LOCAL MULTIFRACTAL PROPERTIES OF A SPATIALLY EXTENDED CHAOTIC SYSTEM THROUGH TIME DELAY COORDINATE RECONSTRUCTION

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The MST algorithm is applied to calculate generalized dimensions and singularity spectra of attractors of the Bloch magnetic domain wall — a spatially extended nonlinear dynamical system. It is shown that the states of the Bloch wall are multifractal and that the time delay coordinate reconstruction enables access to the local multifractal properties of the system. Depending on the size of the system, for some states of the wall, the multifractal properties are uniform throughout the wall. For others, the multifractal properties are different at different points in the wall. In particular, the spatial distribution of the correlation dimension found earlier by means of the Grassberger-Proccacia algorithm is repeated in the spatial distribution of the generalised dimensions.

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1. Introduction and theory

Two tools often used for the analysis of finite dimensional chaotic systems described by ordinary differential equations or maps are: generalized dimensions D(q) and the thermodynamic formalism of singularity spectra $f(\alpha)$ [1, 2]. They have been successfully calculated for many such systems, for instance for the Lorenz attractor [3] or for the Hénon map [4, 5]. The question open now is whether these methods may be applied for the much more complicated spatially extended chaotic systems which possess an infinite number of degrees of freedom.

Usually, the analysis of a one dimensional signal from experimental data starts with the delay time reconstruction of the finite dimensional phase space of the system — a method proposed by Takens [6]. For a spatially extended system, one can obtain a whole set of sampled signals from different locations in the real space occupied by the system and does not have to use the Takens method. All these signals considered together form a high but finite dimensional phase space — an experimentalist's approximation to the infinite dimensional phase space of the system. However, calculations of the generalised dimensions and singularity spectra performed in this high dimensional phase space give us only the spatial average values for the whole attractor [7]. The local behaviour of the system is often more interesting and this requires calculations for the single time series from a given point in the real space.

Dimension densities [8–10] have been used in the past to study the effect of the decay of the effect of the dynamics at one point in the system on the dynamics at a remotely situated point in the system. This technique has been shown to work well at least for the class of systems for which it was designed, *i.e.* such that the mean amplitude dynamics does not depend on position [10]. Calculation of dimension densities [10] as well as the results of Grassberger [11] show that the effect of the local dynamics on the dynamics at other points on the system does decay with the distance — often rather rapidly.

With this effect in mind, we propose the use of the concept of a "local phase space" as applied in [12], *i.e.* all degrees of freedom needed to describe the local behaviour of the small area of a system near a given point in the real space. The effect of the nonlinearity in a spatially extended system is that neighbouring parts of the system interact together mutually influencing their behaviour. Due to this, the dimension of the local phase space may be greater than the number of local variables at the given point in the system, the additional dimensions describing the nonlinear interaction. In order to reconstruct the local phase space we applied the well-known Takens method [6] to a one dimensional signal from a given point in the real space.

As will be shown below, applying the time-delay embedding to selected local variables of the spatially extended system yields interesting and nontrivial results for the system we used as a model: the Bloch domain wall. We do not possess a rigorous mathematical proof of the general validity of our procedure. The generalization of the Takens theorem to spatially extended systems is still the matter of future investigations for mathematicians. At this point we simply aim to manifest an effect that is clearly seen in numerical calculations by the methods used in the present paper and in the results of some previous reports obtained by other methods [13–15].

2. Application

We chose the strange attractors of the Bloch domain walls of the uniaxial thin films [12–14] to study the possibility of calculating the local values of dimensions $D(q)$ and $f(\alpha)$ curves for a spatially extended system. Recently published results [13] regarding the spatial behaviour of the correlation dimension in the local phase space of this system indicated a possible success of such calculations.

The Bloch domain wall in a thin magnetic film [12–14] is a transition zone between two magnetic domains bounded by an interdomain distance Δ and the thickness of the magnetic material h (see Fig. 1). At a given point along the Bloch wall there are two variables describing the system: $q(z,t)$ is the local position of the wall and $\phi(z,t)$ is the angle between the magnetization vector and the plane of the wall. The z axis lies in the plane of the wall and is parallel to the easy axis of the anisotropy — here perpendicular to the surface of the magnetic material. The equations of motion of the wall consist of two nonlinear partial differential equations yielding only numerical solutions in the interval of parameters considered here [12–14]:

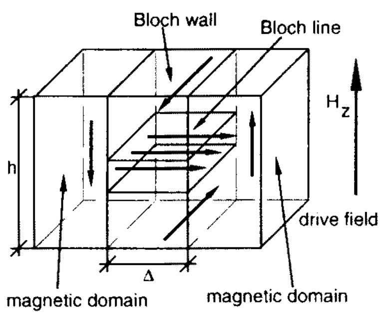

Fig. 1. Bloch wall between two domains in a thin magnetic film with uniaxial perpendicular anisotropy.

$$\dot{q} = \gamma\Delta\left[2\pi M \sin 2\phi - \frac{2A}{M}\frac{\partial^2\phi}{\partial z^2}\right] + \alpha\Delta\dot{\phi}\,,$$

$$\dot{\phi} = \gamma\left[H_z + \frac{2A}{\Delta M}\frac{\partial^2 q}{\partial z^2}\right] - \alpha\frac{\dot{q}}{\Delta}\,,$$

where the material parameters are: A is the exchange constant, γ is the gyromagnetic ratio, $4\pi M$ is the saturation magnetization, α is the Gilbert damping constant and H_z is the constant drive field. Above a critical value of the external magnetic drive field H_w (the Walker critical field) chaotic states emerge. The Bloch wall divides itself into parts separated by the so called Bloch lines (solitary waves like kinks) moving along the surface of the wall in opposite directions. The numerical method of calculation of the solutions of these differential equations describing the system was a version of the well known DuFort-Frenkel method rewritten in vector form [12]. The material parameters chosen were the same as in [12].

We analysed Bloch walls 3.5 μm and 5.0 μm in height. The values of the variables q and ϕ were calculated at a time interval of 1 ns at a finite number of points along the wall (52 points for the 3.5 μm wall and 88 points for the 5.0 μm wall). For the 3.5 μm wall height, the values of the variable ϕ at 15 grid points 1, 4, 8, 12, 16, 20, 24, 26, 28, 32, 36, 40, 44, 48 and 52 (from the top surface to the bottom of the film) were used to reconstruct the local phase spaces of dimension 5 at these points with a time delay of 3 ns. For the 5.0 μm wall height 13 variables ϕ from grid points 1, 4, 16, 24, 28, 36, 44, 52, 60, 68, 76, 84 and 88 (from the surface to the base of the magnetic film) were taken. Local phase spaces of dimension 6 were reconstructed using a time delay of 3 ns. 15000 data points for each time series were used.

The particular choice of the fixed embedding dimensions 5 and 6, the time delay and the number of data points is justified by the results of [12] where the correlation dimension was obtained for the same states of the wall and the dependence of the dimension on the reconstruction parameters was considered. It was found there that the value of the correlation dimension saturates at the embedding dimension as low as 4 for all attractors considered. The value of the correlation dimension found in [13] for the 3.5 μm high wall was almost constant at 2.15 ± 0.1 throughout the wall height. For the 5.0 μm high wall the value of the correlation dimension in this distribution did not exceeded 2.6. Thus, in the present calculation, we fixed embedding dimensions for the two attractors considered here accordingly.

We performed calculations of the $D(q)$ curves by means of the recently proposed minimal spanning tree (MST) algorithm described in detail in [3, 5]. We chose this particular algorithm from among other approaches (*e.g.* density reconstruction [17]) as it promised valid results for a minimum number of points required (15000 were sufficient for the Lorenz attractor [3]) and to remain as close as possible to the mathematical definition of the generalized dimensions [3]. We cut-off the branches of the minimal spanning trees that were longer than 10 times and shorter than 0.01 of the average branch length in order to get rid of anomalous lengths distorting the partition function, as suggested in [3]. Note, that in [13] the data set contained 5000 points for the correlation dimension calculation, while in the present calculation 15000 data points per set was used with the MST algorithm.

The minimal spanning trees were calculated for the number of reference points varying from 0.4% to 40% of the number of points in the set. Because of the large number of data series to analyse we did not apply any automatic numerical procedures [18] to detect the scaling region inside the fixed 0.4%-40% interval. Instead, we discarded the results for which the linear regression method returned the correlation coefficient below the value fixed at 0.9. Since most of the results obtained had their correlation coefficient above that value, we inferred that the number of reference points was chosen correctly. For the given number of the minimal spanning tree nodes, the calculations of partition functions were averaged 10 times to improve the quality of the fit. In view of the fact that in the MST algorithm random reference points are chosen, the entire calculation was repeated for each data set six times to assess the spread and repeatability of the method.

The Legendre transform from the dimensions $D(q)$ to the singularity spectra $f(\alpha)$ was performed with the help of the cubic spline interpolation for the $D(q)$. For that interpolation the analytical first derivatives are known [16]. This released us from the tedious and often precarious numerical differentiation of the $D(q)$ curves.

3. Error estimates

The MST method requires randomly distributed reference points. To estimate the degree of influence of this randomness on the numerical results, we carried out the same calculations for the same data set six times. The uncertainties of the individual calculations returned by the linear regression method for the values of $D(q)$ for the variable q from interval $(-5, 5)$ were of order 0.1 %. This is evidence of wide and smooth scaling regions. However, the dispersion of the results obtained from the repetition of the calculation is much greater than these uncertainties. This in turn suggests that, at least in the case of the MST method, one must be cautious in order to obtain the true estimation of errors from a single calculation.

4. Results

Typical examples of the $D(q)$ curves obtained in this series of calculations are shown in Fig. 2 and Fig. 3 for $h = 3.5\ \mu$m and $h = 5.0\ \mu$m, respectively. The linear regression errors on $D(q)$ curves grow rapidly outside the interval of small values of q (approximately $(-5, 5)$) indicating the loss of scaling properties of the distribution of points. On some of the $D(q)$ curves we noticed a nonmonotonicity in the neighbourhood of point $q = 2$. The same nonmonotonicity of $D(q)$ curves for the Lorenz attractor was reported in [3]. The authors explain this as a peculiar property of the specified

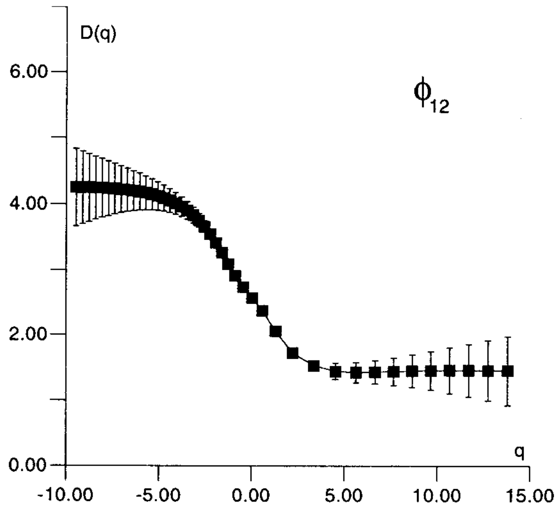

Fig. 2. $D(q)$ curve from signal $\phi(t)$ at point no.12 in 3.5 μm Bloch wall.

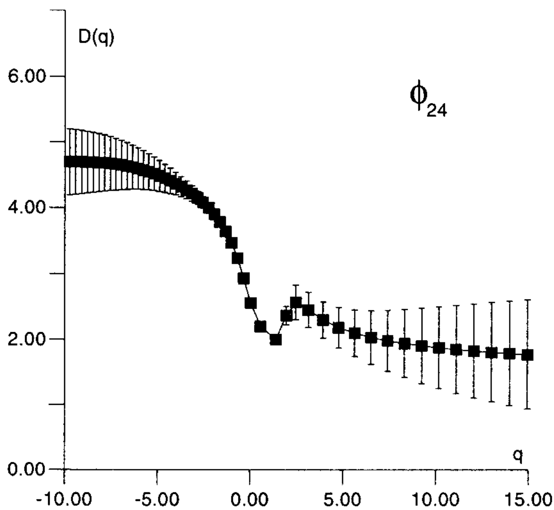

Fig. 3. $D(q)$ curve from signal $\phi(t)$ at point no.24 in 5.0 μm Bloch wall.

attractor. In the light of our results, we consider it a general property of at least the MST method for calculating generalized dimensions.

Typical examples of $f(\alpha)$ curves are shown in Figs 4 and 5 for the attractor at $h = 3.5$ μm and $h = 5.0$ μm respectively. The same loss of scaling properties as in the $D(q)$ curves for large q values was observed. With the rapidly growing linear regression errors at their ends, they acquired unphysical, divergent "whiskers" we had to cut off manually. The dispersion of individual results over and above the intervals of the very small linear regression error was especially visible in the case of the $f(\alpha)$ curves. We believe this to be the effect of the random choice of reference points as made in the present work and as prescribed by the authors of the method [5]. It can be seen that the large dispersion is obtained mainly for the right-hand branches of the $f(\alpha)$ curves. This unfortunately seems to be a common property of many other methods also.

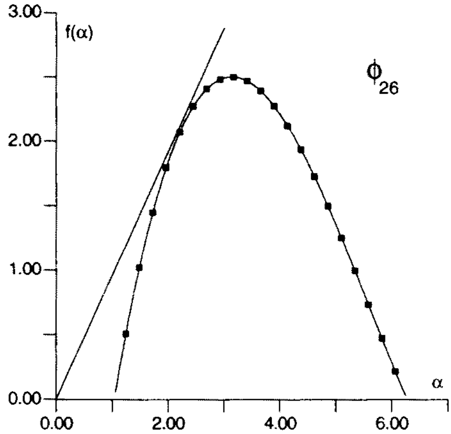

Fig. 4. $f(\alpha)$ curve from signal $\phi(t)$ at point no.26 in 3.5 μm Bloch wall.

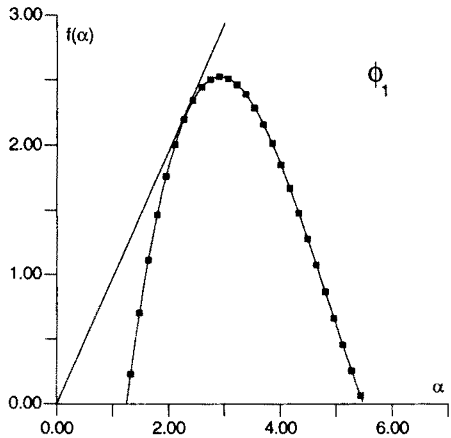

Fig. 5. $f(\alpha)$ curve from signal $\phi(t)$ at point no.1 in 5.0 μm Bloch wall.

For the same set of material parameters as used in the present discussion, strong indications of spatial correlations were obtained, for the case of the Bloch wall 5.0 μm in height, by means of the correlation dimension (Grassberger-Proccacia algorithm) [13] and with a nonlinear correlation coefficient [15]. In contrast, the dynamics in the case of the Bloch wall 3.5 μm in height was found to be much more spatially uniform [13, 15]. Fig. 6 and

Fig. 7 depicts the spatial distributions of $D(0)$, $D(1)$ and $D(2)$ as found by means of the MST method for the 5.0 μm wall height and for the 3.5 μm wall height, respectively. In the latter case the change of fractal dimension along the wall did not exceed 0.25 which is not very much larger than the error (Fig. 7). Also the value of all fractal dimensions is larger at grid point number 1 than at the other end of the wall — at the opposite surface of the film. In [13] the value of the correlation dimension (equivalent of $D(2)$ here) obtained from a slightly modified Grassberger-Proccacia algorithm was 2.25 at grid point 1 and decreased practically monotonically to 2.05 at grid point 52 (Fig. 9). Similarly as in the MST calculations presented here there was a certain amount of scatter in the correlation dimension at the midplane of the wall (points marked by $\times$ in Fig. 9) due to a secondary scaling region.

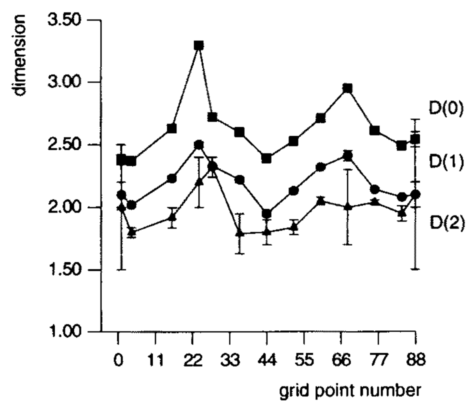

Fig. 6. Spatial dependence of the dimensions $D(0)$, $D(1)$ and $D(2)$ in the Bloch wall with height 5.0 μm. For the convenience of the eye we have connected the points with straight lines.

On the other hand, the spatial distribution of the correlation dimension found in [13] for the 5.0 μm wall height (Fig. 8) strongly resembles that found for $D(0)$, $D(1)$ and $D(2)$ (Fig. 6). Except for $D(0)$ both fractal dimensions $D(1)$ and $D(2)$ show a certain increase of value at grid point number 1. There is a maximum of all three fractal dimensions at grid points 22 and 66, there is a minimum at the midplane and there is an increase of the value of all three fractal dimensions at the opposite end of the wall (grid point 88).

It can be seen that the results presented here reflect, at least qualitatively, the main features of the spatial distribution of the correlation dimension as found in [13] by a complete different numerical method: for the 3.5 μm case, the dynamics is almost uniform throughout the system while for the 5.0 μm case all dimensions calculated here have a spatial distribution of a definite symmetry. Note, however, that the values of correlation dimension distribution found in [13] are somewhat larger than the values of $D(2)$

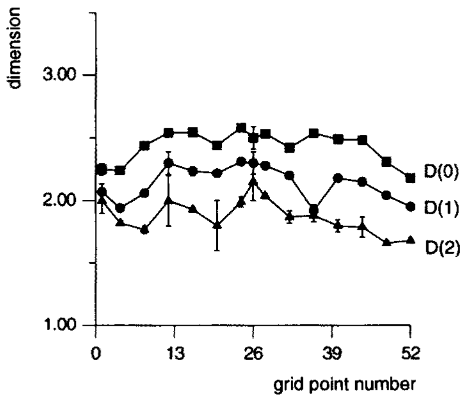

Fig. 7. Spatial dependence of the dimensions $D(0)$, $D(1)$ and $D(2)$ in the Bloch wall with height 3.5 μm. For the convenience of the eye we have connected the points with straight lines.

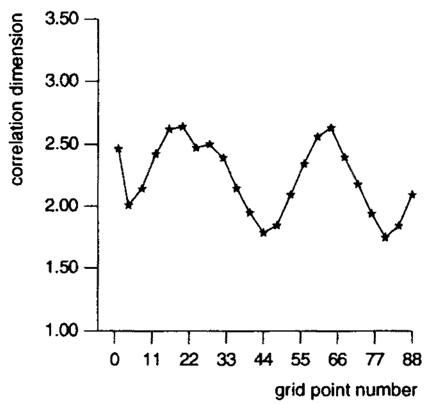

Fig. 8. Spatial dependence of the correlation dimension in the Bloch wall with height 5.0 μm as found in ref.[13].

found here due to a difference in the definition of fractal dimensions used in the MST algorithm and in the Grassberger-Proccacia algorithm (see [3]).

In [15] it was shown that the symmetry found for the attractor at wall height 5.0 μm is associated with the existence of a specific state of spatial intermittency in which chaotic domains of similar chaotic dynamics situated along the wall height are interlaced with domains of different chaotic dynamics. It was shown there also that the domains of similar dynamics, remotely separated in space, show strong correlation with each other. The fact that a third method — the MST algorithm, a completely different method of fractal analysis, shows the same features of the $D(q)$ and $f(\alpha)$ spatial distributions as found in [13] and [15] further strengthens the case

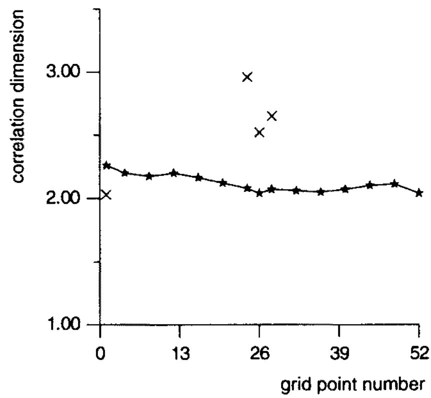

Fig. 9. Spatial dependence of the correlation dimension in the Bloch wall with height 3.5 μm as found in ref.[13].

for the new kind of spatial intermittency as defined in [15].

Finally, a two-point correlation dimension similar to the one applied by Kurz and Meyer-Kress [9, 10] to calculate dimension densities was also used in [15]. This quantity was not calculated per unit length of the system (*i.e.* as a density). Instead, a spatial distribution of the two-point correlation dimension was obtained. The shape of this distribution confirms the spatial intermittency found by means of the nonlinear coefficient and manifesting itself in the data presented here in the spatial distribution of the fractal dimensions. The basic difference between the results obtained from the two-point correlation dimension and those found through the local phase space here and in [13, 15] is that the two-point correlation dimension changes much more rapidly with the distance indicating a very fast spatial decay of correlations in the system. As seen in Fig. 6 and Fig. 8 (as well as by the behaviour of the nonlinear coefficient in [15]) the correlations between remotely situated areas of the system seems to be rather long range.

5. Summary

The concept of the local phase space proposed here and in [13–15] serves as a good model for the transfer to spatially extended nonlinear systems of some of the tools for time series analysis developed earlier for chaotic systems with finite degrees of freedom. In the case of the Bloch domain wall, we were able to demonstrate its local multifractality and verify the spatial correlations in the system. The calculations performed here show all the features of the new kind of spatial intermittency found earlier in [12].

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