag{17}$$

for every $k \in E_B$. We must point out that in (17) the wave vector k is the independent nonnegative variable from the domain E_B and the distribution $f_{E'(k)}$ is a function of k, because the random variables $E'(k)$ depend on the choice of k. We present the plot of the probability density function of $E'(k)$ in Figure 1, putting $M_0 = 1, \sigma_m = 1, k = 1$ and $\hbar = 1$. The derived statistics (17) is governed by non-power law:

$$f_{E'(k)}(o_k) \sim o_k^{-2} \exp(-\frac{(\frac{\hbar^2 k}{o_k} - M_0)^2}{2\sigma_m^2}) \tag{18}$$

which is completely new result.

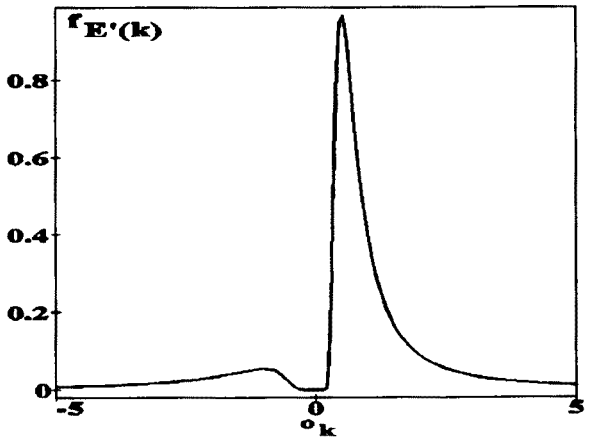

Fig. 1. The probability density function of the first derivative of energy $f_{E'(k)}$.

Finally, we compute the distribution of the absolute value (or the length) $A(k)$ of the first derivative $E'(k)$ for every $k \in E_B$. Let us define the random variable $A(k)$ as follows:

$$A(k) : P \ni p_k \to A(k)(p_k) \in \boldsymbol{R}\,, \tag{19}$$

where P is the sample space and p_k is the sample point. Then we define the random function $\bar{A}$

$$\bar{A} : E_B \times P \ni (k, \bar{p}) \to \bar{A}(k, \bar{p}) = A(k)(\bar{p}) \in \boldsymbol{R},$$

where $\bar{p}$ is the sample point, P is the sample space and the map

$$P \ni \bar{p} \to \bar{A}(k, \bar{p}) \in \boldsymbol{R}$$

is the random variable $A(k)$ for every $k \in E_B$. The map transforming $E'(k)$ to $A(k)$ is the module map:

$$g : \boldsymbol{R} \ni x \to |x| \in \boldsymbol{R},$$

which is not a homeomorphism of class $C^1(\boldsymbol{R})$. The map g is a homeomorphism of class C^1 on its sub-domains $\boldsymbol{R}_+$ and $\boldsymbol{R}\backslash\boldsymbol{R}_+$, where $\boldsymbol{R}_+$ is the set of the nonnegative real numbers. We can restrict g to these two sub-domains obtaining two homeomorphisms

$$\begin{aligned} &g_1 : E_B \ni x \to g(x) \in \boldsymbol{R}, \\ &g_2 : \boldsymbol{R}\backslash E_B \ni x \to g(x) \in \boldsymbol{R} \end{aligned} \tag{20}$$

and use twice the change of the variable formula [12] as follows:

$$f_{A(k)}(p_k) = \begin{cases} |\operatorname{jac} g_1^{-1}(p_k)| f_{E'(k)}(g_1^{-1}(p_k)) & \\ +|\operatorname{jac} g_2^{-1}(p_k)| f_{E'(k)}(g_2^{-1}(p_k)) & \text{for } g_1^{-1}(p_k) \in \boldsymbol{R}_+, \\ 0 & \text{for } g_2^{-1}(p_k) \in \boldsymbol{R}\backslash\boldsymbol{R}_+, \end{cases} \tag{21}$$

where $\operatorname{jac} g_1^{-1}$ is the Jacobian transformation of g_1^{-1} and $jac\, g_2^{-1}$ is the Jacobian transformation of g_2^{-1}. It follows from (20) and (21) that the distribution of $A(k)$ fulfils the following equation:

$$f_{A(k)}(p_k) = \begin{cases} f_{E'(k)}(p_k) + f_{E'(k)}(-p_k) & \text{for } p_k \geq 0, \\ 0 & \text{for } p_k < 0. \end{cases} \tag{22}$$

Therefore from (6) and (8) one easily obtains that

$$f_{A(k)}(p_k) = \begin{cases} \frac{\hbar^2 k}{\sqrt{2\pi}\sigma_m p_k^2} \left(\exp\left(-\frac{(\frac{\hbar^2 k}{p_k} - M_0)^2}{2\sigma_m^2}\right) + \exp\left(-\frac{(\frac{\hbar^2 k}{-p_k} - M_0)^2}{2\sigma_m^2}\right)\right) & \text{for } p_k > 0, \\ 0 & \text{for } p_k \leq 0, \end{cases} \tag{23}$$

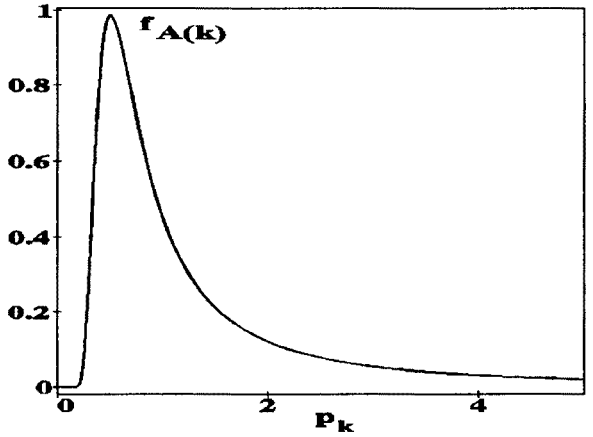

Fig. 2. The probability density function of the absolute value of the first derivative of energy $f_{A(k)}$.

for every $k \in E_B$. We present the plot of the distribution $f_{A(k)}$ in Figure 2, putting $M_0 = 1, \sigma_m = 1, k = 1$ and $\hbar = 1$.

We derive the distribution of $B(k)$. It follows from (6) and (8) that

$$B(k) = \frac{k}{2} A(k) .$$

Thence, from the change of the variable formula [12] one easily obtains that

$$f_{B(k)}(q_k) = \begin{cases} \frac{\hbar^2 k^2}{2\sqrt{2\pi}\sigma_m q_k^2} \left(\exp\left(-\frac{\left(\frac{\hbar^2 k^2}{2q_k} - M_0\right)^2}{2\sigma_m^2}\right) + \exp\left(-\frac{\left(\frac{\hbar^2 k^2}{-2q_k} - M_0\right)^2}{2\sigma_m^2}\right)\right) & \text{for } q_k > 0, \\ 0 & \text{for } q_k \leq 0, \end{cases} \tag{24}$$

for every $k \in E_B$.

4. Integrable model

Let us assume a pure integrable one dimensional quantum system and derive the distribution of $s(k)$. The system is described by the Schrödinger equation with the random effective mass m^* (compare (4)). The energy $E(k)$ of a particle in an energy band is a continuous function E of wave vector k

$$E : E_B \ni k \to E(k) = \frac{\hbar^2 k^2}{2\, m^*} \in E_R , \tag{25}$$

where E_B is its continuous (non discrete) domain set, E_R is its continuous range set (compare (5)). In order to delete the double degeneration of the energy $E(k)$ with respect to k we restrict its domain E_B to the set of nonnegative real numbers:

$$E_B = \{k : k \geq 0\}\,.$$

We assume that the random effective mass m^* is always positive. Therefore energy levels $E(k)$ will be ordered increasingly and we could introduce the continuous analogue of spacing. The absolute value $B(k)$ of $E(k)$ is given by:

$$B : E_B \ni k \to B(k) = |E(k)| = E(k)\,. \tag{26}$$

The function B is monotonously increasing and the absolute values of energy levels $B(k)$ are ordered into an ascending sequence. Now, we are ready to introduce the continuous analogue $A(k)$ of spacing. We compute the first derivative of the function E with respect to k:

$$E'(k) = \frac{dE(k)}{dk} = \frac{\hbar^2 k}{m^*}\,. \tag{27}$$

Since the random effective mass is always positive, the absolute value $A(k)$ of first derivative $E'(k)$ is equal to the first derivative of B and $E(k)$:

$$A(k) = |E'(k)| = \frac{dB(k)}{dk} = E'(k) = \frac{\hbar^2 k}{m^*}\,. \tag{28}$$

Let us assume that m^* is uniformly distributed in the interval $[a, b]$, (where $0 < a < b$):

$$m^* : M \ni m \to m^*(m) \in \boldsymbol{R}\,, \tag{29}$$

where M is the sample space and m is the sample point. This uniform distribution is an analogue to the quantum integrable system of discrete energy levels. We have assumed domain of the random mass m^* to be the interval $[a, b]$, which corresponds to positive and finite effective masses, where a and b are model parameters. The model on the finite interval is unique one with normalizable uniform distribution.

The probability density function g_{m^*} is given by the following formula

$$g_{m^*}(m) = \begin{cases} \frac{1}{b-a} & \text{for} \quad m \in [a, b]\,, \\ 0 & \text{for} \quad m \in \boldsymbol{R}\backslash[a, b]\,. \end{cases} \tag{30}$$

Its inverse $X = 1/m^*$ is a random variable

$$X : X_D \ni x \to X(x) \in \boldsymbol{R}\,, \tag{31}$$

where X_D is the sample space and x is the sample point. From the change of the variable formula [12] one easily obtains that the inverse X has the following distribution

$$g_X(x) = \begin{cases} \frac{1}{(b-a)x^2} & \text{for} \quad m \in [\frac{1}{b}, \frac{1}{a}], \\ 0 & \text{for} \quad m \in \boldsymbol{R} \backslash [\frac{1}{b}, \frac{1}{a}]. \end{cases} \tag{32}$$

It follows from (25), (32), and from the change of the variable formula [12] that the distribution of the random variable $E(k)$ is

$$g_{E(k)}(\omega_k) = \begin{cases} \frac{\hbar^2 k^2}{2(b-a)\omega_k^2} & \text{for} \quad \omega_k \in [\frac{\hbar^2 k^2}{2b}, \frac{\hbar^2 k^2}{2a}], \\ 0 & \text{for} \quad \omega_k \in \boldsymbol{R} \backslash [\frac{\hbar^2 k^2}{2b}, \frac{\hbar^2 k^2}{2a}], \end{cases} \tag{33}$$

for every $k \in E_B$.

The absolute value of energy $B(k)$ has the same distribution as $E(k)$

$$g_{B(k)}(\omega_k) = g_{E(k)}(\omega_k),$$

for every $k \in E_B$.

The first derivative $E'(k)$ is also the random variable

$$E'(k) : O \ni o_k \to E'(k)(o_k) \in \boldsymbol{R},$$

for every $k \in E_B$, where o_k is the sample point and O is the sample space. Hence we define the random function $\bar{E}'$

$$\bar{E}' : E_B \times O \ni (k, \bar{o}) \to \bar{E}(k, \bar{o}) = E(k)(\bar{o}) \in \boldsymbol{R},$$

where $\bar{o}$ is the sample point, and the map

$$O \ni \bar{o} \to \bar{E}(k, \bar{o}) \in \boldsymbol{R}$$

is the random variable $E'(k)$ for every $k \in E_B$. From (27), (32) and from the change of the variable formula [12] one easily obtains that its distribution is

$$g_{E'(k)}(o_k) = \begin{cases} \frac{\hbar^2 k}{(b-a)o_k^2} & \text{for} \quad o_k \in [\frac{\hbar^2 k}{b}, \frac{\hbar^2 k}{a}], \\ 0 & \text{for} \quad o_k \in \boldsymbol{R} \backslash [\frac{\hbar^2 k}{b}, \frac{\hbar^2 k}{a}], \end{cases} \tag{34}$$

for every $k \in E_B$. We must point out that in (34) the wave vector k is the independent nonnegative variable from the domain E_B and the distribution $g_{E'(k)}$ is a function of k, because the random variables $E'(k)$ depend on the

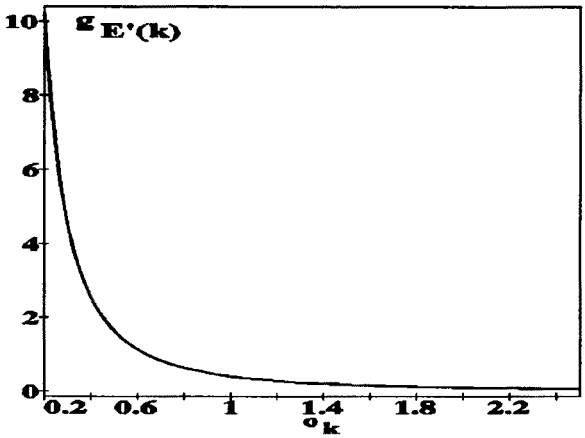

Fig. 3. The probability density function of the first derivative of energy $g_{E'(k)}$.

choice of k. We present the plot of the probability density function of $E'(k)$ in Figure 3, putting $a = 0.1, b = 5, k = 1$ and $\hbar = 1$.

The absolute value (or the length) $A(k)$ of the first derivative $E'(k)$ has the same distribution as $E'(k)$:

$$g_{A(k)}(o_k) = g_{E'(k)}(o_k)\,, \tag{35}$$

for every $k \in E_B$.

5. Discussion

One can extend the Schrödinger equation (4) by adding potential energy of the external forces. If this splits energy spectrum into continuous bands, then the statistics of every band will lead to the results similar to the result obtained already. For example, a spectrum of the three dimensional particle in external homogeneous magnetic field H reads:

$$E_n(k_z) = \frac{\hbar^2 k_z^2}{2m^*} + \frac{e\hbar}{2m^* c} H(2n+1)\,, \tag{36}$$

where the quantum number n labels energy bands. For fixed n

$$\frac{dE_n(k_z)}{dk_z} = \frac{\hbar^2 k_z}{m^*}\,. \tag{37}$$

This is identical to (7) and leads to the same results.

The first derivative $E'(k)$ is the group velocity $v_g(k)$ of excitation. Hence, the distribution of the group velocity corresponds to the derived

results (17), (34). We encourage the experimentalists to measure its distribution by the positron annihilation technique, which would be a confirmation of existence of quantum chaos in solid states.

Now we interpret the obtained results. For the discrete quantum systems the following observations have been made: if the discrete system is quantum integrable, then the Hamiltonian eigenvalues are randomly distributed and are uncorrelated (their distribution is uniform). In this case the population of small spacings is dominant and the eigenvalues do not repel each other. The distribution of the nearest neighbour spacings between the eigenvalues is the Poisson one

$$P_P(s) = \begin{cases} \frac{1}{D}\exp(-\frac{s}{D}) & \text{for} \quad s \geq 0 \\ 0 & \text{for} \quad s < 0\,. \end{cases} \tag{38}$$

We present the plot of the Poisson probability density function P_P in Figure 4, putting $D = 1$. The spacing equal to zero is the most probable.

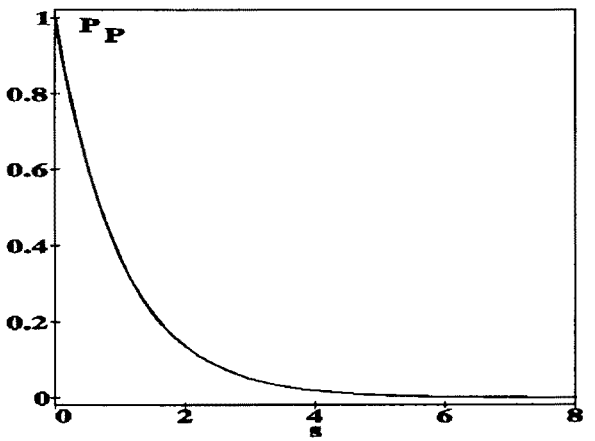

Fig. 4. The Poisson probability density function of the spacing P_P.

Wigner in his works [13–16] and also Landau and Smorodinsky [17] assumed a statistical hypothesis for the many body Hamiltonian to explain observed nuclear spectra. In an ascending sequence of energy levels of the many level quantum chaotic system one can compute the nearest neighbour spacing and obtain a shortage of small spacings. To explain this phenomenon Wigner assumed the GOE(2) (the Gaussian Orthogonal Ensemble of dimension 2) and thus the Hamiltonian matrix elements were independent random variables. It led him to the conclusion that the distribution of the nearest neighbour spacings between the eigenvalues of the Hamiltonian

matrix is

$$P_W(s) = \begin{cases} \frac{\pi s}{2D^2} \exp(-\frac{\pi s^2}{4D^2}) & \text{for} \quad s \geq 0\,, \\ 0 & \text{for} \quad s < 0\,, \end{cases} \tag{39}$$

where s is the spacing and D is the average spacing throughout the sequence. We present the plot of the Wigner probability density function P_W in Figure 5, putting $D = 1$. Thus the effect of repulsion of the eigenvalues was appropriately explained. The energy levels repel each other and they are correlated. The population of small spacings equals nearly zero and the maximum of the Wigner distribution is at positive s. The spacing equal to zero is improbable.

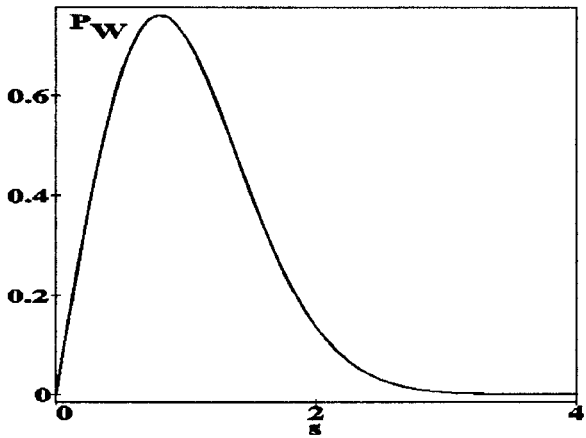

Fig. 5. The Wigner probability density function of the spacing P_W.

For the discrete quantum integrable system the Poisson distribution predicts that the following six equivalent conditions are the most probable:

1. The discrete function $f(i) = E_i$ (compare section Introduction) is constant for some discrete set of its indices $i_1, \ldots, i_N : E_{i_1} = \ldots = E_{i_N} = \bar{E}$, where the energy $\bar{E}$ is degenerated with respect to its index.
2. The inversion of the function f cannot be made for the energy $\bar{E}$, the inverse $i = i(E) = f^{-1}(E)$ is not a function at $\bar{E}$ (it is multivalued function or relation).
3. The discrete density of states $\rho(E_i) = \frac{\Delta^1 i}{\Delta^1 E_i} = \frac{1}{s_i}$ assumes infinite values at $\bar{E} : |\rho(\bar{E})| = +\infty$ (because $\Delta^1 E_i = E_{i+1} - E_i = 0$ for $i = i_1, \ldots, i_{N-1}$).
4. The spacing is equal to zero at $(N-1)$ points: $s_{i_1} = \ldots = s_{i_{N-1}} = 0$.
5. The cumulative distribution function $n = n(E_i) = \sum_{E \leq E_i} 1$ has a positive jump greater than one at $\bar{E} : \Delta n(\bar{E}) = n(\bar{E}+) - n(\bar{E}-) > 1$.

6. The energies tend to "cluster" in the domain of indices.

The Poisson distribution means that the fulfillment of above six conditions is the most probable.

For the discrete quantum chaotic system the Wigner distribution predicts, on the contrary, the improbability of the fulfillment of the above six conditions. Therefore their contraries hold:

1. The discrete function $f(i) = E_i$ is not constant for any discrete set of its indices and the energy E_i is not degenerate with respect to its index.
2. The inversion of the function f can be made for every energy.
3. The discrete density of states $\rho(E_i)$ assumes finite values at every energy.
4. The spacing is not equal to zero.
5. The cumulative distribution function $n = n(E_i)$ has no jumps greater than one.
6. The energies tend to repel each other.

For the continuous quantum system the two models have been presented (compare Section 3 and Section 4). The distribution $f_{E'(k)}$ (17) is a continuous analogue to the Wigner distribution P_W (39) (compare Figures 2 and 5), while the distribution $g_{E'(k)}$ (34) is a continuous analogue to the Poisson distribution P_P (38) (compare Figures 3 and 4). Both distributions $f_{E'(k)}$ and $g_{E'(k)}$ deal with the continuous analogy of spacing: the first derivative $s(k) = E'(k)$.

Let us study the continuous quantum integrable system. The distribution of $E'(k)$ assumes its maximum value at the origin of its domain (compare (34) and Figure 3). Also the Poisson distribution assumes its maximum value at the origin of its domain (compare (38) and Figure 4). If we assume that the upper band b of the possible values of random effective mass m^* tends to infinity: $b \to +\infty$, then the zero value of $E'(k)$ is the most probable (see (34)). This fact is analogous to the above description of the discrete quantum integrable system. The distribution $g_{E'(k)}$ (34) predicts that the fulfillment of the following six equivalent conditions is the most probable:

1. The continuous function $E(k) = E_k$ (compare the Introduction) is constant for some compact set of its continuous indices $k_b \leq k \leq k_t$: $E(k_b) = \ldots = E(k) = \ldots = E(k_t) = \bar{E}$, where the energy $\bar{E}$ is degenerate with respect to its continuous index k.
2. The inversion of the function E cannot be made for the energy $\bar{E}$, the inverse $k = k(E)$ is not a function at $\bar{E}$ (it is multivalued function or relation).

3. The continuous density of states

$$\rho(E(k)) = \frac{dk}{dE(k)} = \frac{1}{s(k)}$$

assumes infinite values at $\bar{E}$: $|\rho(\bar{E})| = +\infty$ (because $E'(k) = 0$).
4. The first derivative is equal to zero at the continuous set of k: $E'(k_b) = \ldots = E'(k) = \ldots = E'(k_t) = 0$.
5. The cumulative distribution function $n = n(E(k)) = \int_{E \leq E(k)} dk$ has a positive jump at $\bar{E}$: $\Delta n(\bar{E}) = n(\bar{E}+) - n(\bar{E}-) > 0$.
6. The energies tend to "cluster" in the domain of k.

The distribution $g_{E'(k)}$ fulfils the above six conditions like the Poisson distribution fulfils its six ones.

Let us study the continuous quantum chaotic system. The zero value of $E'(k)$ is improbable (see (17)). This fact is analogous to the above description of the discrete quantum chaotic system. The distribution $f_{E'(k)}$ (17) predicts the improbability of the fulfillment of the above six conditions. Therefore their contraries hold:

1. The continuous function $E(k) = E_k$ is not constant at its domain and the energy $E(k)$ is not degenerate with respect to its continuous index k.
2. The inversion of the function E can be made for every energy.
3. The continuous density of states $\rho(E(k))$ assumes finite values at every energy.
4. The first derivative is not equal to zero.
5. The cumulative distribution function $n = n(E(k))$ has no jumps.
6. The energies tend to repel each other in the domain of k.

The distribution $f_{E'(k)}$ fulfils the above six conditions like the Wigner distribution fulfils its six ones.

Having drawn these analogies we state that the continuous quantum systems in which the first derivatives $E'(k)$ are governed by $f_{E'(k)}$ are chaotic and the continuous quantum systems in which the first derivatives $E'(k)$ are governed by $g_{E'(k)}$ are integrable. We call former systems CCE — Continuous Chaotic Ensembles, and the latter CIE — Continuous Integrable Ensembles.

To end we study the distribution of the jumps $\Delta n(\bar{E})$. For the discrete systems the jump is:

$$\Delta n(\bar{E}) = \sum_{i(E_b) \leq i \leq i(E_t)} 1 . \tag{40}$$

Therefore for the continuous systems the jump is:

$$\Delta n(\bar{E}) = \int\limits_{k(E_b)\leq k\leq k(E_t)} d\,k\,. \tag{41}$$

Since the energy is a random process (compare (5), (11), (29)), therefore the jump is also a random process. The distribution of the jump for CCE is

$$f_{\Delta n(\bar{E})}(w) = \begin{cases} \dfrac{\hbar^2 w}{\sqrt{2\pi}\sigma_m(\sqrt{E_t}-\sqrt{E_b})^2} & \\ \times \exp\left(-\dfrac{\left(\frac{\hbar^2 w^2}{2(\sqrt{E_t}-\sqrt{E_b})^2}-M_0\right)^2}{2\sigma_m^2}\right) & \text{for} \quad w \geq 0\,, \\ 0 & \text{for} \quad w < 0\,, \end{cases} \tag{42}$$

and for CIE is

$$g_{\Delta n(\bar{E})}(z) = \begin{cases} \dfrac{\hbar^2 z}{(b-a)(\sqrt{E_t}-\sqrt{E_b})^2} & \text{for } z \in [\frac{\sqrt{2a(E_t-E_b)}}{\hbar}, \frac{\sqrt{2b(E_t-E_b)}}{\hbar}]\,, \\ 0 & \text{for } z \in \boldsymbol{R}\backslash[\frac{\sqrt{2a(E_t-E_b)}}{\hbar}, \frac{\sqrt{2b(E_t-E_b)}}{\hbar}]\,, \end{cases} \tag{43}$$

(compare (12), (30), (41) and the change of the variable formula [12]. In obtaining (42) we perform the steps analogue to those which were performed in obtaining (24)).

We find the distribution of $E'(k)$ to be the proper tool for description of quantum chaos in continuous spectrum.

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