

THE BICOVARIANT DIFFERENTIAL CALCULUS ON THE THREE-DIMENSIONAL κ -POINCARÉ GROUP*

P. KOSIŃSKI, M. MAJEWSKI

Department of Theoretical Physics
University of Łódź
ul. Pomorska 149/153, 90-236 Łódź, Poland

P. MAŚLANKA

Department of Functional Analysis
University of Łódź
ul. St. Banacha 22, 90-238 Łódź, Poland

(Received February 9, 1996)

The bicovariant differential calculus on three-dimensional κ -Poincaré group and the corresponding Lie algebra structure are described. The equivalence of this Lie algebra structure and the three-dimensional κ -Poincaré algebra is proved.

PACS numbers: 11.25.Hf

1. Introduction

Recently, considerable interest has been paid to the deformations of groups and algebras of space-time symmetries [1]. In particular, an interesting deformation of the Poincaré algebra [2] as well as group [3] has been introduced which depend on dimensionful deformation parameter κ ; the relevant objects are called κ -Poincaré algebra and κ -Poincaré group, respectively. Their structure was studied in some detail and many of their properties are now well understood. The κ -Poincaré algebra and group for the space-time of any dimension has been defined [4], the realizations of the algebra in terms of differential operators acting on commutative Minkowski as well as momentum spaces were given [5]; the unitary representations of the deformed group were found [6]; the deformed universal covering

* Supported by KBN grant 2 P 302 217 06 p 02.

$ISL(2, \mathbb{C})$ was constructed [7]; the bicrossproduct [8] structure, both of the algebra and group was revealed [9]. The proof of formal duality between κ -Poincaré group and κ -Poincaré algebra was also given, both in two [10] as well as in four dimensions [11].

One of the important problems is the construction of the bicovariant differential calculus on κ -Poincaré group. Using an elegant approach due to Woronowicz [12], the differential calculi on four-dimensional Poincaré group [13], as well as on the Minkowski space [14] were constructed.

In the present paper we briefly sketch the construction of the differential calculus on the three-dimensional κ -Poincaré group. Apart from possible applications to the three dimensional field theory the calculus presented here is very interesting on its own and significantly differs from the calculi defined on four-dimensional [13] and two-dimensional [15] κ -Poincaré group.

It is well known that in most cases of bicovariant differential calculi on the quantum groups the space of (say) left-invariant 1-forms has larger dimensionality than its classical counterpart. However, in many cases it is sufficient to add one additional biinvariant form; the corresponding left-invariant vector field reduces, in the classical limit, to quadratic Casimir operator [15, 16]. In the case under consideration it appears that it is necessary to add further invariant form related to Pauli-Lubanski invariant.

The paper is organized as follows. The remaining part of the introduction is devoted to the description of κ -Poincaré algebra and group and their formal duality. In Section 2 the bicovariant $*$ -calculus on three-dimensional κ -Poincaré group is constructed. In Section 3 we obtain the corresponding Lie algebra and prove its equivalence to the κ -Poincaré algebra. Finally, in Section 4 some conclusions are given. Some calculations are relegated to the Appendix.

The three-dimensional κ -Poincaré group \mathcal{P}_κ is the Hopf $*$ -algebra defined as follows [3]. Consider the universal $*$ -algebra with unity, generated by selfadjoint elements $\Lambda^\mu{}_\nu$, x^μ subject to the following relations

$$\begin{aligned} [x^\mu, x^\nu] &= \frac{i}{\kappa} (\delta_0^\mu x^\nu - \delta_0^\nu x^\mu), \\ [\Lambda^\mu{}_\nu, x^\rho] &= -\frac{i}{\kappa} ((\Lambda^\mu{}_0 - \delta_0^\mu) \Lambda^\rho{}_\nu + (\Lambda^0{}_\nu - \delta_\nu^0) g^{\mu\rho}), \end{aligned} \quad (1.1)$$

here $g_{\mu\rho} = g^{\mu\rho} = \text{diag}(+, -, -)$ is the metric tensor.

The comultiplication, antipode and counit are defined as follows

$$\begin{aligned} \Delta(\Lambda^\mu{}_\nu) &= \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu, \\ \Delta(x^\mu) &= \Lambda^\mu{}_\nu \otimes x^\nu + x^\mu \otimes I, \\ S(\Lambda^\mu{}_\nu) &= \Lambda_\nu{}^\mu, \\ S(x^\mu) &= -\Lambda_\nu{}^\mu x^\nu, \end{aligned}$$

$$\begin{aligned}\varepsilon(\Lambda^\mu{}_\nu) &= \delta^\mu{}_\nu, \\ \varepsilon(x^\mu) &= 0.\end{aligned}\tag{1.2}$$

It is easy to see [9] that \mathcal{P}_κ has the bicrossproduct structure [8]

$$\mathcal{P}_\kappa = T^* \triangleright \blacktriangleleft C(SO(2, 1)), \tag{1.3}$$

where $C(SO(2, 1))$ is the standard Hopf algebra of functions defined over the Lorentz group, while T^* is defined by the relations

$$\begin{aligned}[x^\mu, x^\nu] &= \frac{i}{\kappa}(\delta_0{}^\mu x^\nu - \delta_0{}^\nu x^\mu), \\ \Delta(x^\mu) &= x^\mu \otimes I + I \otimes x^\mu, \\ S(x^\mu) &= -x^\mu, \\ \varepsilon(x^\mu) &= 0.\end{aligned}\tag{1.4}$$

Indeed, it is sufficient to define the following structure functions

$$\begin{aligned}\beta(x^\mu) &= \Lambda^\mu{}_\nu \otimes x^\nu, \\ \Lambda^\mu{}_\nu \triangleleft x^\rho &= [\Lambda^\mu{}_\nu, x^\rho].\end{aligned}\tag{1.5}$$

The three-dimensional κ -Poincaré algebra $\tilde{\mathcal{P}}_\kappa$ was first introduced in the third paper of [2]. We present it below in the Majid and Ruegg basis [9]. It is a quantized universal enveloping algebra in the sense of Drinfeld [18] described by the following relations

$$\begin{aligned}[P_\mu, P_\nu] &= 0, \\ [M, P_0] &= 0, \\ [M, P_k] &= i\varepsilon_{kl}P_l, \\ [M, N_k] &= i\varepsilon_{kl}N_l, \\ [N_1, N_2] &= -iM, \\ [N_i, P_j] &= i\delta_{ij}\left(\frac{\kappa}{2}(1 - e^{-2P_0/\kappa}) + \frac{1}{2\kappa}\vec{P}^2\right) - \frac{i}{\kappa}P_iP_j, \\ \Delta P_0 &= P_0 \otimes I + I \otimes P_0, \\ \Delta P_i &= P_i \otimes e^{-P_0/\kappa} + I \otimes P_i, \\ \Delta M_i &= M_i \otimes I + I \otimes M_i, \\ \Delta N_i &= I \otimes N_i + N_i \otimes e^{-P_0/\kappa} + \frac{1}{\kappa}\varepsilon_{ij}M \otimes P_j, \\ S(P_0) &= -P_0, \\ S(P_i) &= -e^{P_0/\kappa}P_i, \\ S(M_i) &= -M_i, \\ S(N_i) &= -N_i e^{P_0/\kappa} + \frac{1}{\kappa}\varepsilon_{ij}M P_j e^{P_0/\kappa}, \\ \varepsilon(X) &= 0, \quad X = P_\mu, M, N_i,\end{aligned}\tag{1.6}$$

here $\mu = 0, 1, 2$ and $i, j, k = 1, 2$.

Again, we can write

$$\widetilde{\mathcal{P}}_\kappa = T \blacktriangleright \blacktriangleleft U(so(2, 1)) \, , \tag{1.7}$$

where $U(so(2, 1))$ is classical envelopping algebra of $so(2, 1)$ while T is defined as follows

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \, , \\ \Delta P_0 &= P_0 \otimes I + I \otimes P_0 \, , \\ \Delta P_i &= P_i \otimes e^{-P_0/\kappa} + I \otimes P_i \, , \\ S(P_0) &= -P_0 \, , \\ S(P_i) &= -e^{P_0/\kappa} P_i \, , \\ \varepsilon(P_\mu) &= 0 \, . \end{aligned} \tag{1.8}$$

In order to show that (1.7) holds it is sufficient to define [9]

$$\begin{aligned} M \triangleright P_\mu &= [M, P_\mu] \, , \\ N_i \triangleright P_\mu &= [N_i, P_\mu] \, , \\ \delta(M) &= M \otimes I \, , \\ \delta(N_i) &= N_i \otimes e^{-P_0/\kappa} + \frac{1}{\kappa} \varepsilon_{ij} M \otimes P_j \, . \end{aligned} \tag{1.9}$$

It has been shown recently [11] that \mathcal{P}_κ and $\widetilde{\mathcal{P}}_\kappa$ are formally dual to each other. The relevant duality relations read

$$\begin{aligned} \langle \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_n}{}_{\nu_n}, M_{\alpha\beta} \rangle &= i \sum_{k=1}^n (\delta^{\mu_k}{}_{\alpha} g_{\nu_k\beta} - \delta^{\mu_k}{}_{\beta} g_{\nu_k\alpha}) \prod_{l \neq k} \delta^{\mu_l}{}_{\nu_l} \, , \\ \langle : F(x^\mu) :, f(p_\nu) \rangle &= f\Big(i \frac{\partial}{\partial x^\nu}\Big) F(x^\mu)|_{x=0} \, . \end{aligned} \tag{1.10}$$

The normal product $: F(x^\mu) :$ is defined as the one in which all x^0 factors stand leftmost. The variables x^μ on the right-hand side of (1.10) are viewed as commuting ones.

2. Bicovariant \ast -calculus on κ -Poincaré group

Let us recall the main result of Woronowicz theory. Given a Hopf algebra \mathcal{A} and $a \in \mathcal{A}$, we write

$$(\Delta \otimes I) \circ \Delta(a) = (I \otimes \Delta) \circ \Delta(a) = \sum_k a_k \otimes b_k \otimes c_k \tag{2.1}$$

and define the adjoint action on a

$$\mathrm{ad}(a) = \sum_k b_k \otimes S(a_k) c_k. \quad (2.2)$$

According to Woronowicz [12] a bicovariant \ast -calculus is uniquely defined by the choice of right ideal $\mathcal{R} \subset \ker \varepsilon$ which has the following properties

- (i) \mathcal{R} is ad-invariant, i.e. $\mathrm{ad}(\mathcal{R}) \subset \mathcal{R} \otimes \mathcal{A}$,
- (ii) for any $a \in \mathcal{R}$, $S(a)^* \in \mathcal{R}$.

The standard calculus in the commutative case is obtained by choosing $\mathcal{R} = (\ker \varepsilon)^2$. Below (Theorem 1) we construct the relevant ideal for κ -Poincaré group. It can be obtained as follows. One starts with $(\ker \varepsilon)^2$. However, due to the noncommutativity, it fails to satisfy (i). It appears that this can be cured by adding to the generators of $(\ker \varepsilon)^2$ some terms linear in generators of $\ker \varepsilon$ (and proportional to $1/\kappa$); new generators form a (not completely reducible) multiplet under the adjoint action of \mathcal{P}_κ . Moreover, (ii) is also satisfied. The whole procedure would be rather straightforward were it not for the fact that the ideal obtained is too large — it produces the calculus which has smaller dimensionality than its classical counterpart. Therefore, we are forced to consider a smaller ideal. This can be achieved by subtracting some subrepresentations from the representation spanned by the generators under consideration. We performed this subtraction in the most economical way: one trace and one completely antisymmetric representations have been subtracted.

The final result can be summarized as follows.

Let us introduce the following notation

$$\begin{aligned} \Delta^\mu{}_\nu &= \Lambda^\mu{}_\nu - \delta^\mu{}_\nu, \\ \Delta^\mu{}_\nu{}^\alpha &\equiv x^\alpha (\Lambda^\mu{}_\nu - \delta^\mu{}_\nu) - \frac{i}{\kappa} (\delta^\alpha{}_\nu (\Lambda^{\mu\alpha} - g^{\mu\alpha}) \\ &\quad + \Lambda^{\mu 0} (\Lambda^\alpha{}_\nu - \delta^\alpha{}_\nu)), \\ x^{\mu\nu} &\equiv x^\mu x^\nu + \frac{i}{\kappa} (g^{\mu\nu} x^0 - g^{0\mu} x^\nu). \end{aligned} \quad (2.3)$$

Then the following theorem holds

Theorem I *Let $\mathcal{R} \subset \ker \varepsilon$ be the right ideal generated by the following elements*

$$\begin{aligned}\Delta^\alpha{}_\beta \Delta^\mu{}_\nu, \\ \tilde{\Delta}^{\mu\nu\alpha} &\equiv \Delta^{\mu\nu\alpha} - \frac{1}{6} \varepsilon^{\mu\nu\alpha} \varepsilon_{\rho\sigma\gamma} \Delta^{\rho\sigma\gamma}, \\ \tilde{x}^{\mu\nu} &\equiv x^{\mu\nu} - \frac{1}{3} g^{\mu\nu} x^\alpha{}_\alpha.\end{aligned}$$

Then \mathcal{R} has the following properties

- (i) \mathcal{R} is ad-invariant, $\text{ad}(\mathcal{R}) \subset \mathcal{R} \otimes \mathcal{P}_\kappa$,
- (ii) for any $a \in \mathcal{R}$, $S(a)^* \in \mathcal{R}$.
- (iii) $\ker \varepsilon / \mathcal{R}$ is spanned by the following elements

$$\begin{aligned}x^\mu; \quad \Delta^\mu{}_\nu, \quad \mu < \nu; \quad \varphi \equiv x^\alpha{}_\alpha = x^2 + \frac{2i}{\kappa} x^0, \\ \Delta \equiv \varepsilon_{\mu\nu\alpha} \Delta^{\mu\nu\alpha}.\end{aligned}$$

We shall omit the proof of this theorem which goes along the same lines as in the four-dimensional case (see the second paper of [13]) and is long. Let us only note the following:

- (a) $\Delta^\mu{}_\nu{}^\alpha$ and $x^{\mu\nu}$ are 'improved' generators of $(\ker \varepsilon)^2$ ($\Delta^\alpha{}_\beta \Delta^\mu{}_\nu$ is the same as in the classical case) which form the (not completely reducible) multiplet under adjoint action of \mathcal{P}_κ (see Appendix);
- (b) the ideal generated by $\Delta^\alpha{}_\beta \Delta^\mu{}_\nu$, $\Delta^\mu{}_\nu{}^\alpha$, $x^{\mu\nu}$ equals $\ker \varepsilon$; in order to obtain reasonable (in the sense that it contains all differentials dx^μ , $d\Lambda^\alpha{}_\beta$) calculus we have subtracted the trace of $x^{\mu\nu}$ and completely antisymmetric part of $\Delta^\mu{}_\nu{}^\alpha$;
- (c) it is easy to conclude from (iii) that our calculus is eight-dimensional.

Having established the structure of \mathcal{R} we can now follow closely the Woronowicz construction. First, we define the left-invariant 1-forms. The basis of the space of left-invariant 1-forms is spanned by the elements of the form $\pi r^{-1}(I \otimes a_i)$, where $\{a_i\}$ is a basis in $\ker \varepsilon / \mathcal{R}$, the operation r^{-1} is defined by

$$r^{-1}(a \otimes b) = (a \otimes I)(S \otimes I)\Delta(b) \tag{2.6}$$

and the operation π is defined by

$$\pi \left(\sum a_k \otimes b_k \right) = \sum a_k db_k, \tag{2.7}$$

where

$$\mathcal{P}_\kappa \otimes \mathcal{P}_\kappa \ni \sum a_k \otimes b_k$$

is such an element that

$$\sum a_k b_k = 0.$$

We readily obtain

$$\begin{aligned}\omega^\mu{}_\nu &\equiv \pi r^{-1}(I \otimes (\Lambda^\mu{}_\nu - \delta^\mu{}_\nu)) = \Lambda_\alpha{}^\mu d\Lambda^\alpha{}_\nu, \\ \omega^\alpha &\equiv \pi r^{-1}(I \otimes x^\alpha) = \Lambda_\mu{}^\alpha dx^\mu, \\ \omega &\equiv \pi r^{-1}(I \otimes \varphi) = d\varphi - 2x_\mu dx^\mu, \\ \Omega &\equiv \pi r^{-1}(I \otimes \Delta) = \varepsilon_{\nu\alpha\beta} \Lambda_\sigma{}^\nu \omega^\beta \Lambda^{\sigma\alpha} + \frac{2i}{\kappa} \varepsilon_{o\alpha\beta} \omega^{\alpha\beta}.\end{aligned}\quad (2.8)$$

The next step is to find the commutation rules between the invariant forms and the generators of \mathcal{P}_κ . This is again straightforward and can be briefly explained as follows.

We write

$$b\omega_i - \omega_i b = \pi r^{-1} r[(b \otimes I)r^{-1}(I \otimes a_i) - r^{-1}(I \otimes a_i)(I \otimes b)],$$

where

$$r(a \otimes b) = (a \otimes I)\Delta(b).$$

Due to the formulae [12]

$$r((a \otimes I)q) = (a \otimes I)r(q), \quad r(q(I \otimes a)) = r(q)\Delta(a)$$

we get

$$r[(b \otimes I)r^{-1}(I \otimes a_i) - r^{-1}(I \otimes a_i)(I \otimes b)] = b \otimes a_i - (I \otimes a_i)\Delta(b). \quad (2.9)$$

Then we use the relations describing our ideal \mathcal{R} to simplify the right-hand side and apply again $\pi \circ r^{-1}$. The detailed calculations results in the following formulae

$$\begin{aligned}[\Lambda^\mu{}_\nu, \omega^\alpha{}_\beta] &= 0, \\ [x^\alpha, \omega^\mu{}_\nu] &= -\frac{i}{\kappa} (\delta_\nu^0 \Lambda^\alpha{}_\rho \omega^{\mu\rho} + \delta_0^\mu \Lambda^\alpha{}_\rho \omega^\rho{}_\nu - \Lambda^\alpha{}_\nu \omega^\mu{}_0 - \Lambda^{\alpha\mu} \omega^0{}_\nu) \\ &\quad - \frac{1}{6} \varepsilon^\mu{}_\nu{}^\beta \Lambda^\alpha{}_\beta \Omega, \\ [\Lambda^\mu{}_\nu, \omega^\alpha] &= -\frac{i}{\kappa} (\delta_\nu^0 \Lambda^\mu{}_\rho \omega^{\rho\alpha} + \Lambda^\mu{}_0 \omega^\alpha{}_\nu) - \frac{1}{6} \varepsilon^\rho{}_\nu{}^\alpha \Lambda^\mu{}_\rho \Omega, \\ [x^\mu, \omega^\alpha] &= -\frac{1}{3} \Lambda^{\mu\alpha} \omega + \frac{i}{\kappa} \Lambda^{\mu\alpha} \omega^0 - \frac{i}{\kappa} \delta_0^\alpha \Lambda^\mu{}_\rho \omega^\rho, \\ [\Lambda^\mu{}_\nu, \omega] &= \frac{3}{\kappa^2} \Lambda^\mu{}_\rho \omega^\rho, \\ [x^\mu, \omega] &= \frac{3}{\kappa^2} \Lambda^\mu{}_\rho \omega^\rho{}_\nu, \\ [\Lambda^\mu{}_\nu, \Omega] &= 0, \\ [x^\mu, \Omega] &= \frac{3}{\kappa^2} \varepsilon_{\beta\rho\sigma} \Lambda^{\mu\beta} \omega^{\rho\sigma}.\end{aligned}\quad (2.10)$$

Let us now define the right action of \mathcal{P}_κ on 1-forms [12]

$$\Gamma \Delta(ad b) = \Delta(a)(d \otimes I) \Delta(b) .$$

Simple calculations give

$$\begin{aligned} \Gamma \Delta(\omega^\mu{}_\nu) &= \omega^\rho{}_\sigma \otimes \Lambda_\rho{}^\mu \Lambda^\sigma{}_\nu , \\ \Gamma \Delta(\omega^\mu) &= \omega^\rho{}_\sigma \otimes \Lambda_\rho{}^\mu x^\sigma + \omega^\rho \otimes \Lambda_\rho{}^\mu , \\ \Gamma \Delta(\omega) &= \omega \otimes I , \\ \Gamma(\Omega) &= \Omega \otimes I \end{aligned} \tag{2.11}$$

which gives the following right-invariant forms

$$\begin{aligned} \eta^\mu{}_\nu &= \omega^\beta{}_\gamma \Lambda^\mu{}_\beta \Lambda_\nu{}^\gamma , \\ \eta^\mu &= -\omega^\beta{}_\gamma \Lambda_\rho{}^\mu x^\rho \Lambda^\mu{}_\beta + \omega^\beta \Lambda^\mu{}_\beta , \\ \eta &= \omega , \\ \Theta &= \Omega . \end{aligned} \tag{2.12}$$

This concludes the description of bimodule Γ of 1-forms on \mathcal{P}_κ . External algebra can be now constructed as follows [12]. On $\Gamma^{\otimes 2}$ we define a bimodule homomorphism σ such that

$$\sigma(\omega \otimes_{\mathcal{P}_\kappa} \eta) = \eta \otimes_{\mathcal{P}_\kappa} \omega \tag{2.13}$$

for any left-invariant $\omega \in \Gamma$ and any right-invariant $\eta \in \Gamma$. Then by definition

$$\Gamma^{\wedge 2} = \frac{\Gamma^{\otimes 2}}{\ker(\mathbf{f} - \sigma)} . \tag{2.14}$$

Higher external power of Γ can be constructed in a similar way [12]. The result of action of σ on our forms is given in Appendix. Finally, after long analysis we obtain the following set of relations

$$\begin{aligned} \omega^\mu{}_\nu \wedge \omega^\alpha{}_\beta + \omega^\alpha{}_\beta \wedge \omega^\mu{}_\nu &= 0 , \\ \omega \wedge \omega &= 0 , \\ \Omega \wedge \Omega &= 0 , \\ \omega \wedge \Omega + \Omega \wedge \omega &= 0 , \\ \Omega \wedge \omega^\mu{}_\nu + \omega^\mu{}_\nu \wedge \Omega &= 0 , \\ \omega^\mu{}_\nu \wedge \omega + \omega \wedge \omega^\mu{}_\nu - \frac{3}{\kappa^2} \omega^\sigma{}_\nu \wedge \omega_\sigma{}^\mu &= 0 , \\ \omega^\mu{}_\nu \wedge \omega^\alpha + \omega^\alpha \wedge \omega^\mu{}_\nu + \frac{i}{\kappa} \delta^0{}_\nu \omega^\alpha{}_\rho \wedge \omega^{\mu\rho} + \frac{i}{\kappa} \delta_0{}^\mu \omega^\alpha{}_\rho \wedge \omega^{\rho\nu} &= 0 , \end{aligned}$$

$$\begin{aligned}
& -\frac{i}{\kappa}\omega^\alpha{}_\nu \wedge \omega^\mu{}_0 - \frac{i}{\kappa}\omega^{\alpha\mu} \wedge \omega_{0\nu} = 0, \\
& \omega^\mu \wedge \Omega + \Omega \wedge \omega^\mu + \frac{3}{\kappa^2}\varepsilon^\mu{}_{\rho\beta}\omega_\nu{}^\beta \wedge \omega^{\rho\nu} = 0, \\
& \omega \wedge \omega^\mu + \omega^\mu \wedge \omega - \frac{3}{\kappa^2}\omega^{\mu\rho} \wedge \omega_\rho = 0, \\
& \omega^\mu \wedge \omega^\nu + \omega^\nu \wedge \omega^\mu + \frac{i}{\kappa}\delta_0{}^\nu\omega^{\mu\rho} \wedge \omega_\rho + \frac{i}{\kappa}\delta_0{}^\mu\omega^{\nu\rho} \wedge \omega_\rho = 0. \quad (2.15)
\end{aligned}$$

The basis in $\Gamma^{\wedge 2}$ consists of the following elements

$$\begin{aligned}
& \omega^{\alpha\beta} \wedge \omega^{\mu\nu} \quad (\alpha < \beta, \quad \mu < \nu, \quad (\alpha\beta) \neq (\mu\nu), \quad \alpha < \mu), \\
& \omega^{\alpha\beta} \wedge \omega^\mu, \quad \omega^{\alpha\beta} \wedge \omega, \quad \omega^{\alpha\beta} \wedge \Omega, \quad \omega^\alpha \wedge \omega^\mu \quad (\alpha < \mu), \\
& \omega^\alpha \wedge \omega, \quad \omega^\alpha \wedge \Omega, \quad \omega \wedge \Omega
\end{aligned}$$

and $\dim \Gamma^{\wedge 2} = \binom{8}{2}$.

The next step is to introduce the $*$ -operation. Due to Theorem I this can be done consistently; moreover, it is sufficient to consider 1-forms. One gets

$$\begin{aligned}
(\omega^\mu{}_\nu)^* &= \omega^\mu{}_\nu, \\
(\omega^\mu)^* &= \omega^\mu - \frac{i}{\kappa}\omega^\mu{}_0, \\
\omega^* &= -\omega, \\
\Omega^* &= -\Omega. \quad (2.16)
\end{aligned}$$

To complete our exterior calculus we derive the Cartan–Maurer equations. The existence of external derivative is granted by Theorem 4.1 of [12]. (2.8) imply then the following formulae for external derivatives of left-invariant forms

$$\begin{aligned}
d\omega^\mu{}_\nu &= \omega_\rho{}^\mu \wedge \omega^\rho{}_\nu, \\
d\omega^\mu &= \omega_\rho{}^\mu \wedge \omega^\rho, \\
d\omega &= 0, \\
d\Omega &= 0. \quad (2.17)
\end{aligned}$$

3. The Lie algebra-like structure

Let us derive the counterpart of the classical Lie algebra. To this end we introduce the counterparts of the left-invariant fields. They are defined by the formula

$$da = \frac{1}{2}(\chi_{\mu\nu} * a)\omega^{\mu\nu} + (\chi_\mu * a)\omega^\mu + (\chi * a)\omega + (\lambda * a)\Omega \quad (3.1)$$

where, for any linear functional φ on \mathcal{P}_κ ,

$$\varphi * a \equiv (I \otimes \varphi) \Delta(a). \quad (3.2)$$

The product of two functionals φ_1, φ_2 is defined by the standard duality relation

$$\varphi_1 \varphi_2(a) \equiv (\varphi_1 \otimes \varphi_2) \Delta(a). \quad (3.3)$$

In order to find the Lie algebra structure, we apply the external derivative to both sides of (3.1), we use $d^2a = 0$ on the left-hand side and calculate the right-hand side using (2.17) and again (3.1). Nullifying the coefficients in front of basis elements of $\Gamma^{\wedge 2}$, we find the quantum Lie algebra

$$\begin{aligned} [m, l_i] &= \left(1 + \frac{i}{\kappa} \chi_0 - \frac{3}{\kappa^2} \chi\right) \varepsilon_{ik} l_k - \frac{i}{\kappa} \chi_i m + \frac{6}{\kappa^2} \lambda \chi_i, \\ [m_i, l_j] &= -\left(1 + \frac{2i}{\kappa} \chi_0 - \frac{3}{\kappa^2} \chi\right) \varepsilon_{ij} m + \frac{6}{\kappa^2} \varepsilon_{ij} \lambda \chi_0, \\ [m, \chi_0] &= 0, \\ [m, \chi_i] &= \left(1 + \frac{i}{\kappa} \chi_0 - \frac{3}{\kappa^2} \chi\right) \varepsilon_{ik} \chi_k, \\ [l_i, \chi_0] &= \left(1 + \frac{i}{\kappa} \chi_0 - \frac{3}{\kappa^2} \chi\right) \chi_i, \\ [l_i, \chi_k] &= \delta_{ik} \left(1 + \frac{i}{\kappa} \chi_0 - \frac{3}{\kappa^2} \chi\right) \chi_0, \\ [\chi_\mu, \chi_\nu] &= 0 \end{aligned} \quad (3.4)$$

while λ and χ commute with all generators; here $m = \chi_{12}$, $l_i = \chi_{i0}$.

Finally, the involution for χ 's and λ can be defined using (2.16) together with the Woronowicz duality relations [12], which in our case, read

$$\begin{aligned} \langle \omega^{\mu\nu}, \chi_{\alpha\beta} \rangle &= \delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu, \\ \langle \omega^\nu, \chi_\mu \rangle &= \delta_\mu^\nu, \\ \langle \omega, \chi \rangle &= 1, \\ \langle \Omega, \lambda \rangle &= 1. \end{aligned} \quad (3.5)$$

the remaining brackets being vanishing. If one puts

$$\langle \psi, \eta^* \rangle = -\langle \psi^*, \eta \rangle$$

one readily gets

$$\begin{aligned} \chi_\mu^* &= -\chi_\mu, \\ m^* &= -m, \\ l_i^* &= -l_i - \frac{i}{\kappa} \chi_i, \\ \chi^* &= \chi, \\ \lambda^* &= \lambda. \end{aligned} \quad (3.6)$$

Having our quantum Lie algebra constructed, we can now pose the question what is the relation between our functionals and the elements of the κ -Poincaré algebra $\tilde{\mathcal{P}}_\kappa$.

It has been shown [11] that \mathcal{P}_κ and $\tilde{\mathcal{P}}_\kappa$ are formally dual. Therefore we expect our functionals to be expressible in terms of the elements of $\tilde{\mathcal{P}}_\kappa$.

To show this let us note that, in the notation introduced in (1.6) the following substitutions reproduce (3.4)

$$\begin{aligned}\chi_0 &= -i\left(\kappa \operatorname{sh}\left(\frac{P_0}{\kappa}\right) + \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa}\right), \\ \chi_i &= -ie^{P_0/\kappa} P_i, \\ \chi &= -\frac{1}{6}\left(4\kappa^2 \operatorname{sh}^2\left(\frac{P_0}{2\kappa}\right) - \bar{P}^2 e^{P_0/\kappa}\right), \\ \lambda &= -\frac{1}{6}\left(\left(\kappa \operatorname{sh}\left(\frac{P_0}{\kappa}\right) + \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa}\right)M + (P_1 N_2 - P_2 N_1) e^{P_0/\kappa}\right), \\ m &= -ie^{P_0/\kappa} M - \frac{6i}{\kappa} \lambda, \\ l_i &= -ie^{P_0/\kappa} N_i.\end{aligned}\tag{3.7}$$

Then, from the Woronowicz theory, it follows that the coproducts of the functionals φ_i ($\varphi_i = \chi_{\mu\nu}, \chi_\nu, \chi, \lambda$) can be written in the form

$$\Delta\varphi_i = \sum_j \varphi_j \otimes f_{ji} + I \otimes \varphi_i,\tag{3.8}$$

where f_{ji} are the functionals entering commutation rules between the left-invariant forms and elements of \mathcal{P}_κ

$$\omega_j a = \sum_i (f_{ji} * a) \omega_i.\tag{3.9}$$

They are therefore calculable, in principle at least, in terms of what we already know. In order to check (3.7) we have first to compare the coproducts. Using (1.6) and (3.7) one obtains

$$\begin{aligned}\Delta\chi_0 &= \chi_0 \otimes \left(\operatorname{ch}\left(\frac{P_0}{\kappa}\right) + \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa}\right) + \chi_i \otimes \frac{1}{\kappa} P_i \\ &\quad + \chi \otimes \frac{3i}{\kappa} \left(\operatorname{sh}\left(\frac{P_0}{\kappa}\right) + \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa}\right) + I \otimes \chi_0 \\ \Delta\chi_i &= \chi_i \otimes I + \chi_0 \otimes \frac{1}{\kappa} P_i e^{P_0/\kappa} + \chi \otimes \frac{3i}{\kappa^2} P_i e^{P_0/\kappa} + I \otimes \chi_i\end{aligned}$$

$$\begin{aligned}
\Delta\chi &= \chi_0 \otimes \left(-\frac{i}{3} \right) \left(\kappa \operatorname{sh} \left(\frac{P_0}{\kappa} \right) + \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa} \right) + \chi_i \otimes \left(-\frac{i}{3} \right) P_i \\
&\quad + \chi \otimes \left(\operatorname{ch} \left(\frac{P_0}{\kappa} \right) + \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa} \right) + I \otimes \chi \\
\Delta\lambda &= \lambda \otimes \left(\operatorname{ch} \left(\frac{P_0}{\kappa} \right) - \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa} \right) + \chi_j \otimes \left(-\frac{i}{6\kappa} P_j M - \frac{i}{6} \varepsilon_{jk} N_k \right) \\
&\quad + \chi_0 \otimes \left[-\frac{i}{6} \left(\kappa \operatorname{ch} \left(\frac{P_0}{\kappa} \right) + \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa} \right) M - \frac{i}{6\kappa} \varepsilon_{jk} P_j N_k e^{P_0/\kappa} \right] \\
&\quad + \chi \otimes \frac{1}{2\kappa^2} \left[\left(\kappa \operatorname{sh} \left(\frac{P_0}{\kappa} \right) + \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa} \right) M + \varepsilon_{jk} P_j N_k e^{P_0/\kappa} \right] \\
&\quad + l_k \otimes \left(-\frac{i}{6} \right) \varepsilon_{jk} P_j + m \otimes \left(-\frac{i}{6} \right) \left(\kappa \operatorname{sh} \left(\frac{P_0}{\kappa} \right) - \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa} \right) \\
&\quad + I \otimes \lambda \\
\Delta l_i &= l_i \otimes I + \chi_0 \otimes \frac{1}{\kappa} e^{P_0/\kappa} N_i + \chi \otimes \frac{3i}{\kappa^2} e^{P_0/\kappa} N_i + m \otimes \frac{1}{\kappa} \varepsilon_{ik} e^{P_0/\kappa} P_k \\
&\quad + \lambda \otimes \frac{6i}{\kappa^2} \varepsilon_{ik} e^{P_0/\kappa} P_k + I \otimes l_i \\
\Delta m &= m \otimes \left(\operatorname{ch} \left(\frac{P_0}{\kappa} \right) + \frac{\bar{P}^2}{2\kappa^2} e^{P_0/\kappa} \right) \\
&\quad + \chi_0 \otimes \left[\frac{1}{\kappa} \left(\operatorname{ch} \left(\frac{P_0}{\kappa} \right) - \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa} \right) M - \frac{1}{\kappa^2} \varepsilon_{jk} P_j N_k e^{P_0/\kappa} \right] \\
&\quad + \chi \otimes \frac{3i}{\kappa^3} \left(\left(\kappa \operatorname{ch} \left(\frac{P_0}{\kappa} \right) - \frac{\bar{P}^2}{2\kappa} e^{P_0/\kappa} \right) M - \varepsilon_{jk} P_j N_k e^{P_0/\kappa} \right) \\
&\quad + \lambda \otimes \left(\frac{6i}{\kappa} \right) \left(\operatorname{sh} \left(\frac{P_0}{\kappa} \right) + \frac{\bar{P}^2}{2\kappa^2} e^{P_0/\kappa} \right) + \chi_j \otimes \left(-\frac{1}{\kappa^2} P_j M - \frac{1}{\kappa} \varepsilon_{jk} N_k \right) \\
&\quad + l_k \otimes \left(-\frac{1}{\kappa} \right) \varepsilon_{jk} P_j + I \otimes m. \tag{3.10}
\end{aligned}$$

Formulae (3.10) have the expected form (3.9). To show that the functionals f_{ji} in (3.8), (3.9) coincide with those defined by (3.10), let us note the following. The coassociativity implies the relations [12]

$$\Delta f_{ji} = \sum_k f_{jk} \otimes f_{ki}. \tag{3.11}$$

The same relations must hold true for the functionals appearing on the right-hand sides of (3.10) (the coproduct for $\tilde{\mathcal{P}}_\kappa$ is consistent and coassociative

as well). Therefore it is sufficient to check that the two sets of functionals f_{ji} defined above take the same values on generators of \mathcal{P}_κ . The relevant values for the functionals defined in (3.9) are readily obtained by using the explicit form of the commutation rules (2.10). On the other hand, for the functionals defined by (3.10) we use the duality $\mathcal{P}_\kappa \Longleftrightarrow \tilde{\mathcal{P}}_\kappa$ established in [11]. As it was mentioned in the introduction, both \mathcal{P}_κ and $\tilde{\mathcal{P}}_\kappa$ have the bicrossproduct structure. From the bicrossproduct theory it follows that the generic element of \mathcal{P}_κ (resp. $\tilde{\mathcal{P}}_\kappa$) can be written as $X \otimes \Lambda$ (resp. $P \otimes M$) where X (resp. P) is an arbitrary element of T^* (resp. T) while Λ (resp. M) — an arbitrary element of $C(SO(2, 1))$ (resp. $U(so(2, 1))$). The duality relations can be written as

$$\langle X \otimes \Lambda, P \otimes M \rangle = \langle X, P \rangle \langle \Lambda, M \rangle. \quad (3.12)$$

Using (3.12) together with (1.10) we have checked that both definitions of functionals f_{ji} indeed coincide. In order to complete the proof of (3.7) it is now sufficient to compare the values of both sides on generators of \mathcal{P}_κ . For the left-hand side we use (3.1) while the right-hand side is calculated with the help of duality relations (1.10) and (3.12).

Thus we have shown that the functionals appearing in the Woronowicz formulation of differential calculus are formally expressible in terms of elements of $\tilde{\mathcal{P}}_\kappa$. In particular, χ is proportional to the first Casimir operator, while λ provides a deformation of Pauli-Lubanski invariant.

As in the four-dimensional case the relation between χ and χ_μ does not follow from Cartan-Maurer equations. However, contrary to the four-dimensional case, the relation

$$\lambda = \frac{1}{12} \varepsilon^{\mu\nu\alpha} \chi_\mu \chi_{\nu\alpha}, \quad (3.13)$$

which is readily obtainable from (3.7), is not derivable from Cartan-Maurer equations in contrast with its four-dimensional counterpart.

4. Conclusions

We have constructed bicovariant \ast -calculi on three-dimensional κ -Poincaré group. The starting point was the Woronowicz theory of differential calculi on quantum groups. The main ingredience of this approach is the choice of right ideal in $\ker \varepsilon$ which is invariant under the adjoint action of the group. In the classical case the ideal under consideration is $(\ker \varepsilon)^2$. In order to obtain as slight as possible deformation of classical calculus we have started with the generators of $(\ker \varepsilon)^2$. However, it appeared that they do not form a multiplet under the action of the κ -Poincaré group. To cure this we

have modified them by adding the appropriate, κ -dependent terms. Due to the noncommutativity, the ideal generated in this way coincided with the whole $\ker \varepsilon$. Therefore, it appeared necessary to subtract from the new generators some ad-invariant terms. As a consequence the resulting calculus contains more invariant forms than its classical counterpart. This results in increasing the dimensions of the relevant *Lie algebra*. The additional elements provide the deformations of mass squared invariant and Pauli-Lubanski invariant. The Woronowicz theory provides us with a unique generalization of the notion of left-invariant vector fields. We have shown that the relevant functionals are expressible in terms of generators of κ -Poincaré algebra, the number of the latter being equal to the dimension of the classical Poincaré algebra. Therefore, there must be relation between the Woronowicz functionals. In our case the functionals χ and λ are expressible in terms of the functionals χ_μ and $\chi_{\mu\nu}$.

Appendix

A.1. In this part of the Appendix we give the explicit formulae for the adjoint action of the \mathcal{P}_κ on the elements Δ^α_β , Δ^μ_ν , $\Delta^{\mu\nu\alpha}$ and $x^{\alpha\beta}$

$$\begin{aligned} \text{ad}(\Delta^\alpha_\beta \Delta^\mu_\nu) &= \Delta^\rho_\sigma \Delta^\gamma_\delta \otimes \Lambda_\rho^\mu \Lambda^\sigma_\nu \Lambda_\gamma^\alpha \Lambda^\delta_\beta, \\ \text{ad}(\Delta^{\mu\nu\alpha}) &= \Delta^\rho_\sigma \Delta^\beta_\gamma \otimes \Lambda_\rho^\mu \Lambda_\beta^\alpha \Lambda^{\sigma\nu} x^\gamma \\ &\quad + \Delta^{\rho\sigma\beta} \otimes \Lambda_\rho^\mu \Lambda_\beta^\alpha \Lambda_\sigma^\nu, \\ \text{ad}(x^{\alpha\beta}) &= x^{\mu\nu} \otimes \Lambda_\mu^\alpha \Lambda_\nu^\beta + (\Delta^{\mu\rho\nu} + \Delta^{\nu\rho\mu}) \otimes \Lambda_\mu^\alpha \Lambda_\nu^\beta x_\rho \\ &\quad + \Delta^\mu_\rho \Delta^\nu_\sigma \otimes \Lambda_\mu^\alpha \Lambda_\nu^\beta \left(x^\rho x^\sigma - \frac{i}{\kappa} g^{\rho\sigma} x^0 + \frac{i}{\kappa} g^{0\sigma} x^\rho \right). \quad (\text{A1}) \end{aligned}$$

From (A1) we conclude that $(\Delta^\alpha_\beta \Delta^\mu_\nu, \Delta^{\mu\nu\alpha}, x^{\mu\nu})$ span a linear ad-invariant set. Moreover, with respect to the Lorentz part of κ -Poincaré group they transform as the corresponding tensors; note also that only the symmetric (with respect to μ, ν) part of $\Delta^{\mu\alpha\nu}$ enters the transformation rule for $x^{\alpha\beta}$. Therefore the linear set spanned by $\Delta^\alpha_\beta \Delta^\mu_\nu$, $\tilde{\Delta}^{\mu\nu\alpha}$ and $\tilde{x}^{\mu\nu}$ which is obtained by subtracting the completely antisymmetric subrepresentation from $\Delta^{\mu\nu\alpha}$ and the scalar (trace) subrepresentation from $x^{\mu\nu}$ is also ad-invariant. By Lemma 1.7 of [12] the ideal \mathcal{R} is ad-invariant.

A.2. The action of σ is

$$\begin{aligned} \sigma(\omega^\mu_\nu \otimes \omega^\alpha_\beta) &= \omega^\alpha_\beta \otimes \omega^\mu_\nu, \\ \sigma(\omega^\mu_\nu \otimes \omega) &= \omega \otimes \omega^\mu_\nu, \\ \sigma(\omega^\mu \otimes \omega) &= \omega \otimes \omega^\mu, \end{aligned}$$

$$\sigma(\omega \otimes \omega) = \omega \otimes \omega,$$

$$\sigma(\Omega \otimes \omega) = \omega \otimes \Omega,$$

$$\sigma(\Omega \otimes \Omega) = \Omega \otimes \Omega,$$

$$\sigma(\omega \otimes \Omega) = \Omega \otimes \omega,$$

$$\sigma(\omega^\mu{}_\nu \otimes \Omega) = \Omega \otimes \omega^\mu{}_\nu,$$

$$\sigma(\omega^\mu \otimes \Omega) = \Omega \otimes \omega^\mu,$$

$$\sigma(\Omega \otimes \omega^\mu{}_\nu) = \omega^\mu{}_\nu \otimes \Omega,$$

$$\sigma(\Omega \otimes \omega^\mu) = \omega^\mu \otimes \Omega - \frac{3}{\kappa^2} \varepsilon_{\beta\rho\nu} \omega^{\mu\beta} \otimes \omega^{\rho\nu},$$

$$\begin{aligned} \sigma(\omega^\mu{}_\nu \otimes \omega^\alpha) &= \omega^\alpha \otimes \omega^\mu{}_\nu + \frac{i}{\kappa} (\delta_\nu^0 \omega^\alpha{}_\rho \otimes \omega^{\mu\rho} \\ &\quad + \delta^\mu{}_0 \omega^\alpha{}_\rho \otimes \omega^{\rho\nu} - \omega^\alpha{}_\nu \otimes \omega^{\mu 0} - \omega^{\alpha\mu} \otimes \omega^0{}_\nu) \\ &\quad + \frac{1}{6} \varepsilon^\mu{}_\nu{}^\sigma \omega^\alpha{}_\sigma \otimes \Omega, \end{aligned}$$

$$\begin{aligned} \sigma(\omega^\alpha \otimes \omega^\mu{}_\nu) &= \omega^\mu{}_\nu \otimes \omega^\alpha + \frac{i}{\kappa} (\delta_\nu^0 \omega^\mu{}_\sigma \otimes \omega^{\sigma\alpha} + \omega^\mu{}_0 \otimes \omega^\alpha{}_\nu \\ &\quad + \delta_0^\mu \omega^\sigma{}_\nu \otimes \omega_{\sigma}{}^\alpha + \omega_{0\nu} \otimes \omega^{\alpha\mu}) + \frac{1}{6} \varepsilon^\sigma{}_\nu{}^\alpha \omega^\mu{}_\sigma \otimes \Omega \\ &\quad + \frac{1}{6} \varepsilon^{\sigma\mu\alpha} \omega_{\sigma\nu} \otimes \Omega, \end{aligned}$$

$$\begin{aligned} \sigma(\omega \otimes \omega^\mu) &= \omega^\mu \otimes \omega + \frac{3i}{\kappa^3} (\omega^{\rho\mu} \otimes \omega_{\rho 0} + \omega_{\rho 0} \otimes \omega^{\rho\mu} - \delta_0^\mu \omega^\rho{}_\sigma \otimes \omega_{\rho}{}^\sigma) \\ &\quad - \frac{3}{\kappa^2} (\omega^{\mu\rho} \otimes \omega_\rho + \omega_\rho \otimes \omega^{\rho\mu}) + \frac{1}{2\kappa^2} \varepsilon^{\mu\rho\sigma} \omega_{\rho\sigma} \otimes \Omega, \end{aligned}$$

$$\sigma(\omega \otimes \omega^\mu{}_\nu) = \omega^\mu{}_\nu \otimes \omega - \frac{3}{\kappa^2} (\omega^\sigma{}_\nu \otimes \omega_\sigma{}^\mu - \omega_\sigma{}^\mu \otimes \omega^\sigma{}_\nu),$$

$$\begin{aligned} \sigma(\omega^\mu \otimes \omega^\nu) &= \omega^\nu \otimes \omega^\mu + \frac{1}{3} \omega^{\nu\mu} \otimes \omega + \frac{i}{\kappa} (\omega^0 \otimes \omega^{\mu\nu} + \omega^{\mu\nu} \otimes \omega^0) \\ &\quad + \frac{1}{\kappa^2} (\omega_0{}^\nu \otimes \omega^\mu{}_0 + \omega_0{}^\mu \otimes \omega^0{}_\nu) \\ &\quad + \frac{1}{\kappa^2} (\delta_0^\nu \omega^{\rho\mu} \otimes \omega_{\rho 0} + \delta_0^\mu \omega_{\rho 0} \otimes \omega^{\rho\nu}) \\ &\quad + \frac{i}{\kappa} (\delta_0^\nu \omega^\rho \otimes \omega_{\rho}{}^\mu + \delta_0^\mu \omega^\nu{}_\rho \otimes \omega^\rho) \\ &\quad - \frac{i}{\kappa^2} \delta_0^\nu \delta_0^\mu \omega^\rho{}_\sigma \otimes \omega_{\rho}{}^\sigma - \frac{i}{6\kappa} \varepsilon^{\nu\mu\sigma} \omega_{0\sigma} \otimes \Omega \\ &\quad - \frac{1}{6} \varepsilon^\nu{}_\rho{}^\mu \omega^\rho \otimes \Omega \\ &\quad + \frac{1}{2\kappa^2} \varepsilon^\nu{}_\rho{}^\mu \varepsilon_{\sigma\tau\lambda} \omega^{\rho\sigma} \otimes \omega^{\tau\lambda}. \end{aligned}$$

The action of σ seems to be as much complicated as in four-dimensional case [13]. In spite of that the exterior calculus appears to be simpler: the dimension of $\Gamma^{\wedge 2}$ equals $\binom{\dim \Gamma}{2}$.

REFERENCES

- [1] W.B. Schmidke, J. Weiss, B. Zumino, *Z. Phys.* **52**, 472 (1991); U. Carow-Watamura, M. Schliecker, M. Scholl, S. Watamura, *Int. J. Mod. Phys. A* **6**, 3081 (1991); S.L. Woronowicz, *Comm. Math. Phys.* **136**, 399 (1991); O. Ogievetsky, W.B. Schmidke, J. Weiss, B. Zumino, *Comm. Math. Phys.* **150**, 495 (1992); M. Chaichian, A.P. Demichev, Proceedings of the Workshop: "Generalized Symmetries in Physics", Clausthal 1993 ; V. Dobrev, *J. Phys. A* **26**, 1317 (1993); L. Castellani, in "Quantum Groups" Proceedings of XXX Karpacz Winter School of Theoretical Physics, Karpacz 1994, PWN 1995, p. 13.
- [2] J. Lukierski, A. Nowicki, H. Ruegg, V. Tolstoy, *Phys. Lett. B* **264**, 331 (1991); J. Lukierski, A. Nowicki, H. Ruegg, *Phys. Lett. B* **293**, 344 (1993); S. Giller, P. Kosiński, J. Kunz, M. Majewski, P. Maślanka, *Phys. Lett. B* **286**, 57 (1992).
- [3] S. Zakrzewski, *J. Phys. A* **27**, 2075 (1994).
- [4] J. Lukierski, H. Ruegg, *Phys. Lett. B* **329**, 189 (1994); P. Maślanka, *J. Phys. A* **26**, L1251 (1993).
- [5] S. Giller, P. Kosiński, J. Kunz, M. Majewski, P. Maślanka, *Phys. Lett. B* **286**, 57 (1992); S. Giller, C. Gonera, P. Kosiński, J. Kunz, P. Maślanka, *Mod. Phys. Lett. A* **8**, 3785 (1993); J. Lukierski, H. Ruegg, W. Rühl, *Phys. Lett. B* **313**, 357 (1993).
- [6] P. Maślanka, *J. Math. Phys.* **34**, 6025 (1993).
- [7] P. Maślanka, *J. Phys. A* **27**, 7099 (1994).
- [8] S. Majid, *Pacific J. Math.* **141**, 311 (1990); S. Majid, *Int. J. Mod. Phys. A* **5**, 1 (1990); S. Majid, *J. Algebra* **130**, 17 (1990).
- [9] S. Majid, H. Ruegg, *Phys. Lett. B* **334**, 348 (1994); Ph. Zaugg, preprint MIT-CTP-2353 (1994).
- [10] A. Ballesteros, E. Celeghini, R. Giachetti, E. Sorace, M. Tarlini, *J. Phys. A* **26**, 7495 (1993); P. Maślanka, *J. Math. Phys.* **35**, 1976 (1994); Ph. Zaugg, preprint MIT-CTP-2353 (1994).
- [11] P. Kosiński, P. Maślanka, preprint IMUL3/94.
- [12] S.L. Woronowicz, *Comm. Math. Phys.* **122**, 125 (1989).
- [13] P. Kosiński, P. Maślanka, J. Sobczyk, talk given at the IV Colloquium on Quantum Groups and Integrable Systems, Praga 1995; P. Kosiński, P. Maślanka, Differential Calculus on the Fourdimensional κ Poincaré Group, to be published.
- [14] C. Gonera, P. Kosiński, P. Maślanka, Differential Calculi on Quantum Minkowski space, to be published; P. Kosiński, P. Maślanka, J. Sobczyk talk given at the IV Colloquium on Quantum Groups and Integrable Systems, Praga 1995.
- [15] S. Giller, C. Gonera, P. Kosiński, P. Maślanka, Differential Calculus on Deformed $E(2)$ Group, to be published.

- [16] B. Jurco, *Lett. Math. Phys.* **22**, 177 (1991); P. Aschieri, L. Castellani, *Int. J. Mod. Phys.* **A8**, 1167 (1993).
- [17] P. Stachura, *Lett. Math. Phys.* **25**, 175 (1992).
- [18] V.G. Drinfeld, in *Proc. Int. Congr. Math.*, Berkeley 1986, p. 789.