GEOMETRY AND DUALITY OF GAUGED SU(3) WESS–ZUMINO–WITTEN MODEL

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We show that the gauged SU(3) Wess–Zumino–Witten model can be classified into several classes by its target space metrics. This fact implies the appearance of target space transition and two target space dualities. We also consider the gauged SU(2,1) Wess–Zumino–Witten model as an analogy of SL(2,$\boldsymbol{R}$) black hole. We show also that these Wess–Zumino–Witten models are connected continuously.

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1. Introduction

Since the string is one dimensional extended object, it can wrap around on a circle, which gives a distinction between particle field theory and string theory. This fact leads us to so-called target space duality [1]. Target space duality is regarded to be peculiar to string theory, so we may think of it as a stringy effect. The most famous one of target space dualities may be $R \to 1/R$ duality which was found in circle compactification, in bosonic string theory. Under this duality, the winding number and the momentum number are exchanged. What this duality means is that a theory on radius R is equivalent to that of $1/R$. Of course, target space duality is not restricted to this case, it appears in many places such as toroidal compactification, two dimensional black hole theory [2–5], axial-vector type duality as a gauge symmetry, topology change and in some sense, mirror symmetry [6]. In particular, one of the most interesting dualities may be mirror symmetry, because it includes target space duality (in fact, mirror symmetry of torus is nothing other than $R \to 1/R$ duality). We know that mirror symmetry gives a powerful tool for Calabi-Yau compactifications in super string theory and that the notion is extended to other fields.

For example, Giveon and Witten [7] showed that the mirror symmetry of $N = 2$ SU(3) Wess–Zumino–Witten (WZW) model was a gauge symmetry, namely the axial-vector type duality, and they conjectured that their construction could be extended to $A_{2n}(n = 1, \cdots)$ WZW models. In [7], they employed SU(3) WZW model as an example and explained their algebraic construction. But their discussion was (mainly) based on algebraic aspects of $N = 2$ super conformal algebra, we can not say that discussions about geometrical aspects and realizations by Lagrangian formalism of such mirror symmetric WZW models are filled, so that it is necessary for us to fill the gap. Of course, though they gave another example SU(2)× U(1), this example is extremely well-known in string theory, so that even if we should apply the idea of mirror symmetry as a gauge symmetry to that example, we could not get any new discovery, *i.e.*, the model is "trivial". Therefore in order to establish the idea in the framework of WZW model, we must study another WZW models which are mirror symmetric. In that case, we may get some new aspects of mirror symmetry as a gauge symmetry. However, it is unnecessary for us to investigate all A_{2n} WZW models. It will be enough to study the simplest case, *i.e.*, SU(3) WZW model corresponding to A_2. Therefore, it is interesting for us to investigate SU(3) WZW model, although supersymmetric WZW model is a little complicated to treat in contrast with bosonic one. However, since these mirror symmetric WZW models can be understood in the framework of current-current marginal perturbation and $N = 2$ current-current perturbation $J_2\bar{J}_2$ has the following form

$$J_2\bar{J}_2 = J_0\bar{J}_0 + (\text{terms including fermions}),$$

where index denotes the number of supersymmetry and $J_N(\bar{J}_N)$ is a left (right) handed current, mirror transformation $J_2 \to -J_2$ induces similar transformation to the bosonic part $J_0 \to -J_0$. Obviously, mirror transformation has an effect on the bosonic part. In other words, in $N = 0$ part of $N = 2$ SU(3) WZW model, target space duality will be expected to appear by bosonic "mirror" transformation $J_0 \to -J_0$. However, during the investigation of $N = 0$ SU(3) WZW model and related problem such as gauging, we have found a curious phenomenon about its geometry. The aim of the paper is to report on the geometrical aspects and dualities of gauged SU(3) WZW model, although we will treat only bosonic theory.

The paper consits of the following sections. In section 2, we derive the SU(3) WZW action and then we gauge it. This gauged model is based on SU(3)/(U(1)× U(1)) which is Kaehlerian as shown in [8], so that our investigation in this paper will be helpful when we extend our discussions to $N = 2$ theory. Of course, though there are several choices for gauging except SU(3)/(U(1)× U(1)), as our gauging is the most fundamental one, we employ this gauging. In section 3, we discuss the metrics of the gauged

SU(3) WZW model and their target space duality transformation. It is shown that the target space has (basically) two classes (phases) in the point of view of the geometry of the target space of SU(3)/(U(1)× U(1)) WZW model. Therefore when we move from one phase to another one, target space transition should occur. This transition happens continuously. According to these facts, we can see that there exists two target space dualities. We will consider analytic continuation from SU(3) to SU(2,1) as an analogy of the transformation from SU(2)/U(1) to SL(2,$\boldsymbol{R}$)/U(1) black hole which are well-known examples of two dimensional black hole. Our models contain two 2-dimensional black holes. In addition to this, we also show that these WZW models are connected. Five appendices are added.

2. Gauged SU(3) WZW action

In this section, we derive SU(3) WZW action using Euler angle parametrization, then we perform its gauging.

General worldsheet WZW action is defined by

$$S = \frac{k}{4\pi}\int d^2\sigma\,\mathrm{Tr}[U_\mu(g)U^\mu(g)] + \frac{ik}{6\pi}\int d^3\sigma\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}[U_\mu(g)U_\nu(g)U_\rho(g)]\,, \tag{2.1}$$

where g is an element of a Lie group G (for notations, see also Appendix A) [1–3].

Suppose G=SU(3) and let us introduce real coodinates θ_1^i and θ_2^i, where θ_1^i and θ_2^i are related to another coordinates θ^i and $\tilde\theta^i$ such as

$$\begin{aligned} \theta^i &= \theta_2^i - \theta_1^i \mod 2\pi\,, \\ \tilde\theta^i &= \theta_2^i + \theta_1^i \mod 2\pi\,, \end{aligned} \tag{2.2}$$

where $i = 3$ or 8 in the case of SU(3) [9]. The Euler angle parametrization for $g \in$ SU(3) (see Appendix C) gives

$$g = \mathrm{e}^{i(\theta_1^3 H^3 + \theta_1^8 H^8)}\cdot \mathrm{e}^{i(x_4T^4 + x_5T^5 + x_6T^6 + x_7T^7)}\cdot \mathrm{e}^{i(\theta_2^3 H^3 + \theta_2^8 H^8)}\,, \tag{2.3}$$

where H^3 and H^8 are generators of the Cartan sub-algebra, T^k are Gell–Mann matrices (see Appendix D) and x_k are real parameters. Then after some calculations (see Appendix B), (2.1) will be[1]

$$\begin{aligned} S = &-\frac{k}{4\pi}\int d^2\sigma(\partial_\mu\theta_1^i\partial^\mu\theta_1^i + \partial_\mu\theta_2^i\partial^\mu\theta_2^i \\ &+ \partial_\mu x_4\partial^\mu x_4 + \partial_\mu x_5\partial^\mu x_5 + \partial_\mu x_6\partial^\mu x_6 + \partial_\mu x_7\partial^\mu x_7) \\ &- \frac{k}{2\pi}\int d^2\sigma P_-^{\mu\nu}\partial_\mu\theta_1^i\partial_\nu\theta_2^j M_{ij}\,. \end{aligned} \tag{2.4}$$

[1] Repeated roman indices are also to be summed.

We can rewrite this action as

$$S = -\frac{k}{2\pi}\int d^2z(\partial\theta_1^i\bar\partial\theta_1^i + \partial\theta_2^i\bar\partial\theta_2^i + 2M_{ij}\bar\partial\theta_1^i\partial\theta_2^j$$
$$+ \partial x_4\bar\partial x_4 + \partial x_5\bar\partial x_5 + \partial x_6\bar\partial x_6 + \partial x_7\bar\partial x_7)\,, \qquad (2.5)$$

where we have introduced a complex coodinate and its complex conjugate

$$z = \sigma^1 + i\sigma^2, \qquad \bar z = \sigma^1 - i\sigma^2\,. \qquad (2.6)$$

Then, differential operators are

$$\partial = \tfrac{1}{2}(\partial_1 - i\partial_2)\,, \quad \bar\partial = \tfrac{1}{2}(\partial_1 + i\partial_2)\,. \qquad (2.7)$$

Note that we have used the integration measure

$$d^2z := |dzd\bar z| = 2d^2\sigma\,. \qquad (2.8)$$

The action (2.5) describes SU(3) WZW model.

Next, let us consider the gauging of (2.5). We can add a total derivative term S_{tot}

$$S_{\text{tot}} = -\frac{k}{2\pi}\int d^2z(\partial\theta_1^i\bar\partial\theta_2^i - \partial\theta_2^i\bar\partial\theta_1^i)\,, \qquad (2.9)$$

which gives a topological contribution and is needed for gauge invariance of gauged action. Then the parent action S' will be

$$S' = S + S_{\text{tot}}. \qquad (2.10)$$

This action is invariant under $U(1)^2_{\text{L}} \times U(1)^2_{\text{R}}$ affine symmetry generated by chiral currents

$$J^i = -\frac{k}{2}\left[-(I-M)_{ij}\partial\theta^j + (I+M)_{ij}\partial\tilde\theta^j\right] \qquad (2.11)$$

and anti-chiral currents

$$\bar J^i = -\frac{k}{2}\left[(I-M)_{ij}\bar\partial\theta^j + (I+M)_{ij}\bar\partial\tilde\theta^j\right]\,, \qquad (2.12)$$

where I is a 2×2 unit matrix. Of course, there are several currents related to x_k directions, but they are not required here because including them does not give any interesting problems.

Now, let us gauge our action (2.10) by the following gauge transformation

$$\partial\tilde\theta^i \longrightarrow \partial\tilde\theta^i + A^i,$$
$$\bar\partial\tilde\theta^i \longrightarrow \bar\partial\tilde\theta^i + \bar A^i, \qquad (2.13)$$

where A^i and $\bar{A}^i$ are $U(1)$ gauge fields. The gauge invariant action will be

$$S_{\rm gauged} = S' + \frac{1}{2\pi}\int d^2z\left[A^i\bar{J}^i + \bar{A}^iJ^i - \frac{kA^i\bar{A}^j}{2}(I+M)_{ij}\right]. \qquad (2.14)$$

Since there is no kinetic term of $A^i(\bar{A}^i)$, we can eliminate them by integration in the path integral. And a finite correction coming from the measure in the integration over these gauge fields gives a dilaton field $\phi(x)$. Thus the complete action will be

$$S_{\rm gauged} = S' + S[x] + \frac{1}{2\pi k}\int d^2z\ 2J^i\bar{J}^j(I+M)^{-1}_{ij}, \qquad (2.15)$$

where

$$S[x] = -\frac{1}{2\pi}\int d^2z\ \frac{1}{4}\phi(x)R^{(2)}, \qquad (2.16)$$

and $R^{(2)}$ is a worldsheet scalar curvature. This type of action is used to construct a curved background in 6-dimensions which is independent of 2 coordinates as an abelian quotient of $(6+2)$ dimensional background. Substituting (2.11) and (2.12) into (2.15), terms including $\tilde{\theta}^i$ vanish and therefore the final result will be

$$\begin{aligned} S_{\rm gauged} = &-\frac{k}{2\pi}\int d^2z[\mathcal{G}_{ij}\partial\theta^i\bar{\partial}\theta^j + \partial x_4\bar{\partial}x_4 \\ &+ \partial x_5\bar{\partial}x_5 + \partial x_6\bar{\partial}x_6 + \partial x_7\bar{\partial}x_7] \\ &- \frac{1}{8\pi}\int d^2z\phi(x)R^{(2)}, \end{aligned} \qquad (2.17)$$

where $\mathcal{G}_{ij}$ is a part of background metric with anti-symmetric tensor B_{ij}. It is given by

$$\begin{aligned} \mathcal{G}_{ij} &= g_{ij} + B_{ij} \\ &= (I-M)_{ik}(I+M)^{-1}_{kj}. \end{aligned} \qquad (2.18)$$

Here g_{ij} is a part of target space metric, in other words, symmetric part of $\mathcal{G}_{ij}$. We give an explicit form of g_{ij} and B_{ij} below,

$$\begin{aligned} g_{ij} &= \begin{pmatrix} g_{33} & g_{38} \\ g_{83} & g_{88} \end{pmatrix} \\ &= \frac{\tan^2\sqrt{X^2-Y^2}}{2(X^2-Y^2)}\begin{pmatrix} a & \sqrt{3}b \\ \sqrt{3}b & 3c \end{pmatrix}, \\ B_{ij} &= \begin{pmatrix} B_{33} & B_{38} \\ B_{83} & B_{88} \end{pmatrix} \\ &= \frac{\tan^2\sqrt{X^2-Y^2}}{2(X^2-Y^2)}\begin{pmatrix} 0 & \sqrt{3}Y^2 \\ -\sqrt{3}Y^2 & 0 \end{pmatrix}, \end{aligned} \qquad (2.19)$$

where

$$\begin{aligned} a &= x_4^2 + x_5^2 + x_6^2 + x_7^2, \\ b &= x_4^2 + x_5^2, \\ c &= x_4^2 + x_5^2 - x_6^2 - x_7^2, \\ X^2 &= 2(x_4^2 + x_5^2), \\ Y^2 &= x_6^2 + x_7^2 . \end{aligned} \tag{2.20}$$

Of course, target space metric itself G_{ab} is

$$G_{ab} = \begin{pmatrix} E & 0 \\ 0 & g_{ij} \end{pmatrix}, \tag{2.21}$$

where E is a 4×4 unit matrix. Since the anti-symmetric tensor is a total derivative, we can drop it. Rewriting $z(\bar{z})$ in terms of σ^i, (2.17) will be

$$\begin{aligned} S_{\text{gauged}} = -\frac{k}{4\pi} \int d^2\sigma \sqrt{\eta}\eta^{\mu\nu} \Big(g_{ij}\partial_\mu \theta^i \partial_\nu \theta^j + \partial_\mu x_4 \partial^\mu x_4 \\ + \partial_\mu x_5 \partial^\mu x_5 + \partial_\mu x_6 \partial^\mu x_6 + \partial_\mu x_7 \partial^\mu x_7 \Big) - \frac{1}{16\pi} \int d^2\sigma \sqrt{\eta} \phi(x) R^{(2)}, \end{aligned} \tag{2.22}$$

where $\eta^{\mu\nu}$ is a worldsheet metric tensor and $\eta = \det \eta^{\mu\nu}$. Though $\eta^{\mu\nu}$ itself is trivial in our case, we included it explicitly here. This action describes SU(3)/(U(1)× U(1)) WZW model.

Now, let us turn to a dilaton. As is shown in [9,10], there is a discrete symmetry group in D-dimensional curved backgrounds which is independent of d-coordinates, and is isomorphic to $O(d, d, \boldsymbol{Z})$ $(d < D)$. This symmetry group relates $(D + d)$-dimensional conformal field theory to D-dimensional one. Then, the dilaton transforms as

$$\phi(x) = \phi_0(x) + \ln \det(I + M), \tag{2.23}$$

where $\phi_0(x)$ is a dilaton field in D-dimensional curved background, at least, at one loop level. Recall that our WZW model is originally defined in 8 dimensions. And the dimension is reduced to 6 by gauging abelian symmetries. Thus in our model, $D = 6$ and $d = 2$, so the symmetry group is $O(2, 2, \boldsymbol{Z})$. Then the dilaton field transforms under the target space duality transformation as

$$\phi'(x) = \phi(x) + \ln \det G_{ab}. \tag{2.24}$$

This is a change of the string coupling constant. This transformation property means that the string coupling constant g_{string} is invariant

$$g_{\text{string}}^{-1} = \langle e^{\phi'(x)} \rangle = \langle \sqrt{\det G_{ab}} e^{\phi(x)} \rangle \tag{2.25}$$

under $O(2, 2, \boldsymbol{Z})$.

3. Metric and duality

In the previous section, we have performed the gauging of the action and have derived the gauged SU(3) WZW action. In this section, we show the detail of the target space metric and the curious duality which we mentioned briefly in the introduction.

We can easily obtain the metric of the target manifold from the action. It is

$$\begin{aligned}(ds)^2 =&(dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2 \\ &+ g_{33}(d\theta^3)^2 + 2g_{38}d\theta^3 d\theta^8 + g_{88}(d\theta^8)^2. \end{aligned} \tag{3.1}$$

Let us assume $X^2 - Y^2 > 0$ and define this region as a region I. Now introduce a new variable $R = X^2 - Y^2$ for later convenience[2] and change notations such as (3.1) is

$$(ds)^2_{\rm I} = (dx)^2 + R_1^2(d\theta^3)^2 + \frac{3R_1^2 + R_2^2}{\sqrt{3}} d\theta^3 d\theta^8 + R_2^2(d\theta^8)^2, \tag{3.2}$$

where we have abbreviated $(dx)^2 \equiv (dx_4)^2 + (dx_5)^2 + (dx_6)^2 + (dx_7)^2$ and

$$\begin{aligned} R_1^2 &= \frac{\tan^2\sqrt{R}}{2R}\left(\frac{X^2}{2} + Y^2\right), \\ R_2^2 &= \frac{3\tan^2\sqrt{R}}{2R}\left(\frac{X^2}{2} - Y^2\right). \end{aligned} \tag{3.3}$$

Strictly speaking, we must consider an overall sign of $X^2/2 - Y^2$. If $R - Y^2 > 0$, then $R > Y^2 > 0$, which is in region I, so we name this region as I-1. If $R - Y^2 < 0$, then we have two possibilities, $Y^2 > R > 0$ and $Y^2 > 0 > R$ (*i.e.*, $0 > R$). Let us name the former as region I-2 and the latter as region II. In this classification, the signature of the metric in I-2 and II is Minkowski whereas it is Euclidean in I-1. However, if we should use these three classes, the result would be a little complicated and might give us uneasy impression to understand. Thus, we employ (the basic) two classes I and II instead of I-1, I-2 and II.

We may think R_1 and R_2 as radii, so, roughly speaking, this space may be regarded as a direct product manifold $\boldsymbol{R}^4 \times T^2$, where T^2 is a (real) two dimensional torus with some additional structure whose existence is implied by non-diagonal components in the metric tensor. This metric behaves singularly due to its own right, for the metric can partially vanish.

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[2] Do not confuse R with the radius itself.

We can easily see that there seem to be several singularities corresponding to $X^2 = 0, X^2 - 2Y^2 = 0, R = 0$ and $R = n^2\pi^2$ $(n = 1, 2, \cdots)$. $R = 0$ can be excluded among these candidates of singularity, because

$$\left.\frac{\tan^2 \sqrt{R}}{2R}\right|_{R\to 0} \longrightarrow \frac{1}{2}. \tag{3.4}$$

Therefore the remainders are the candidates.

In order to see whether these candidates are really singular points or not, one needs certain quantities such as scalar curvature or curvature invariant which are independent of choices of coordinate system. We included the scalar curvature in Appendix E.

At $X^2 = 0$, *i.e.*, at the point $(x_4, x_5) = (0, 0)$, the non-diagonal components of the metric vanish, so this space reduces to the "pure" space with two radii R_1 and R_2, in other words, this space reduces to $\boldsymbol{R}^4 \times T^2$ exactly. However, as can be easily seen from the expression of the scalar curvature, $X^2 = 0$ is a regular point of the scalar curvature, so it is simply a coordinate singularity which can be absorbed by a coordinate redefinition.

At $X^2 - 2Y^2 = 0$, *i.e.*, the points which are solutions of $x_4^2 + x_5^2 - x_6^2 - x_7^2 = 0$, R_2 is zero, so this space is the space whose one radius vanished. Since the scalar curvature at these points is regular, those are coordinate singularities.

The constraint $R = n^2\pi^2$ originates from the following fact. Since the radii R_1 and R_2 contain tangent function, they are quantized with a periodicity π, *i.e.*,

$$\sqrt{X^2 - Y^2} \bmod \pi\,. \tag{3.5}$$

At $R = n^2\pi^2$ $(n = 1, 2, \cdots)$,

$$\left.\frac{\tan^2 \sqrt{R}}{2R}\right|_{R=n^2\pi^2} \longrightarrow 0, \tag{3.6}$$

so these may be expected to give singular points. In fact, since the scalar curvature diverges to infinity at these points, these are "really" singularities.

To complete the discussion, let us consider the determinant of the metric tensor of (3.2), which is

$$\det G_{ab} = -\frac{3Y^4 \tan^4 \sqrt{R}}{4R^2}\,. \tag{3.7}$$

Since X^2 and Y^2 always belong to real positive numbers, (3.7) is always negative, which means our choice of Euler angle coordinate for SU(3) is a negative coordinate, although this is not important because the sign could be absorbed if we should use another coordinate system to parametrize SU(3). From (3.7), we can easily see that $Y^2 = 0$ and (3.5) give zeros of the determinant. Thus $Y^2 = 0$ and (3.5) mean that the volume form of the target space is zero. But since the scalar curvature at $Y^2 = 0$ is regular, it is a coordinate singularity.

It is interesting to note that if we should further cut the space with $Y^2 = 0$ with $x_5 = 0$ or $x_4 = 0$ then the target space metric would be

$$\begin{aligned} (ds)^2_{\mathrm{I}\ x_5=0}^{\ Y^2=0} &= (dx_4)^2 + \frac{\tan^2\sqrt{2}x_4}{4}(d\theta^3)^2 \\ &\quad + \frac{\sqrt{3}\tan^2\sqrt{2}x_4}{2} d\theta^3 d\theta^8 + \frac{3\tan^2\sqrt{2}x_4}{4}(d\theta^8)^2, \\ (ds)^2_{\mathrm{I}\ x_4=0}^{\ Y^2=0} &= (dx_5)^2 + \frac{\tan^2\sqrt{2}x_5}{4}(d\theta^3)^2 \\ &\quad + \frac{\sqrt{3}\tan^2\sqrt{2}x_5}{2} d\theta^3 d\theta^8 + \frac{3\tan^2\sqrt{2}x_5}{4}(d\theta^8)^2. \end{aligned} \tag{3.8}$$

The first two terms, for example, are nothing other than SU(2)/U(1) black hole[3] up to constant factor whereas the fourth is S^1. Thus we can identify the reduced space is a product space SU(2)/U(1) black hole $\times S^1$ with some additional structure which correponds to the third term. Accordingly, the gauged SU(3) WZW model includes SU(2)/U(1) black holes in a sense. However, this fact is not surprising because SU(2) is a sub-group of SU(3) and thus the appearance of SU(2) is naturally expected. From this simple reason, roughly speaking, we can see that our cuttings correspond to the "break down" from SU(3) to SU(2). The reason why we have used a term not black hole but black "holes" is due to the freedom $x_4 = 0$ or $x_5 = 0$. On the other hand, at $X^2 = 0$ with $x_6 = 0$ or $x_7 = 0$, SL(2,$\boldsymbol{R}$)/U(1) black holes will appear which can be shown repeating a similar discussion above! Therefore we can say that the gauged SU(3) model contains two black hole theories. However, it is unclear whether the target space (3.2) itself is also black hole or not. In addition to this, as we could not find a convenient coordinate such as Kruskal coordinate, we do not discuss the causality of this space time here. So physical meanings are not understood at present. But, since we already know that the target space has singularities which can not be absorbed by coordinate redefinitions, we conjecture that the space may be a six dimensional black hole.

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[3] The appearance of the number $\sqrt{2}$ in front of the argument of tangent is due to our normalization for Gell–Mann matrices. In the literature [2–5,10], since the normalization $\mathrm{Tr}\, T^i T^i = 2$ is taken, the number in front of the argument is 1.

The dilaton in the region I is given by

$$\phi_{\rm I}(x) = \phi_0^{\rm I} + \ln 4\cos^2\sqrt{R}, \tag{3.9}$$

where $\phi_0^{\rm I}$ is a constant dilaton in the region I.

We have assumed as far that we are in the region I. Next, let us turn to the region II ($R < 0$). The discussion on the null line $R = 0$ will be treated later.

In this region, (3.2) reduces to another metric, because $\tan\sqrt{-1}x = \sqrt{-1}\tanh x$, *i.e.*, $\tan^2 ix = -\tanh^2 x$. Therefore we obtain the following metric

$$(ds)^2_{\rm II} = (dx)^2 + R_3^2(d\theta^3)^2 + \frac{3R_3^2 + R_4^2}{\sqrt{3}}d\theta^3 d\theta^8 + R_4^2(d\theta^8)^2, \tag{3.10}$$

where

$$\begin{aligned} R_3^2 &= \frac{\tanh^2\sqrt{R'}}{2R'}\left(\frac{X^2}{2} + Y^2\right), \\ R_4^2 &= \frac{3\tanh^2\sqrt{R'}}{2R'}\left(\frac{X^2}{2} - Y^2\right), \end{aligned} \tag{3.11}$$

and $R' = Y^2 - X^2$. Note that the factor $(X^2/2 - Y^2)$ in R_4^2 is actually always negative in the region II. The derivation of the above metric requires a little care. For example, notice that R_1^2 is a function of X and Y. If we used R, it could be written as $(X^2/2+Y^2)\tan^2\sqrt{R}/2R$. In this expression, the first factor should be unchanged (or it is unnecessary to rewrite it under $R \to -R = R'$) when we pass from I to II because we can always move freely from I to II continuously preserving $X^2, Y^2 > 0$.

Now we can see that the theories are classified by a cone $X^2 - Y^2 = 0$. In the region I, the metric behaves tangential whereas it behaves hyperbolic tangential in the region II. Accordingly, we can say that the transition of the target space of the gauged SU(3) WZW model appears when passing through the null line. These two metrics are connected on the null line. This is easy to check (see below). These features can not be seen in SU(2)/U(1) black hole solution or other space-time manifolds. For example, in the case of two dimensional black hole solution, it has only one class due to its triviality, which means that the target space has just one type metric in the full domain. However, the target space of the gauged SU(3) WZW model has now (basically) two types of metrics. We will summarize the discussion in the figure 1.

In this figure, T. means that the metric is "tangential" whereas H.T. "hyperbolic tangential", whose meanings should be obvious.

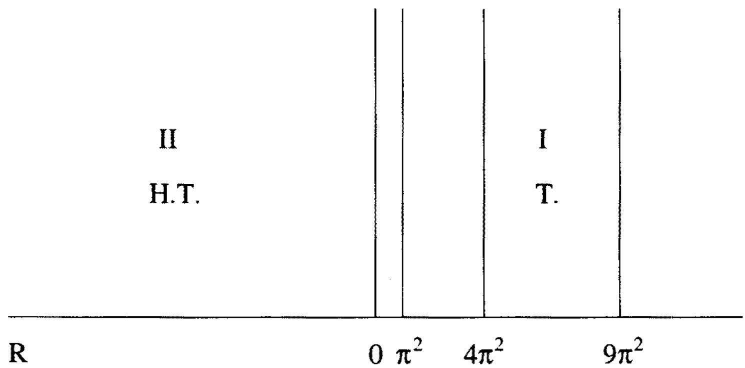

Fig. 1. Classification of target space of gauged SU(3) WZW model.

The dilaton in the region II is given by

$$\phi_{\rm II}(x) = \phi_0^{\rm II} + \ln 4\cosh^2\sqrt{R'} \tag{3.12}$$

where $\phi_0^{\rm II}$ is a constant dilaton in II. Since the theories in I and II should coincide on the null line, we can get $\phi_0 \equiv \phi_0^{\rm I} = \phi_0^{\rm II}$ from the requirement $\phi_{\rm I}(x)|_{R\to 0} = \phi_{\rm II}(x)|_{R'\to 0}$. Under this situation, the underlying string theory is well-behaved in the full domain of the gauged SU(3) in view of coupling constant. Namely, there is no gap of string coupling constant passing from I to II. Recalling the points on the null line in these target spaces we can find that the target space transition occurs continuously.

Now, let us consider target space duality transformation. To get a dual metric, it is enough to make

$$\begin{aligned} R_1'^2 &= \frac{1}{R_1^2} = \frac{4R}{(X^2+2Y^2)}\cot^2\sqrt{R}, \\ R_2'^2 &= \frac{1}{R_2^2} = \frac{4R}{3(X^2-2Y^2)}\cot^2\sqrt{R}, \end{aligned} \tag{3.13}$$

for (3.2) in the region I, for example. Then, it is easy to write down the dual metric. The dual metric is

$$(ds')_{\rm I}^2 = (dx)^2 + R_1'^2(d\theta^3)^2 + \frac{3R_1'^2 + R_2'^2}{\sqrt{3}}d\theta^3 d\theta^8 + R_2'^2(d\theta^8)^2. \tag{3.14}$$

The dilaton transforms under the target space duality transformation as

$$\phi_{\rm I}'(x) = \phi_0^{\rm I} + \ln\frac{3Y^4\tan^4\sqrt{R}\cos^2\sqrt{R}}{R^2}, \tag{3.15}$$

where we used (2.23), (2.24) and (3.9). The dual metric for (3.10) is

$$(ds')^2_{\rm II} = (dx)^2 + R_3'^2 (d\theta^3)^2 + \frac{3R_3'^2 + R_4'^2}{\sqrt{3}} d\theta^3 d\theta^8 + R_4'^2 (d\theta^8)^2, \tag{3.16}$$

where

$$\begin{aligned} R_3'^2 &= \frac{1}{R_3^2} = \frac{4R'}{(X^2+2Y^2)} \coth^2 \sqrt{R'}, \\ R_4'^2 &= \frac{1}{R_4^2} = \frac{4R'}{3(X^2-2Y^2)} \coth^2 \sqrt{R'}. \end{aligned} \tag{3.17}$$

Target space dualities are now generalized version of so-called cigar and trumpet of two dimensional black hole. It is important to note that we have now two kinds of target space dualities. The dilaton transformation is now given by

$$\phi'_{\rm II}(x) = \phi_0^{\rm II} + \ln \frac{3Y^4 \tanh^4 \sqrt{R'} \cosh^2 \sqrt{R'}}{R'^2}. \tag{3.18}$$

Also in the dual theory, we can get the result $\phi_0^{\rm I} = \phi_0^{\rm II}$, repeating similar discussion as before.

Next, let us consider analytic continuation which is an analogy of the transformation from SU(2)/U(1) black hole to SL(2,$\boldsymbol{R}$/U(1) black hole. This is given by $x_k \to ix_k$. This analytic continuation turns SU(3) into SU(2,1). Then (3.1) in the region I will be

$$(ds)^2 = -(dx)^2 - R_{11}^2 (d\theta^3)^2 - \frac{3R_{11}^2 + R_{22}^2}{\sqrt{3}} d\theta^3 d\theta^8 - R_{22}^2 (d\theta^8)^2, \tag{3.19}$$

where

$$\begin{aligned} R_{11}^2 &= \frac{\tanh^2 \sqrt{R}}{2R} \left(\frac{X^2}{2} + Y^2 \right), \\ R_{22}^2 &= \frac{3\tanh^2 \sqrt{R}}{2R} \left(\frac{X^2}{2} - Y^2 \right). \end{aligned} \tag{3.20}$$

However, since an overall sign of metric is trivial, we may always change the sign of it. This corresponds to $k \to -k$ in the action. It is useful to invert the sign. Thus, we will employ

$$(d\hat{s})^2_{\rm I} = (dx)^2 + R_{11}^2 (d\theta^3)^2 + \frac{3R_{11}^2 + R_{22}^2}{\sqrt{3}} d\theta^3 d\theta^8 + R_{22}^2 (d\theta^8)^2, \tag{3.21}$$

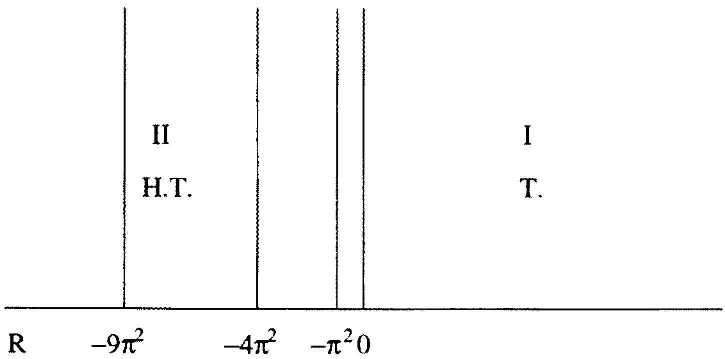

Fig. 2. Classification of target space of gauged SU(2,1) WZW model.

as the metric after analytic continuation, in other words, the metric of SU(2,1)/(U(1)× U(1)) WZW model.

It is interesting to note that if we cut this space with metric (3.21) by the planes $x_5 = x_6 = x_7 = 0$ or $x_4 = x_6 = x_7 = 0$, then the metric of cutted space would coincide with that of SL(2,$\boldsymbol{R}$)/U(1) black hole while $X^2 = 0$ with $x_6 = 0$ or $x_7 = 0$ gives again SU(2)/U(1) black holes! So we can say that this analytically continued WZW model also contains both of SL(2,$\boldsymbol{R}$)/U(1) and SU(2)/U(1) black hole solutions.

On the other hand, in the region II, (3.21) reduces to another metric, because $\tanh^2$ turns into $-\tan^2$. Namely, the metric in II is

$$(d\hat{s})^2_{\rm II} = (dx)^2 + R^2_{33}(d\theta^3)^2 + \frac{3R^2_{33} + R^2_{44}}{\sqrt{3}} d\theta^3 d\theta^8 + R^2_{44}(d\theta^8)^2, \qquad (3.22)$$

where

$$R^2_{33} = \frac{\tan^2\sqrt{R'}}{2R'}\left(\frac{X^2}{2} + Y^2\right),$$
$$R^2_{44} = \frac{3\tan^2\sqrt{R'}}{2R'}\left(\frac{X^2}{2} - Y^2\right). \qquad (3.23)$$

These metrics (3.21) and (3.22) are very similar to that of gauged SU(3) model except interchangings of I and II, R and R'. The radii are again quantized with a periodicity π and give a constraint

$$\sqrt{Y^2 - X^2} \bmod \pi \qquad (3.24)$$

instead of (3.5). By repeating the similar discussion as in the original SU(3)/(U(1)× U(1)) WZW model, we can summarize them in the following figure 2. Notices are the same as before.

In this figure, the reader may ask the sign changing of the constraint, but we have included them in the region II in order to emphasize that (3.24) appears in II. Therefore, $R = -\pi^2$, for example, in the figure means actually $R' = \pi^2$. As the target space duality transformation and dilatons are the same as before except interchanging the regions, $\tan^2$ and $\tanh^2$, R and R', we will not repeat them here.

Before ending this section, let us give a comment on metric on the null line. The metrics have a same form on the null line and is given by

$$(ds)^2_{\mathrm{I,II}} = (d\hat{s})^2_{\mathrm{I,II}} = (dx)^2 + \frac{3Y^2}{4}(d\theta^3)^2 + \frac{\sqrt{3}Y^2}{2} d\theta^3 d\theta^8 - \frac{3Y^2}{4}(d\theta^8)^2 . \quad (3.25)$$

Thus the gauged SU(3) and SU(2,1) WZW models on the null line is just the same! This fact implies that we can move freely from SU(3) theory to SU(2,1) theory passing through the null line continuously and these theories are degenerate on it. Turning to dilaton, though dilatons in each WZW model are different function, they coincide on the null line. Accordingly, we can see that these WZW models are connected. So, if we express this situation roughly, we may say that we can go and go back among these space-times through the null line.

4. Summary

We have described the target space geometry of the gauged SU(3) WZW model and have observed that there are two interesting facts. One is the transition of the target space and the other is the relation between the gauged SU(3) and SU(2,1) WZW models.

The former means the transition of the metric passing through the null line. When we pass the null line, the metric is partially analytically continued from tangential type to hyperbolic tangential one. Of course, since the metric determines the theory, we should regard that these two theories are basically different, although we can pass continuously from the region I to II and vice versa. Accordingly, the null line may be considered as the "horizon" of the gauged SU(3) WZW model in a sense.

And the latter will be important when we perform analytic continuation from SU(3) to SU(2,1). We have seen that the gauged SU(2,1) WZW model has a similar property as SU(3) model. These two WZW models are "reflected" on the "wall" of the null line. Their metrics in any regions are reduced to a certain metric on the null line and thus we may say that the null line is a horizon all the same, also in this case. Therefore, we can interpolate these two WZW models (as well as moving from I to II, and vice versa), while the dynamical mechanism passing through the null line is not revealed at present.

In addition, since SU(2) is a sub-group of SU(3), the discussion about the model in this paper may be seen as an extended version of SU(2)/U(1) black hole. But, as it actually also contains SL(2,$\boldsymbol{R}$)/U(1) black hole, it may be considered as an extended version of these two dimensional black hole theories. However, in the (Lorentzian) SL(2,$\boldsymbol{R}$)/U(1) black hole, there is a Kruskal-like coordinate and so one can discuss causality, but unfortunately, we could not find such convenient coordinate for our WZW models and therefore we avoid the discussion. Note that these two dimensional black holes have only one phase, so the target space transition which we have observed can not appear.

From the discussion throughout the paper, we conjecture that general WZW models have several classes as well as SU(3) model and the similar transition should occur.

To summarize, it will be necessary for us to study further the dynamics passing through the null line in order to clarify physical contents of these gauged SU(3) and SU(2,1) WZW models because such discussions are not given in the paper. It may be interesting to apply our discussions to mirror symmetry, in particular mirror symmetry as a gauge symmetry which is expected to exchange axial and vector gauging, and so on. These topics will be discussed elsewhere.

I would like to express my thanks to Dr. H. Kanno for discussions and T. Yamanoue for workstation service in the department.

Appendix A

In order to avoid ambiguities among references and to clarify notations, we derive Polyakov-Wiegmann's formula [2, 3, 11, 12] in this appendix. We define level k WZW model on a Lie group G $(g \in G)$ as

$$I(g) = \frac{k}{4\pi} I_{NS}(g) + \frac{ik}{6\pi} \Gamma(g), \tag{A.1}$$

where

$$\begin{aligned} I(g)_{NS} &= \int d^2\sigma \ \mathrm{Tr}[U_\mu(g) U^\mu(g)], \\ \Gamma(g) &= \int_B d^3\sigma \ \varepsilon^{\mu\nu\rho} \, \mathrm{Tr}[U_\mu(g) U_\nu(g) U_\rho(g)] \, . \end{aligned} \tag{A.2}$$

σ^i are worldsheet coordinates and $\varepsilon^{\mu\nu\rho}$ is an anti-symmetric tensor ($\varepsilon^{123} = 1$). We set

$$U_\mu(g) = g^{-1}\partial_\mu g, \quad V_\mu(g) = \partial_\mu g \cdot g^{-1}, \tag{A.3}$$

and $\partial B = S^2$. Using $h \in G$,

$$I(gh) = \frac{k}{4\pi} I_{NS}(gh) + \frac{ik}{6\pi} \Gamma(gh) \,. \tag{A.4}$$

Since

$$\begin{aligned} U_\mu(gh) &= h^{-1} g^{-1} \partial_\mu(gh) \\ &= h^{-1} U_\mu(g) h + U_\mu(h), \end{aligned} \tag{A.5}$$

the trace of the product will be

$$\begin{aligned} \mathrm{Tr}[U_\mu(gh) U^\mu(gh)] &= \mathrm{Tr}[h^{-1} U_\mu(g) U^\mu(g) h + h^{-1} U_\mu(g) h U_\mu(h) \\ &\quad + U_\mu(h) h^{-1} U^\mu(g) h + U_\mu(h) U^\mu(h)] \\ &= U_\mu(g) U^\mu(g) + 2U_\mu(g) V^\mu(h) + U_\mu(h) U^\mu(h) \,. \end{aligned} \tag{A.6}$$

Therefore,

$$I_{NS}(gh) = I_{NS}(g) + I_{NS}(h) + 2 \int d^2\sigma \, \mathrm{Tr}[\delta^{\mu\nu} U_\mu(g) V_\nu(h)] \,. \tag{A.7}$$

Similar calculation gives

$$\begin{aligned} \mathrm{Tr}[U_\mu(gh) U_\nu(gh) U_\rho(gh)] &= \mathrm{Tr}[U_\mu(g) U_\nu(g) U_\rho(g) + U_\mu(h) U_\nu(h) U_\rho(h) \\ &\quad + U_\mu(g) V_\nu(h) U_\rho(g) + V_\mu(h) U_\nu(g) V_\rho(h) \\ &\quad + V_\mu(h) V_\nu(h) U_\rho(g) + U_\mu(g) U_\nu(g) V_\rho(h) \\ &\quad + U_\mu(g) V_\nu(h) V_\rho(h) + V_\mu(h) U_\nu(g) V_\rho(h)] \,. \end{aligned} \tag{A.8}$$

Consequently,

$$\begin{aligned} \Gamma(gh) = \Gamma(g) + \Gamma(h) + \int d^3\sigma \, \varepsilon^{\mu\nu\rho} \, \mathrm{Tr}[&U_\mu(g) V_\nu(h) U_\rho(g) + V_\mu(h) U_\nu(g) U_\rho(g) \\ &+ V_\mu(h) V_\nu(h) U_\rho(g) + U_\mu(g) V_\nu(h) V_\rho(h) + U_\mu(g) U_\nu(g) V_\rho(h) \\ &+ V_\mu(h) U_\nu(g) V_\rho(h)] \,. \end{aligned} \tag{A.9}$$

After partial integration, we will obtain

$$\Gamma(gh) = \Gamma(g) + \Gamma(h) - 3 \int d^2\sigma \, \varepsilon^{\nu\rho} \, \mathrm{Tr}[U_\nu(g) V_\rho(h)] \,. \tag{A.10}$$

Then Polyakov–Wiegmann's formula is

$$I(gh) = I(g) + I(h) + \frac{k}{2\pi} \int d^2\sigma P_-^{\mu\nu} \, \mathrm{Tr}[U_\mu(g) V_\nu(h)], \tag{A.11}$$

where

$$P_-^{\mu\nu} = \delta^{\mu\nu} - i\varepsilon^{\mu\nu}. \tag{A.12}$$

Repeating Polyakov–Wiegmann's formula, we will find

$$\begin{aligned} I(ghf) =& I(g) + I(h) + I(f) \\ &+ \frac{k}{2\pi}\int d^2\sigma P_-^{\mu\nu}\,\mathrm{Tr}[U_\mu(g)V_\nu(h) + U_\mu(gh)V_\nu(f)], \end{aligned} \tag{A.13}$$

where $f \in G$. From Euler angle parametrization (see also Appendix B)

$$ghf = e^{i\sum_{i=1}^{r}\alpha_i H^i}\cdot h\cdot e^{i\sum_{i=1}^{r}\gamma_i H^i}, \tag{A.14}$$

where $r = \dim G$, and H^i are bases of Cartan sub-algebra normalized as $\mathrm{Tr}[H^iH^j] = \delta^{ij}$, (A.13) will be reduced to σ-model action.

After some calculations, (A.13) will be written as

$$\begin{aligned} I(ghf) = I(h) &- \frac{k}{4\pi}\int d^2\sigma[\partial_\mu\alpha_i\partial^\mu\alpha_i + \partial_\mu\gamma_i\partial^\mu\gamma_i] \\ &+ \frac{ik}{2\pi}\int d^2\sigma P_-^{\mu\nu}[\partial_\mu\alpha_i\cdot V_\nu^i(h) + \partial_\nu\gamma_i\cdot U_\mu^i(h)] \\ &- \frac{k}{2\pi}\int d^2\sigma P_-^{\mu\nu}\partial_\mu\alpha_i\partial_\nu\gamma_j M_{ij}(h), \end{aligned} \tag{A.15}$$

where repeated roman indices are to be summed and we have used following notations,

$$U_\mu^i(h) = \mathrm{Tr}[H^iU_\mu(h)], \quad V_\mu^i(h) = \mathrm{Tr}[H^iV_\mu(h)] \tag{A.16}$$

and

$$M_{ij}(h) = \mathrm{Tr}[H^ihH^jh^{-1}]. \tag{A.17}$$

Appendix B

In this appendix, we show the calculation of (A.17) in the case of SU(3) because its evaluation is technical. In this case, h is defined by (see Appendix C)

$$h = e^{i(x_4T^4 + x_5T^5 + x_6T^6 + x_7T^7)}, \tag{B.1}$$

where T^k are Gell–Mann matrices. However, in order to simplify the notation, we will drop $i = \sqrt{-1}$ from now on. Its Lie algebra is $[T^i, T^j] = i{f_{ij}}^kT^k$ and the structure constants ${f_{ij}}^k$ are listed below.

TABLE I

Structure constants of SU(3).

i	j	k	$f_{ij}{}^k$	i	j	k	$f_{ij}{}^k$
1	2	3	$2\sqrt{2}$	3	4	5	$\sqrt{2}$
1	4	7	$\sqrt{2}$	3	6	7	$-\sqrt{2}$
1	5	6	$-\sqrt{2}$	4	5	8	$\sqrt{6}$
2	4	6	$\sqrt{2}$	6	7	8	$\sqrt{6}$
2	5	7	$\sqrt{2}$				

$f_{ij}{}^k$ are skew symmetric on its indices.

Let us compute $M_{33}(h)$ as an example. It is defined by

$$M_{33}(h) = \mathrm{Tr}[H^3 h H^3 h^{-1}]\,. \tag{B.2}$$

Now, define

$$f(\alpha) = h(\alpha) H^3 h^{-1}(\alpha) \tag{B.3}$$
$$\equiv e^K H^3 e^{-K}\,, \tag{B.4}$$

where α is a real parameter and

$$K = \alpha(x_4 T^4 + x_5 T^5 + x_6 T^6 + x_7 T^7)\,. \tag{B.5}$$

Of course, $f(\alpha \to 1)$ is what we need. Differentiating (B.4) respect to α, we will obtain

$$\begin{aligned} f' &= i e^K [x_4 T^4 + x_5 T^5 + x_6 T^6 + x_7 T^7, H^3] e^{-K} \\ &= i\sqrt{2} e^K [-x_4 T^5 + x_5 T^4 + x_6 T^7 - x_7 T^6] e^{-K}\,. \end{aligned} \tag{B.6}$$

Differentiating f' again,

$$\begin{aligned} f'' &= 2(x_4^2 + x_5^2 + x_6^2 + x_7^2) e^K H^3 e^{-K} \\ &+ 2\sqrt{3}(x_4^2 + x_5^2 - x_6^2 - x_7^2) e^K H^8 e^{-K} \\ &= 2(x_4^2 + x_5^2 + x_6^2 + x_7^2) f \\ &+ 2\sqrt{3}(x_4^2 + x_5^2 - x_6^2 - x_7^2) e^K H^8 e^{-K}\,. \end{aligned} \tag{B.7}$$

And third derivative gives

$$f''' = 4(2x_4^2 + 2x_5^2 - x_6^2 - x_7^2) f' \tag{B.8}$$

Integrating (B.8) we find that a general solution is

$$f(\alpha) = A + B e^{2\sqrt{2(x_4^2 + x_5^2) - (x_6^2 + x_7^2)}\alpha} + C e^{-2\sqrt{2(x_4^2 + x_5^2) - (x_6^2 + x_7^2)}\alpha}\,. \tag{B.9}$$

Integration constants A, B and C can easily be obtained from initial conditions,

$$\begin{aligned} f(0):\ & H^3 = A + B + C, \\ f'(0):\ & i\sqrt{2}[-x_4T^5 + x_5T^4 + x_6T^7 - x_7T^6] \\ & = 2\sqrt{2(x_4^2 + x_5^2) - (x_6^2 + x_7^2)}(B - C), \\ f''(0):\ & 2\sqrt{3}(x_4^2 + x_5^2 - x_6^2 - x_7^2)H^8 + 2(x_4^2 + x_5^2 + x_6^2 + x_7^2)H^3 \\ & = 2[4(x_4^2 + x_5^2) - 2(x_6^2 + x_7^2)](B + C)\,. \end{aligned} \tag{B.10}$$

After some algebraic works, we will find that

$$\begin{aligned} A =& \frac{(X^2 - 2Y^2)}{4(X^2 - Y^2)}(3H^3 - \sqrt{3}H^8), \\ B =& \frac{(X^2 + 2Y^2)H^3 + \sqrt{3}(X^2 - 2Y^2)H^8}{8(X^2 - Y^2)} \\ & + \frac{i\sqrt{2}(-x_4T^5 + x_5T^4 + x_6T^7 - x_7T^6)}{4\sqrt{R}}, \\ C =& \frac{(X^2 + 2Y^2)H^3 + \sqrt{3}(X^2 - 2Y^2)H^8}{8(X^2 - Y^2)} \\ & - \frac{i\sqrt{2}(-x_4T^5 + x_5T^4 + x_6T^7 - x_7T^6)}{4\sqrt{R}}, \end{aligned} \tag{B.11}$$

where

$$X^2 = 2(x_4^2 + x_5^2),\ Y^2 = x_6^2 + x_7^2\,. \tag{B.12}$$

As we would like to get the quantity M_{33}, we must evaluate the trace of $H^3f(1)$. Namely,

$$M_{33} = \mathrm{Tr}[H^3f(1)]\,. \tag{B.13}$$

Using the matrices in Appendix D, we can arrive at the result

$$\mathrm{Tr}[H^3f(1)] = \frac{1}{4(X^2 - Y^2)}[3(X^2 - 2Y^2) + (X^2 + 2Y^2)\cos 2\sqrt{R}], \tag{B.14}$$

where we have recovered the missing i. The other components of M matrix can be obtained by similar calculations, but we will leave them to the reader as an exercise. The result is

$$\begin{aligned} M_{83} &= \frac{\sqrt{3}(X^2 - 2Y^2)}{4(X^2 - Y^2)}(\cos 2\sqrt{R} - 1), \\ M_{38} &= \frac{\sqrt{3}(X^2 + 2Y^2)}{4(X^2 - Y^2)}(\cos 2\sqrt{R} - 1), \\ M_{88} &= \frac{1}{4(X^2 - Y^2)}[(X^2 + 2Y^2) + 3(X^2 - 2Y^2)\cos 2\sqrt{R}]\,. \end{aligned} \tag{B.15}$$

Appendix C

In this appendix, we explain a little about Euler angle parametrization[10,13]. As is well-known, SU(2) is a rotation group in 3-d Euclid space and is isomorphic to SO(3). SO(3) can be parametrized by Euler angles. In this case, $g \in$ SU(2) can be written (using Pauli matrices σ_i ($i = 1, 2, 3$) as the bases of SU(2) Lie algebra and Euler angles θ_1, θ_2 and θ_3) as

$$g = e^{i\theta_1\sigma_1}e^{i\theta_2\sigma_2}e^{i\theta_3\sigma_1}. \tag{C.1}$$

For another groups G, let $r =$ rank G and N denotes the total number of generators. Then such parametrization (strictly speaking, G is assumed to be an unitary group or its sub-group) is locally

$$g = e^{i\sum_{i=1}^{r}\theta_i H^i} \cdot h(x_a) \cdot e^{i\sum_{j=1}^{r}\varphi_j H^j}, \tag{C.2}$$

where H^i are generators of the Cartan sub-algebra. $h(x_a)$ is any element of G and is independent of θ_i and φ_j. It is convenient to choose

$$h(x_a) = e^{i\sum_{a=1}^{N-2r} x_a T^a}, \tag{C.3}$$

where x_a are real parameters and T^a are specific $N - 2r$ generators outside the Cartan sub-algebra.

Using this form of a group field g, we can write WZW action in the form of σ-model action (in fact, recall that we have performed it in Appendix A).

With the above observation, we set

$$g = e^{i(\theta_1^3 H^3 + \theta_1^8 H^8)}e^{i(x_4T^4 + x_5T^5 + x_6T^6 + x_7T^7)}e^{i(\theta_2^3 H^3 + \theta_2^8 H^8)}, \tag{C.4}$$

in the SU(3) WZW model. The reason why we have chosen $h(x_a) = e^{i(x_4T^4 + x_5T^5 + x_6T^6 + x_7T^7)}$ is that T^4, T^5, T^6 and T^7 give closed commutation relations with H^3 and H^8. The reader may ask whether this parametrization is indeed suitable for SU(3) or not. But, since any elements of SU(3) are in fact obtainable from (C.4) taking certain values for "Euler angles" and are therefore well-parametrized by eight independent parameters, it is unnecessary to worry about it.

Finally, let us comment on the analytic continuation, *i.e.*, $x_k \to ix_k$. Then (C.4) will be

$$g' = e^{i(\theta_1^3 H^3 + \theta_1^8 H^8)}e^{-(x_4T^4 + x_5T^5 + x_6T^6 + x_7T^7)}e^{i(\theta_2^3 H^3 + \theta_2^8 H^8)}, \tag{C.5}$$

where g' is now an element of SU(2,1), which can be shown by direct and lengthy computations.

Appendix D

We take the following Gell–Mann matrices throughout the paper as the bases of SU(3).

$$T^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T^6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$T^3 = H^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^8 = H^8 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

These are normalized as Tr $(T^iT^i) = 1$. H^3 and H^8 generate Cartan sub-algebra.

Appendix E

The scalar curvature is

$$\mathcal{R} = \frac{A+B+C+D}{R^2(R^2+2RY^2-7Y^4)^2}, \tag{E.1}$$

where

$$\begin{aligned} A &= (R+3Y^2)(8R^4-81R^3Y^2-291R^2Y^4-7RY^6+707Y^8), \\ B &= 6\sqrt{R}Y^2(-9R^4+20R^3Y^2+198R^2Y^4 \\ &\quad -116RY^6-413Y^8)\mathrm{cosec}\sqrt{R}\sec\sqrt{R}, \\ C &= 8R(2R+3Y^2)(R^2+2RY^2-7Y^4)^2\sec^2\sqrt{R}, \\ D &= R(2R+3Y^2)(R^2+2RY^2-7Y^4) \\ &\quad \times (3R^2+6RY^2-17Y^4)\sec^2\sqrt{R}\,\mathrm{cosec}^2\sqrt{R}, \\ R &= X^2-Y^2\,. \end{aligned} \tag{E.2}$$

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