A NOTE ON GEOMETRY OF κ-MINKOWSKI SPACE*

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(Received April 9, 1996)

The infinitesimal action of κ-Poincaré group on κ-Minkowski space is computed both for generators of κ-Poincaré algebra and those of Woronowicz generalized Lie algebra. The notion of invariant operators is introduced and generalized Klein–Gordon equation is written out.

PACS numbers: 11.25.Hf

1. Introduction

In this short note we consider some simple properties of differential operators on κ-Minkowski space $\mathcal{M}_\kappa$ — a noncommutative deformation of Minkowski space-time which depends on dimensionful parameter κ [1]. We calculate the infinitesimal action of κ-Poincaré group $\mathcal{P}_\kappa$ [1] on $\mathcal{M}_\kappa$ both for the generators of κ-Poincaré algebra $\widetilde{\mathcal{P}}_\kappa$ [2] (this is done using the duality $\widetilde{\mathcal{P}}_\kappa \Longleftrightarrow \mathcal{P}_\kappa$ described in [3]) and for the elements of Woronowicz generalized Lie algebra [4] of κ-Poincaré group [5]. The result supports the relation between both algebras found in [5]. We introduce also the notion of invariant differential operators on $\mathcal{M}_\kappa$ and write out the generalized Klein–Gordon equation.

Let us conclude this section by introducing the notions of κ-Poincaré group $\mathcal{P}_\kappa$ and algebra $\widetilde{\mathcal{P}}_\kappa$. $\mathcal{P}_\kappa$ is defined by the following relations [1]

$$
\begin{aligned}
&[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu)\,,\\
&[\Lambda^\mu{}_\nu, \Lambda^\alpha{}_\beta] = 0\,,\\
&[\Lambda^\mu{}_\nu, x^\rho] = -\frac{i}{\kappa}((\Lambda^\mu{}_0 - \delta^\mu_0)\Lambda^\rho{}_\nu + (\Lambda^0{}_\nu - \delta^0_\nu)g^{\mu\rho})\,,
\end{aligned}
$$

* Supported by Łódź University grant N° 458.

$$\begin{aligned}
&\Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu\,,\\
&\Delta(x^\mu) = \Lambda^\mu{}_\alpha \otimes x^\alpha + x^\mu \otimes I\,,\\
&S(\Lambda^\mu{}_\nu) = \Lambda_\nu{}^\mu\,,\\
&S(x^\mu) = -\Lambda_\nu{}^\mu x^\nu\,,\\
&\varepsilon(\Lambda^\mu{}_\nu) = \delta^\mu_\nu\,,\\
&\varepsilon(x^\mu) = 0\,.
\end{aligned} \tag{1}$$

The dual structure, the κ-Poincaré algebra $\widetilde{\mathcal{P}}_\kappa$, is, in turn, defined as follows [6]

$$\begin{aligned}
&[P_\mu, P_\nu] = 0\,,\\
&[M_i, M_j] = i\varepsilon_{ijk} M_k\,,\\
&[M_i, N_j] = i\varepsilon_{ijk} N_k\,,\\
&[N_i, N_j] = -i\varepsilon_{ijk} M_k\,,\\
&[M_i, P_0] = 0\,,\\
&[M_i, P_j] = i\varepsilon_{ijk} P_k\,,\\
&[N_i, P_0] = iP_i\,,\\
&[N_i, P_j] = i\delta_{ij}\left(\frac{\kappa}{2}(1 - e^{-2P_0/\kappa}) + \frac{1}{2\kappa}\vec{P}^2\right) - \frac{i}{\kappa} P_i P_j\,,\\
&\Delta(M_i) = M_i \otimes I + I \otimes M_i\,,\\
&\Delta(N_i) = N_i \otimes e^{-P_0/\kappa} + I \otimes N_i - \frac{1}{\kappa}\varepsilon_{ijk} M_j \otimes P_k\,,\\
&\Delta(P_0) = P_0 \otimes I + I \otimes P_0\,,\\
&\Delta(P_i) = P_i \otimes e^{-P_0/\kappa} + I \otimes P_i\,,\\
&S(M_i) = -M_i\,,\\
&S(N_i) = -N_i + \frac{3i}{2\kappa} P_i\,,\\
&S(P_\mu) = -P_\mu\,,\\
&\varepsilon(P_\mu, M_i, N_i) = 0\,.
\end{aligned} \tag{2}$$

Structures (1), (2) are dual to each other, the duality being fully described in [3].

The analysis given below was suggested to two of the authors (P. Kosiński and P. Maślanka) by J. Lukierski.

2. κ-Minkowski space

The κ-Minkowski space $\mathcal{M}_\kappa$ [1] is a universal $*$-algebra with unity generated by four selfadjoint elements x^μ subject to the following conditions

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu)\,. \tag{3a}$$

Equipped with the standard coproduct

$$\Delta x^\mu = x^\mu \otimes I + I \otimes x^\mu \,, \tag{3b}$$

antipode $S(x^\mu) = -x^\mu$ and counit $\varepsilon(x^\mu) = 0$ it becomes a quantum group.

On $\mathcal{M}_\kappa$ one can construct a bicovariant five-dimensional calculus which is defined by the following relations [5]

$$\begin{aligned}
&\tau^\mu \equiv dx^\mu \,, \qquad \tau \equiv d\Big(x^2 + \frac{3i}{\kappa}x^0\Big) - 2x_\mu dx^\mu \,, \\
&[\tau^\mu, x^\nu] = \frac{i}{\kappa}g^{0\mu}\tau^\nu - \frac{i}{\kappa}g^{\mu\nu}\tau^0 + \frac{1}{4}g^{\mu\nu}\tau \,, \\
&[\tau, x^\mu] = -\frac{4}{\kappa^2}\tau^\mu \,, \\
&\tau^\mu \wedge \tau^\nu = -\tau^\nu \wedge \tau^\mu \,, \\
&\tau \wedge \tau^\mu = -\tau^\mu \wedge \tau \,, \\
&(\tau^\mu)^* = \tau^\mu \,, \qquad \tau^* = -\tau \,, \\
&d\tau^\mu = 0 \,, \\
&d\tau = -2\tau_\mu \wedge \tau^\mu \,.
\end{aligned} \tag{4}$$

The κ-Minkowski space carries a left-covariant action of κ-Poincaré group $\mathcal{P}_\kappa$ [1], $\rho_{\mathrm{L}} : \mathcal{M}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$, given by

$$\rho_{\mathrm{L}}(x^\mu) = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes I \,. \tag{5}$$

The calculus defined by (4) is covariant under the action of $\mathcal{P}_\kappa$ which reads

$$\begin{aligned}
\widetilde{\rho}_{\mathrm{L}}(\tau^\mu) &= \Lambda^\mu{}_\nu \otimes \tau^\nu \,, \\
\widetilde{\rho}_{\mathrm{L}}(\tau) &= I \otimes \tau \,.
\end{aligned} \tag{6}$$

3. Derivatives, infinitesimal actions and invariant operators

The product of generators x^μ will be called normally ordered if all x^0 factors stand leftmost. This definition can be used to ascribe a unique element $:f(x):$ of $\mathcal{M}_\kappa$ to any polynomial function of four variables f. Formally, it can be extended to any analytic function f.

Let us now one define the (left) partial derivatives: for any $f \in \mathcal{M}_\kappa$ we write

$$df = \partial_\mu f \tau^\mu + \partial f \tau \,. \tag{7}$$

It is a matter of some boring calculations (using the commutation rules (3a)) to find the following formula

$$
\begin{aligned}
d:f:=: & \Big(\kappa \sin\big(\frac{\partial_0}{\kappa}\big) + \frac{i}{2\kappa} e^{i\frac{\partial_0}{\kappa}} \Delta\Big) f : \tau^0 + : e^{i\frac{\partial_0}{\kappa}} \frac{\partial f}{\partial x^i} : \tau^i \\
& + : \Big(\frac{\kappa^2}{4}\Big(1-\cos\big(\frac{\partial_0}{\kappa}\big)\Big) - \frac{1}{8} e^{i\frac{\partial_0}{\kappa}} \Delta\Big) f : \tau
\end{aligned} \tag{8}
$$

or

$$
\begin{aligned}
\partial_0 : f :=: & \Big(\kappa \sin\big(\frac{\partial_0}{\kappa}\big) + \frac{i}{2\kappa} e^{i\frac{\partial_0}{\kappa}} \Delta\Big) f :, \\
\partial_i : f :=: & \, e^{i\frac{\partial_0}{\kappa}} \frac{\partial f}{\partial x^i} :, \\
\partial : f :=: & \Big(\frac{\kappa^2}{4}\Big(1-\cos\big(\frac{\partial_0}{\kappa}\big)\Big) - \frac{1}{8} e^{i\frac{\partial_0}{\kappa}} \Delta\Big) f : .
\end{aligned} \tag{9}
$$

Let us now define the infinitesimal action of $\mathcal{P}_\kappa$ on $\mathcal{M}_\kappa$. Let X be any element of the Hopf algebra dual to $\mathcal{P}_\kappa$ — the κ-Poincaré algebra $\widetilde{\mathcal{P}}_\kappa$ (*cf.* [3] for the proof of duality). The corresponding infinitesimal action

$$\widehat{X} : \mathcal{M}_\kappa \to \mathcal{M}_\kappa$$

is defined as follows: for any $f \in \mathcal{M}_\kappa$,

$$\widehat{X} f = (X \otimes \mathrm{id}) \circ \rho_{\mathrm{L}}(f) . \tag{10}$$

Using the standard duality rules [3], we conclude that

$$
\begin{aligned}
& \widehat{P}_\mu x^\alpha = i\delta^\alpha_\mu , \\
& \widehat{P}_\mu : x^\alpha x^\beta := i\delta^\beta_\mu x^\alpha + i\delta^\alpha_\mu x^\beta
\end{aligned} \tag{11}
$$

etc. One can show that, in general,

$$\widehat{P}_\mu : f :=: i\frac{\partial f}{\partial x^\mu} : . \tag{12}$$

Also, using the fact that $\tilde{\rho}_{\mathrm{L}}$ is a left action of $\mathcal{P}_\kappa$ on $\mathcal{M}_\kappa$ together with the duality $\mathcal{P}_\kappa \to \widetilde{\mathcal{P}}_\kappa$, we conclude that

$$F(\widehat{P}_\mu) : f :=: F\Big(i\frac{\partial f}{\partial x^\mu}\Big) f : . \tag{13}$$

Formulae (11)–(13) have the following interpretation. In [5] the fifteen-dimensional bicovariant calculus on $\mathcal{P}_\kappa$ has been constructed using the methods developed by Woronowicz [4]. The resulting generalized Lie algebra is also fifteen-dimensional, the additional generators being the generalized mass square operator and the components of generalized Pauli-Lubanski fourvector. All generators of this Lie algebra can be expresses in terms of the generators P_μ, $M_{\alpha\beta}$ of $\widetilde{\mathcal{P}}_\kappa$ [5]. In particular, the translation generators χ_μ as well as the mass squared operator χ are expressible in terms of P_μ only. The relevant expressions are given by formulae (20) of [5]. Comparing them with (9), (13) above, we conclude that

$$\begin{aligned}\widehat{\chi}_\mu &\equiv \partial_\mu\,,\\ \widehat{\chi} &\equiv \partial\,.\end{aligned} \tag{14}$$

These relations, obtained here by explicit computations, follow also from (7) if one takes into account that $\mathcal{M}_\kappa$ is a quantum subgroup of $\mathcal{P}_\kappa$.

It is also not difficult to obtain the action of Lorentz generators. Combining (1) and (3a) with the duality $\mathcal{P}_\kappa \to \widetilde{\mathcal{P}}_\kappa$ described in detail in [5], we conclude first that the action of M_i and N_i coincides with the proposal of Majid and Ruegg [6]; the actual computation is then easy and gives

$$\begin{aligned}\widehat{M}_i : f(x^\mu) &:=: -i\varepsilon_{ijl}x^j \frac{\partial f(x^\mu)}{\partial x^l}:\\ \widehat{N}_i : f(x^\mu) &:=: \left(ix^0\frac{\partial}{\partial x^i} + x^i\left(\frac{\kappa}{2}\left(1-\exp\left(\frac{-2i}{\kappa}\frac{\partial}{\partial x^0}\right)\right) - \frac{1}{2\kappa}\Delta\right)\right.\\ &\qquad \left. + \frac{1}{\kappa}x^k\frac{\partial^2}{\partial x^k \partial x^i}\right) f(x^\mu):\,.\end{aligned} \tag{15}$$

Let us now pass to the notion of invariant operator; $\widehat{C}$ is an invariant operator on $\mathcal{M}_\kappa$ if

$$\rho_{\mathrm{L}} \circ \widehat{C} = (\mathrm{id} \otimes \widehat{C}) \circ \rho_{\mathrm{L}}\,. \tag{16}$$

We shall show that if C is a central element of $\widetilde{\mathcal{P}}_\kappa$, then

$$\widehat{C} f = (C \otimes \mathrm{id}) \circ \rho_{\mathrm{L}}(f) \tag{17}$$

is an invariant operator. To prove this let us take any $Y \in \widetilde{\mathcal{P}}_\kappa$, then

$$YC = CY \tag{18}$$

or, in other words, for any $a \in \mathcal{P}_\kappa$,

$$Y(a_{(1)})C(a_{(2)}) = C(a_{(1)})Y(a_{(2)}) \tag{19}$$

where $\Delta a = a_{(1)} \otimes a_{(2)}$. Let us fix a and write (19) as

$$Y(a_{(1)}C(a_{(2)})) = Y(a_{(2)}C(a_{(1)}))\,. \tag{20}$$

As (20) holds for any $Y \in \widetilde{\mathcal{P}}_\kappa$ we conclude that for any $a \in \mathcal{P}_\kappa$

$$C(a_{(2)})a_{(1)} = C(a_{(1)})a_{(2)} . \tag{21}$$

Now let

$$\begin{aligned} \rho_{\mathrm{L}}(x) &= a_{(1)} \otimes x_{(1)} , \\ \rho_{\mathrm{L}}(x_{(1)}) &= a_{(1)(2)} \otimes x_{(2)} , \\ \Delta a_{(1)} &= a^{(1)}_{(1)} \otimes a^{(2)}_{(1)} . \end{aligned} \tag{22}$$

The identity

$$(\mathrm{id} \otimes \rho_{\mathrm{L}}) \circ \rho_{\mathrm{L}} = (\Delta \otimes \mathrm{id}) \circ \rho_{\mathrm{L}} \tag{23}$$

implies

$$a_{(1)} \otimes a_{(1)(2)} \otimes x_{(2)} = a^{(1)}_{(1)} \otimes a^{(2)}_{(1)} \otimes x_{(1)} . \tag{24}$$

Applying to both sides $\mathrm{id} \otimes C \otimes \mathrm{id}$ and $C \otimes \mathrm{id} \otimes \mathrm{id}$, we get

$$\begin{aligned} C(a_{(1)(2)})a_{(1)} \otimes x_{(2)} &= C(a^{(2)}_{(1)})a^{(1)}_{(1)} \otimes x_{(1)} , \\ C(a_{(1)})a_{(1)(2)} \otimes x_{(2)} &= C(a^{(1)}_{(1)})a^{(2)}_{(1)} \otimes x_{(1)} . \end{aligned} \tag{25}$$

It follows from (21) applied to $a_{(1)}$ that the right-hand sides of (25) are equal. So,

$$C(a_{(1)(2)})a_{(1)} \otimes x_{(2)} = C(a_{(1)})a_{(1)(2)} \otimes x_{(2)} , \tag{26}$$

i.e.

$$(\mathrm{id} \otimes \widehat{C}) \circ \rho_{\mathrm{L}}(x) = \rho_{\mathrm{L}} \circ \widehat{C}(x) . \tag{27}$$

Using the above result we can easily construct the deformed Klein–Gordon equation. Namely, we take as a central element the counterpart of mass squared Casimir operator χ [5]. Due to (14) the generalized Klein–Gordon equation reads

$$\left(\partial + \frac{m^2}{8}\right) f = 0 ; \tag{28}$$

the coefficient $\frac{1}{8}$ is dictated by the correspondence with standard Klein–Gordon equation in the limit $\kappa \to \infty$. Let us note that (28) can be written, due to (9), in the form

$$\left[\partial_0^2 - \partial_i^2 + m^2\left(1 + \frac{m^2}{4\kappa^2}\right)\right] f = 0 ; \tag{29}$$

here ∂_0, ∂_i are the operators given by (9). It seems therefore that the Woronowicz operators χ_μ are better candidates for translation generators than P_μ's. Note that the operators χ_μ already appeared in [7, 8].

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