# EXACTLY INTEGRABLE MODEL OF RELATIVISTIC N-BODY SYSTEM IN THE TWO-DIMENSIONAL VARIANT OF THE FRONT FORM OF DYNAMICS

V. Shpytko

Institute for Condensed Matter Physics
of the Ukrainian National Academy of Sciences,
Svientsitskyj Street 1, Lviv, 290011, Ukraine
E-mail: gaida@omega.icmp.lviv.ua

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Relativistic N-particle exactly integrable system with oscillator-like interactions in the two-dimensional space-time is considered within the framework of the front form of dynamics. Using the Weyl quantization rule we obtain eigenstates and eigenvalues of the mass-squared operator.

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## 1. Introduction

Among various forms of relativistic dynamics [1, 2] the front form [3] in the two-dimensional space-time $\mathbb{M}_2$ [4, 5] takes a special place. Its distinctive feature consists in the fact that the Poincaré transformations are automorphisms of the set of simultaneity surfaces. As a result the Poincaré-invariance conditions [2] allow the existence of the N-particle interaction Lagrangians depending only on the first order derivatives of the particle coordinates [5]. This means that the front form in two-dimensional space-time $\mathbb{M}_2$ is an exception of the no-interaction theorem [6]. Therefore covariant particle coordinates can be used as canonical variables and the transition to the Hamiltonian description may be performed by usual Legendre transformation. This allows us to construct rather simple models with exact solutions in classical [6] as well as in quantum [7] cases.

The aim of this article is an application of the Weyl quantization rule [8] in the front form of dynamics in the two-dimensional space-time [7, 9] in the special case of an N-particle system with oscillator-like interactions. A similar analysis has been carried out in Refs. [7, 9], and [10] for the two- and three-particle cases, respectively. This model constitutes one of possible relativistic generalization of nonrelativistic N-particle system with oscillator interaction. The symmetry group in our case is the Poincaré group $\mathcal{P}(1,1)$. This leads to three conserved quantities: energy, momentum, and boost integral. The model considered here possesses additional integrals of motion in involution. They are responsible for the exact integrability of the system. But the structure of these conserved quantities is quite different than in the nonrelativistic case. The reason is the unusual kinematical part of the Hamiltonian and the momentum dependence of the interaction term which is caused by the Poincaré-invariance condition.

Systems with oscillator-like relativistic interactions are of interest for a variety of reasons. Such models are as a rule exactly solvable in the classical case as well as in the quantum case. They may describe phenomenological aspects of the inner structure of mesons and baryons [3, 11]. Besides, these models may be useful for the verification of different approximation methods, and may be considered as a some approximation of more realistic models. It appears to be significant to explanation of the relativistic effects in the well-established nonrelativistic oscillator-like quark models of hadrons. A number of relativistic models was considered in various formalisms and approaches in the classical case [12] and in the quantum case [13] as well.

Using the Weyl quantization rule (Sec. 2) and Jacobi type inner variables (Sec. 3) we solve the eigenvalue problem for the total mass operator of the N-particle system with oscillator-like interactions and obtain its eigenvalues and eigenfunctions (Sec. 4).

## 2. Weyl quantization in the two-dimensional front form of dynamics

In the two-dimensional space-time $\mathbb{M}_2$ with coordinates $(x^0, x)$ the front form of dynamics corresponds to the foliation of $\mathbb{M}_2$ by isotropic hyperplanes $x^0 + x = \tau$, where $\tau \in \mathbb{R}$ is the evolution parameter [5] (here and henceforth $c = \hbar = 1$). The classical Hamiltonian description of the system of N structureless particles with masses $m_a$ $(a = \overline{1, N})$ leads in this case to the canonical realization of the Poincaré group $\mathcal{P}(1,1)$ in terms of canonical variables $x_a, p_a$ with generators $H, P, K$. They correspond to the energy, momentum, and boost integral. It is convenient to introduce the quantities $P_\pm = H \pm P$, for which the commutation relations of the Poincaré algebra $p(1,1)$ in terms of Poisson brackets are

$$\{P_+, P_-\} = 0, \qquad \{K, P_\pm\} = \pm P_\pm\,. \tag{2.1}$$

The structure of these quantities is determined by the formulae [5]

$$P_+ = \sum_{a=1}^{N} p_a, \qquad K = \sum_{a=1}^{N} x_a p_a , \tag{2.2}$$

$$P_- = \sum_{a=1}^{N} \frac{m_a^2}{p_a} + \frac{1}{P_+} V(rp_b, r_{1c}/r) . \tag{2.3}$$

Poincaré-invariant function $V$ depends on $2N - 1$ indicated arguments, where $r_{ac} = x_a - x_c; r = r_{12}; a, b = \overline{1, N}, c = \overline{2, N}$ . Generators (2.2), (2.3) determine the square of the mass function of the system

$$M^2 = P_+ P_- = P_+ \sum_{a=1}^{N} \frac{m_a^2}{p_a} + V(rp_b, r_{1c}/r) . \tag{2.4}$$

The function $V$ describes the particles interaction, and the first term in Eqs (2.3), (2.4) corresponds to the free-particle system. The special feature of the front of dynamics is the positiveness of the the momentum variables: $p_a > 0$ [3, 9].

As follows from (2.2)–(2.4), the quantization problem reduces to the construction of a Hermitian operator corresponding to the function $V$ in Eqs (2.3),(2.4). It determines the square of mass (inner energy) operator of the system

$$\hat{M}^2 = \hat{M}_f^2 + \hat{V} , \tag{2.5}$$

where $\hat{M}_f^2$ is the free particle part of the square mass operator. Operator (2.5) commutes with all operators which determine unitary realization of the group $\mathcal{P}(1, 1)$.

It is well known that there exists no unique path from the classical description to the quantum description. From the set of known paths for such a transition we choose the Weyl quantization rule [8]. According to this rule, operator $\hat{A}$, which corresponds to the classical function $a(x, p)$ on the 2N-dimensional phase space with coordinates $x_a, p_a$, has the form

$$\hat{A} = \int (ds)(dk)\tilde{a}(k, s) \exp\left( i \sum_{a=1}^{N} (k_a \hat{x}_a + s_a \hat{p}_a) \right) , \tag{2.6}$$

where $(dk) = \prod_{a=1}^{N} dk_a, (ds) = \prod_{a=1}^{N} ds_a$, $\tilde{a}$ is the Fourier transform of the function $a$

$$a(x, p) = \int (ds)(dk)\tilde{a}(k, s) \exp\left( i \sum_{a=1}^{N} (k_a x_a + s_a p_a) \right) , \tag{2.7}$$

and $\hat{x}_a, \hat{p}_a$ are Hermitian operators of coordinate and momentum of the a-th particle in some Hilbert space.

The wave functions $\psi(p) = \langle p|\psi\rangle$ describing the physical (normalized) states in the front form of dynamics constitute the Hilbert space $\mathcal{H}_N^F = \mathcal{L}^2(\mathbb{R}_+^N, d\mu_N^F)$ with inner product [3, 9]

$$(\psi_1, \psi) = \int d\mu_N^F(p)\psi_1^*(p)\psi(p)\,, \tag{2.8}$$

where

$$d\mu_N^F(p) = \prod_{a=1}^N \frac{dp_a}{2p_a}\Theta(p_a) \tag{2.9}$$

is Poincaré-invariant measure, and $\Theta(p_a)$ is the Heaviside function.

An application of the Weyl quantization rule to the classical functions (2.2), (2.3) in the case of the Hilbert space $\mathcal{H}_N^F$ leads to operators [9]

$$\hat{P}_+ = \sum_{a=1}^N p_a, \quad \hat{K} = i\sum_{a=1}^N p_a\partial/\partial p_a, \quad \hat{P}_- = \hat{M}^2/\hat{P}_+\,, \tag{2.10}$$

which are Hermitian with respect to the inner product (2.8). They determine the unitary realization of the group $\mathcal{P}(1.1)$ on the Hilbert space $\mathcal{H}_N^F$. Here $\hat{M}$ is determined by (2.5) where

$$\hat{M}_f^2 = \hat{P}_+ \sum_{a=1}^N \frac{m_a^2}{p_a}\,. \tag{2.11}$$

The operator $\hat{V}$ acts on the wave functions as integral operator

$$(\hat{V}\psi)(p) = \int d\mu_N^F(p)V(p,p')\psi(p') \tag{2.12}$$

with the kernel

$$V(p,p') = \left[\prod_{d=1}^N \sqrt{4p_d p'_d}\right]\delta(P_+ - P'_+)\int_{-\infty}^{\infty} V\left(r\frac{p_b + p'_b}{2}; \frac{r_{1c}}{r}\right) \times \exp\left[i\sum_{a=2}^N r_{1a}(p_a - p'_a)\right]\prod_{a=2}^N \frac{dr_{1a}}{2\pi}\,. \tag{2.13}$$

The general properties of the Weyl transformation [14] ensure that in the classical limit these operators correspond to the functions (2.2), (2.3).

The evolution of the quantum system is described in the front form of dynamics by Schrödinger-type equation

$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi \,, \tag{2.14}$$

where $\Psi \in \mathcal{H}_N^F$ and

$$\hat{H} = \tfrac{1}{2}(\hat{P}_+ + \hat{P}_-) = \tfrac{1}{2}(\hat{P}_+ + \hat{M}^2/\hat{P}_+) \,. \tag{2.15}$$

Putting $\Psi = \chi(t, P_+)\psi$, where $\psi$ is a function of some Poincaré-invariant inner variables, we obtain the stationary eigenvalue problem for the operator $\hat{M}^2$.

## 3. Inner variables

The separation of the motion of the system as a whole may be performed by choice of $P_+$ and $Q = K/P_+$ as new (external) variables. There exist many possibilities of the choice of inner variables. For the model considered below it is convenient to introduce the following inner variables

$$\eta_a = \frac{P_{a+} - p_{a+1}}{2P_{(a+1)+}}; q_a = P_{(a+1)+}(Q_a - x_{a+1}); P_+ = P_{N+}; Q = Q_N \,, \tag{3.1}$$

$$\{q_a, \eta_b\} = \delta_{ab}; \qquad \{Q, P_+\} = 1, \qquad a, b = \overline{1, N-1} \,, \tag{3.2}$$

where

$$P_{a+} = \sum_{i=1}^{a} p_i, \qquad Q_a = P_{a+}^{-1} \sum_{i=1}^{a} x_i p_i \,. \tag{3.3}$$

The Poisson brackets between other pairs are equal to zero. Inverse transformation of variables is determined by

$$p_a = P_+(\tfrac{1}{2} - \eta_{a-1}) \prod_{i=a}^{N} (\tfrac{1}{2} + \eta_i) \,, \tag{3.4}$$

$$x_a = Q + P_+^{-1} \left[ \sum_{i=a}^{N} \frac{(\frac{1}{2} - \eta_i)q_i}{\prod_{j=i+1}^{N}(\frac{1}{2} + \eta_j)} - \frac{(\frac{1}{2} + \eta_{a-1})q_{a-1}}{\prod_{j=a}^{N}(\frac{1}{2} + \eta_j)} \right] , \tag{3.5}$$

where we put $q_0 = q_N \equiv 1/2; \eta_N = -\eta_0 \equiv 1/2$. In the two-particle case variables (3.1) coincide with the variables proposed in Ref. [3].

Using variables (3.1) in Eq.(2.4), we obtain

$$M^2 = M_f^2(\eta) + F(q, \eta) \,, \tag{3.6}$$

where

$$M_f^2(\eta) = \sum_{a=1}^{N} \frac{m_a^2}{1/2 - \eta_{a-1}} \prod_{i=a}^{N} (1/2 + \eta_i)^{-1} \tag{3.7}$$

is the free-particle term and $F$ is the expression of the interaction function $V$ in terms of variables (3.1)

$$V(rp_b, r_{1c}/r) = F(q, \eta)\,. \tag{3.8}$$

Let us note that the positivity of the particle momentum variables $p_a > 0$ gives inequalities $|\eta_k| < 1/2, k = \overline{1, N-1}$. As a consequence of the equality

$$\prod_{a=1}^{N} \frac{dp_a}{p_a} = \frac{dP_+}{P_+} \prod_{k=1}^{N-1} \frac{d\eta_k}{1/4 - \eta_k^2} \tag{3.9}$$

the Hilbert space $\mathcal{H}_N^F$ decomposes into the tensor product $\mathcal{H}_N^F = h_{int} \otimes \mathcal{H}_{ext}$, where "inner" and "external" spaces are realized, correspondingly, by functions $\psi(\eta)$, and $f(P_+)$ with the inner products

$$(f_1, f) = \frac{1}{2} \int_0^\infty \frac{dP_+}{P_+} f_1^*(P_+) f(P_+)\,, \tag{3.10}$$

$$(\psi_1, \psi) = \int_{-1/2}^{1/2} \left( \prod_{k=1}^{N-1} \frac{d\eta_k}{1/2 - 2\eta_k^2} \right) \psi_1^*(\eta)\psi(\eta)\,. \tag{3.11}$$

It is convenient to pass from the functions $\psi$ with inner product (3.11) to the functions

$$\varphi(\eta) = \psi(\eta) \prod_{b=1}^{N-1} (\tfrac{1}{2} - 2\eta_b^2)^{-1/2} \tag{3.12}$$

with inner product

$$(\varphi_1, \varphi) = \int_{-1/2}^{1/2} \varphi^*(\eta)\varphi(\eta) \prod_{a=1}^{N-1} d\eta_a\,. \tag{3.13}$$

The latter differs from the nonrelativistic product only by limits of integration.

As it follows from (2.12) after passing to variables (3.1), the function $V$ in the integral expression for kernel takes the form $V = V(\tilde{\eta}, \tilde{q})$, where

$\tilde{\eta} = \eta((p+p')/2), \tilde{q} = q((p+p')/2, x_a - x_b)$. Therefore it is convenient to use $\tilde{q}$ as new variables of integration in (2.13). In terms of variables (3.1) the quantities $\tilde{q}, \tilde{\eta}$ have the form

$$\tilde{q}_a = \frac{\mathcal{D}_{a+1}}{2\mathcal{D}_a}\left(\frac{\mathcal{D}_a q_a}{\prod_{j=a+1}^{N-1}(\frac{1}{2}+\eta_j)} + \sum_{\nu=1}^{a-1}(\eta'_\nu - \eta_\nu)q_\nu \prod_{j=\nu+1}^{N-1}\frac{\frac{1}{2}+\eta'_j}{\frac{1}{2}+\eta_j}\right), \quad (3.14)$$

$$\tilde{\eta}_a = \mathcal{D}_{a+1}^{-1}\left(\eta_a \prod_{j=a+1}^{N}(\tfrac{1}{2}+\eta_j) + \eta'_a \prod_{j=a+1}^{N}(\tfrac{1}{2}+\eta'_j)\right), \quad (3.15)$$

where

$$\mathcal{D}_a = \prod_{j=a}^{N}(\tfrac{1}{2}+\eta_j) + \prod_{j=a}^{N}(\tfrac{1}{2}+\eta'_j). \quad (3.16)$$

The operator $\hat{M}^2$ acts nontrivially only on $h_{int}$. It is an integral operator, which is determined by the rule

$$(\hat{M}^2\varphi)(\eta) = M_f^2(\eta)\varphi(\eta) + \int_{-1/2}^{1/2} W(\eta,\eta')\varphi(\eta') \prod_{n=1}^{N-1} d\eta_n, \quad (3.17)$$

with the kernel

$$W(\eta,\eta') = \left(\prod_{d=1}^{N-1}\left[(\tfrac{1}{2}+\eta'_d)(\tfrac{1}{2}+\eta_d)\right]^{(d-1)/2}\mathcal{D}_{d+1}^{-1}\right)$$
$$\times \int_{-\infty}^{\infty} F(\tilde{q},\tilde{\eta}) \exp\left(i\sum_{a=1}^{N-1}\tilde{q}_a(\eta_a - \eta'_a)Y_a\right)\prod_{b=1}^{N-1}\frac{d\tilde{q}_b}{\pi}, \quad (3.18)$$

where

$$Y_a = 4\mathcal{D}_{a+1}^{-2}\prod_{j=a+1}^{N}(\tfrac{1}{2}+\eta_j)(\tfrac{1}{2}+\eta'_j). \quad (3.19)$$

As it follows from Eqs (3.15), (3.16), (3.19), the kernel $W$ satisfies the conditions

$$W^*(\eta,\eta') = W(\eta',\eta), \quad (3.20)$$

which provides the Hermiticity of the integral operator (3.17).

## 4. System with oscillator-like interaction

Let us consider the system with the interaction function

$$V = \omega^2 \sum_{a<b}\sum r_{ab}^2 p_a p_b , \quad \omega^2 > 0 . \tag{4.1}$$

The function (4.1) gives N-particle generalization of two-particle interaction (see Ref. [9]), as well as one of possible relativistic generalization of N-particle oscillator potential. There exists no unique relativistic generalization of the nonrelativistic oscillator potential. For instance oscillator-like model has been considered by Staruszkiewicz [15] in Lagrangian formalism for the two-particle system in the two-dimensional space-time. It leads, in the Hamiltonian description, to the interaction function $V = \omega^2 r^2 (p_1 + p_2)^2$.

In terms of variables (3.1) the interaction function $V$ takes the form

$$V = F(q, \eta) = \omega^2 \sum_{a=1}^{N-1} \left(\tfrac{1}{4} - \eta_a^2\right) q_a^2 \prod_{j=a+1}^{N} \left(\tfrac{1}{2} + \eta_j\right)^{-1} . \tag{4.2}$$

The system with interaction (4.1) has $N-2$ additional integrals of motion $\lambda_j$, which mutually commute

$$\{\lambda_i, \lambda_k\} = 0, \quad i, k = \overline{2, N-1} . \tag{4.3}$$

In terms of variables (3.1) they have the form

$$\lambda_{j+1}^2 = \sum_{d=1}^{j} \frac{m_d^2}{\frac{1}{2} - \eta_{d-1}} \prod_{i=d}^{j} \left(\tfrac{1}{2} + \eta_i\right)^{-1} + \frac{m_{j+1}^2}{\frac{1}{2} - \eta_j}$$
$$+ \omega^2 \sum_{d=1}^{j-1} \left(\tfrac{1}{4} - \eta_d^2\right) q_d^2 \prod_{i=d+1}^{j} \left(\tfrac{1}{2} + \eta_i\right)^{-1} + \omega^2 \left(\tfrac{1}{4} - \eta_j^2\right) q_j^2 , \tag{4.4}$$

where $\lambda_N^2 = M^2, j = \overline{1, N-1}$. They can be represented by means of recurrence relations

$$\lambda_{j+1}^2 = \frac{\lambda_j^2}{\frac{1}{2} + \eta_j} + \frac{m_{j+1}^2}{\frac{1}{2} - \eta_j} + \omega^2 \left(\tfrac{1}{4} - \eta_j^2\right) q_j^2 , \tag{4.5}$$

where we denote $\lambda_1^2 = m_1^2$. As it follows from Eqs (4.4),(4.5), the integral of motion $\lambda_i$ does not contain $\eta_k, q_k, k \geq i$ and it plays the role of particle mass for integral $\lambda_{i+1}$.

Consider now the eigenvalue problem for the total mass operator

$$(\hat{M}^2\varphi)(\eta) = M^2\varphi(\eta) \tag{4.6}$$

using the representation (3.13). Substituting interaction terms of the integral $\lambda_{j+1}^2$ (they contain $\omega$) into kernel (3.18) we obtain expression for operator $\hat{\lambda}_{j+1}^2$

$$\begin{aligned}\hat{\lambda}_{j+1}^2 = &\sum_{d=1}^{j} \frac{m_d^2}{\frac{1}{2}-\eta_{d-1}} \prod_{i=d}^{j}(\tfrac{1}{2}+\eta_i)^{-1} + \frac{m_{j+1}^2}{\frac{1}{2}-\eta_j}\\ &-\omega^2\sum_{d=1}^{j-1}\left[-\frac{1}{4} - 2\eta_d\frac{\partial}{\partial\eta_d} + (\tfrac{1}{4}-\eta_d^2)\frac{\partial^2}{\partial\eta_d^2}\right]\prod_{k=d+1}^{j}(\tfrac{1}{2}+\eta_k)^{-1}\\ &-\omega^2\left[-\frac{j+1}{4} - 2\eta_j\frac{\partial}{\partial\eta_j} + (\tfrac{1}{4}-\eta_j^2)\frac{\partial^2}{\partial\eta_j^2}\right]\end{aligned} \tag{4.7}$$

and boundary conditions

$$\lim_{\eta_j\to\pm1/2}(\tfrac{1}{4}-\eta_j^2)\frac{\partial\varphi_j}{\partial\eta_j} = \lim_{\eta_j\to\pm1/2}\varphi_j(\eta_j) = 0\,, \tag{4.8}$$

which ensure the Hermiticity of (4.7). Putting $j = N-1$ we have the expression of total mass operator. As it follows from Eq.(4.7) the operator $\hat{\lambda}_i$ does not contain $\eta_k$, $k \geq i$ and derivatives with respect to $\eta_k$, $k \geq i$. The operators (4.7) can be determined by means of the following recurrence relations

$$\begin{aligned}\hat{\lambda}_{j+1}^2 = &\frac{\hat{\lambda}_j^2 - \omega^2(j-1)/4}{\frac{1}{2}+\eta_j} + \frac{m_{j+1}^2}{\frac{1}{2}-\eta_j}\\ &-\omega^2\left(-\frac{j+1}{4} - 2\eta_j\frac{\partial}{\partial\eta_j} + (\tfrac{1}{4}-\eta_j^2)\frac{\partial^2}{\partial\eta_j^2}\right).\end{aligned} \tag{4.9}$$

Operators (4.7) commute between themselves and therefore they have a common set of eigenfunctions. Let $\varphi(\eta)$ be the eigenfunction of $\hat{M}^2 = \hat{\lambda}_N^2$. Putting $\varphi(\eta) = \prod_{i=1}^{N-1}\varphi_i(\eta_i)$ gives us the system of $N-1$ differential equation of the hypergeometric type

$$\begin{aligned}&\left[\frac{\lambda_j^2 - \omega^2(j-1)/4}{\frac{1}{2}+\eta_j} + \frac{m_{j+1}^2}{\frac{1}{2}-\eta_j}\right.\\ &\left.-\omega^2\left(\frac{\lambda_{j+1}}{\omega^2} - \frac{j+1}{4} - 2\eta_j\frac{\partial}{\partial\eta_j} + (\tfrac{1}{4}-\eta_j^2)\frac{\partial^2}{\partial\eta_j^2}\right)\right]\varphi_j(\eta_j) = 0\,,\end{aligned} \tag{4.10}$$

where $\prod_{i=1}^{j}\varphi_i(\eta_i)$ is an eigenfunction and $\lambda^2_{j+1}$ is an eigenvalue of operator $\hat{\lambda}^2_{j+1}$ and $\lambda^2_j$ is an eigenvalue of operator $\hat{\lambda}^2_j$. The substitution

$$\varphi_j(\eta_j) = (\tfrac{1}{2} - \eta_j)^{a_j}(\tfrac{1}{2} + \eta_j)^{b_j} y(\eta_j)\,, \tag{4.11}$$

where $a^2_j = m^2_{j+1}/\omega^2, b^2_j = \lambda^2_j/\omega^2 - (j-1)/4$, reduces Eqs (4.10) to the known equation for Jacobi polynomial $P^{(2a_j,2b_j)}_{n_j}(2\eta_j)$ [16]. The Eqs (4.10) have nontrivial solutions, which are bounded and square integrable on the interval $(-\frac{1}{2}, \frac{1}{2})$ under the conditions

$$\sqrt{\lambda^2_{j+1} - \omega^2 j/4} = \sqrt{\lambda^2_j - \omega^2(j-1)/4} + m_{j+1} + \omega(n_j + \tfrac{1}{2}),\,, \tag{4.12}$$

where $n_j = 0, 1, 2, \ldots$ . Here we consider

$$a_j > 0, \qquad b_j > 0\,. \tag{4.13}$$

Solving recurrence relations (4.12) we obtain the mass spectrum

$$M^2_n = \left[\sum_{a=1}^{N} m_a + \omega \sum_{b=1}^{N-1} (n_b + \tfrac{1}{2})\right]^2 + \frac{N-1}{4}\omega^2\,. \tag{4.14}$$

The normalized solutions of the Eqs (4.10) give the wave functions

$$\varphi_n(\eta) = C_{n_j} \prod_{j=1}^{N-1} (1/2 - \eta_j)^{a_j}(\tfrac{1}{2} + \eta_j)^{b_j} P^{(2a_j,2b_j)}_{n_j}(2\eta_j)\,, \tag{4.15}$$

where

$$a_j = m_{j+1}/\omega, \qquad b_j = \frac{1}{\omega}\sum_{k=1}^{j} m_k + \sum_{k=1}^{j-1}(n_k + \tfrac{1}{2})\,. \tag{4.16}$$

The constants $C_{n_j}$ are determined by the normalization conditions $(\varphi_{n_j}, \varphi_{n'_j}) = \delta_{n_j n'_j}$ which give

$$|C_{n_j}|^2 = \frac{n_j!(2n_j + 1 + 2a_j + 2b_j)\Gamma(n_j + 1 + 2a_j + 2b_j)}{\Gamma(n_j + 1 + 2a_j)\Gamma(n_j + 1 + 2b_j)}\,. \tag{4.17}$$

Let us note that conditions of square integrability for $\varphi_{n_j}(\eta_j)$ are fulfilled under weaker limitations on numbers $a_j, b_j$ than (4.13), namely $2a_j + 1 > 0, 2b_j + 1 > 0$. This could give three other branches of the spectrum, which differ from the Eq. (4.14) by signs of $m_a$. But corresponding wave functions do not satisfy the boundary conditions (4.8).

Interaction function (4.1) may be generalized by adding terms which are linear in the coordinates

$$V \to \tilde{V} = V + \alpha \sum\sum_{a<b} r_{ab}(p_a - p_b)\,. \tag{4.18}$$

Such a system has also additional integrals of motion. They may be determined by the recurrence relations

$$\begin{aligned}\lambda_{j+1}^2 =& \frac{\lambda_j^2}{\frac{1}{2}+\eta_j} + \frac{m_{j+1}^2}{\frac{1}{2}-\eta_j} + \omega^2(\tfrac{1}{4} - \eta_j^2)q_j^2 \\ &+ \alpha[(1-j)/2 + (1+j)\eta_j]q_j\,,\end{aligned} \tag{4.19}$$

and satisfy the Eqs. (4.3). The integral $\lambda_N^2$ coincides with the square of the total mass $M^2$ of the system . Therefore, similarly to the previous case, we obtain the wave functions

$$\varphi_n(\eta) = C_{n_j} \prod_{j=1}^{N-1} (\tfrac{1}{2} - \eta_j)^{a_j + \frac{i\alpha}{2\omega^2}} (\tfrac{1}{2} + \eta_j)^{b_j + \frac{i\alpha j}{2\omega^2}} P_{n_j}^{(2a_j, 2b_j)}(2\eta_j) \tag{4.20}$$

and the mass spectrum

$$M_n^2 = \left[\sum_{a=1}^{N} \sqrt{m_a^2 - \frac{\alpha^2}{4\omega^2}} + \omega \sum_{b=1}^{N-1} (n_b + \tfrac{1}{2})\right]^2 + \frac{N-1}{4}\omega^2 + \frac{\alpha^2 N^2}{4\omega^2}\,, \tag{4.21}$$

where

$$a_j = \sqrt{\frac{m_{j+1}^2}{\omega^2} - \frac{\alpha^2}{4\omega^4}}, \qquad b_j = a_0 + \sum_{k=1}^{j-1} (a_k + n_k + \tfrac{1}{2}) \tag{4.22}$$

and constants $C_{n_j}$ are determined by Eq. (4.17). Discrete spectrum exists only for real $a_j$ . This leads to inequality $|\alpha| \le 2\omega \min\{m_a\}, a = \overline{1,N}$. In the nonrelativistic limit $M \to m + E/c^2$ we obtain the energy spectrum

$$E_n = \omega \sum_{k=1}^{N-1} \left(n_k + \tfrac{1}{2} - \frac{\alpha_k^2}{2\omega^2 \mu_k}\right)\,. \tag{4.23}$$

The expression (4.23) corresponds to the spectrum of the N-particle system, which is described by the Hamiltonian

$$\tilde{H} = \sum_{k=1}^{N-1} \left( \frac{\xi_k^2}{2\mu_k} + \frac{\omega^2 \mu_k \rho_k^2}{2} + \alpha_k \rho_k \right) , \tag{4.24}$$

where masses $\mu_k$ and constants $\alpha_k$ are related to parameters of original relativistic problem by equalities

$$\mu_k = \frac{m_{k+1} \sum_{j=1}^{k} m_j}{\sum_{j=1}^{k+1} m_j}, \quad a_k = \frac{\alpha \left( \sum_{j=1}^{k} m_j - k m_{j+1} \right)}{2 \sum_{j=1}^{k+1} m_j}, \quad k = \overline{1, N-1}. \tag{4.25}$$

Expression (4.24) may be obtained from the nonrelativistic limit of the Hamiltonian of the system with interaction (4.18)

$$H_{c\to\infty} = \sum_{a=1}^{N} \frac{\tilde{p}_a^2}{2m_a} + \frac{\omega^2}{2m} \sum\sum_{a<b} r_{ab}^2 m_a m_b + \frac{\alpha}{2m} \sum\sum_{a<b} r_{ab}(m_a - m_b) = \frac{\tilde{P}^2}{2m} + H_{in} , \tag{4.26}$$

where $m = \sum_{a=1}^{N} m_a; a, b = \overline{1, N}; k = \overline{1, N-1}$ and $\tilde{p}_a$ are nonrelativistic particle momenta. After canonical transformation

$$\xi_k = \frac{m_{k+1} \sum_{j=1}^{k} \tilde{p}_j - \tilde{p}_{k+1} \sum_{j=1}^{k} m_j}{\sum_{j=1}^{k+1} m_j}; \quad \rho_k = \frac{\sum_{j=1}^{k} x_j m_j}{\sum_{j=1}^{k} m_j} - x_{k+1};$$

$$\xi_N = \tilde{P} = \sum_{j=1}^{N} \tilde{p}_j; \quad \rho_N = \frac{\sum_{j=1}^{N} x_j m_j}{\sum_{j=1}^{N} m_j}; \quad \{\rho_a, \xi_b\} = \delta_{ab} \tag{4.27}$$

the inner Hamiltonian takes the form (4.24). In the case of equal particle masses, $m_1 = m_2 = ... = m_N$ , the "nonrelativistic" constants of linear interaction are equal to zero, $\alpha_k = 0$ .

## 5. Conclusions

We have considered the quantization of exactly integrable model of N–particle relativistic system with oscillator-like interaction in the two-dimensional variant of the front form of dynamics. Using the Weyl quantization rule in the case of Hilbert space $\mathcal{H}_N^F$ and inner variables (3.1) we have solved the eigenvalue problem for square of mass (inner energy) operator $\hat{M}^2$ . Generalization of the oscillator-like potential (4.1) by addition of terms which are linear in the coordinates permits, to find the exact solution. In contrast to the nonrelativistic case, the value of the coupling constant of linear interaction must lie in the interval $|\alpha| \leq 2\omega min\{m_a\}, a = \overline{1, N}$. This condition is necessary for the existence of a discrete spectrum. Other values of $\alpha$ lead to continuous spectrum.

There are many quantization methods. As a result of Poincaré -invariance conditions, the relativistic Hamiltonians usually cannot be represented as a sum of terms that depend only on commutative operators. Therefore different quantization methods may lead to different expressions for observable quantities. We have applied the Weyl quantization rule because it leads to the unitary representation of the Poincaré group in the two-dimensional space-time [9], is in agreement with the quantum results, and preserves, in the case of oscillator-like potential (4.2), the commutation relations between additional integrals of motion: $[\hat{\lambda}_i, \hat{\lambda}_j] = 0$. This means that Weyl quantization rule preserves additional symmetries which are responsible for integrability of this model.

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