

## THE KINEMATICS AND DYNAMICS OF FLAVOR MIXING\*

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In view of the observed strong hierarchy of the quark and lepton masses and of the flavor mixing angles, it is argued that the description of flavor mixing must take this into account. One particularly interesting way to describe the flavor mixing, which, however, is not the one used today, emerges, which is particularly suited for models of quark mass matrices based on flavor symmetries. We conclude that the unitarity triangle important for  $B$  physics should be close to or identical to a rectangular triangle. CP violation is maximal in this sense.

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The phenomenon of flavor mixing, which is intrinsically linked to CP violation, is an important ingredient of the Standard Model of Basic Interactions. Yet unlike other features of the Standard Model, *e.g.* the mixing of the neutral electroweak gauge bosons, it is a phenomenon which can merely be described. A deeper understanding is still lacking, but most theorists would agree that it is directly linked to the mass spectrum of the quarks — the possible mixing of lepton flavors will not be discussed here. Furthermore there is a general consensus that a deeper dynamical understanding would require to go beyond the physics of the Standard Model. In this talk I shall not go thus far. Instead I shall demonstrate that the observed properties of the flavor mixing, combined with our knowledge about the quark mass spectrum, suggest specific symmetry properties which allow to fix the flavor mixing parameters with high precision, thus predicting the outcome of the experiments which will soon be performed at the  $B$ -meson factories.

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In the standard electroweak theory, the phenomenon of flavor mixing of the quarks is described by a  $3 \times 3$  unitary matrix, the Cabibbo–Kobayashi–Maskawa (CKM) matrix [1, 2]. This matrix can be expressed in terms of four parameters, which are usually taken as three rotation angles and one phase. A number of different parametrizations have been proposed in the literature [2–5]. Of course, adopting a particular parametrization of flavor mixing is arbitrary and not directly a physical issue. Nevertheless it is quite likely that the actual values of flavor mixing parameters (including the strength of CP violation), once they are known with high precision, will give interesting information about the physics beyond the standard model. Probably at this point it will turn out that a particular description of the CKM matrix is more useful and transparent than the others. For this reason, let me first analyze all possible parametrizations and point out their respective advantages and disadvantages.

In the standard model the quark flavor mixing arises once the up- and down-type mass matrices are diagonalized. The generation of quark masses is intimately related to the phenomenon of flavor mixing. In particular, the flavor mixing parameters do depend on the elements of quark mass matrices. A particular structure of the underlying mass matrices calls for a particular choice of the parametrization of the flavor mixing matrix. For example, in Ref. [6] it was noticed that a rather special form of the flavor mixing matrix results, if one starts from Hermitian mass matrices in which the (1,3) and (3,1) elements vanish. This has been subsequently observed again in a number of papers [7]. Recently we have studied the exact form of such a description from a general point of view and pointed out some advantages of this type of representation in the discussion of flavor mixing and CP-violating phenomena [5], which will be discussed later.

In the standard model the weak charged currents are given by

$$\overline{(u, c, t)}_L \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L, \quad (1)$$

where  $u, c, \dots, b$  are the quark mass eigenstates,  $L$  denotes the left-handed fields, and  $V_{ij}$  are elements of the CKM matrix  $V$ . In general  $V_{ij}$  are complex numbers, but their absolute values are measurable quantities. For example,  $|V_{cb}|$  primarily determines the lifetime of  $B$  mesons. The phases of  $V_{ij}$ , however, are not physical, like the phases of quark fields. A phase transformation of the  $u$  quark ( $u \rightarrow u e^{i\alpha}$ ), for example, leaves the quark mass term invariant but changes the elements in the first row of  $V$  (i.e.,  $V_{uj} \rightarrow V_{uj} e^{-i\alpha}$ ). Only a common phase transformation of all quark fields leaves all elements of  $V$  invariant, thus there is a five-fold freedom to adjust the phases of  $V_{ij}$ .

In general the unitary matrix  $V$  depends on nine parameters. Note that in the absence of complex phases  $V$  would consist of only three independent parameters, corresponding to three (Euler) rotation angles. Hence one can describe the complex matrix  $V$  by three angles and six phases. Due to the freedom in redefining the quark field phases, five of the six phases in  $V$  can be absorbed and we arrive at the well-known result that the CKM matrix  $V$  can be parametrized in terms of three rotation angles and one CP-violating phase. The question about how many different ways to describe  $V$  may exist was raised some time ago [8]. Below we shall reconsider this problem and give a complete analysis.

If the flavor mixing matrix  $V$  is first assumed to be a real orthogonal matrix, it can in general be written as a product of three matrices  $R_{12}$ ,  $R_{23}$  and  $R_{31}$ , which describe simple rotations in the (1,2), (2,3) and (3,1) planes:

$$\begin{aligned}
 R_{12}(\theta) &= \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
 R_{23}(\sigma) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\sigma & s_\sigma \\ 0 & -s_\sigma & c_\sigma \end{pmatrix}, \\
 R_{31}(\tau) &= \begin{pmatrix} c_\tau & 0 & s_\tau \\ 0 & 1 & 0 \\ -s_\tau & 0 & c_\tau \end{pmatrix}, \tag{2}
 \end{aligned}$$

where  $s_\theta \equiv \sin \theta$ ,  $c_\theta \equiv \cos \theta$ , etc. Clearly these rotation matrices do not commute with each other. There exist twelve different ways to arrange products of these matrices such that the most general orthogonal matrix  $R$  can be obtained [8]. Note that the matrix  $R_{ij}^{-1}(\omega)$  plays an equivalent role as  $R_{ij}(\omega)$  in constructing  $R$ , because of  $R_{ij}^{-1}(\omega) = R_{ij}(-\omega)$ . Note also that  $R_{ij}(\omega)R_{ij}(\omega') = R_{ij}(\omega + \omega')$  holds, thus the product  $R_{ij}(\omega)R_{ij}(\omega')R_{kl}(\omega'')$  or  $R_{kl}(\omega'')R_{ij}(\omega)R_{ij}(\omega')$  cannot cover the whole space of a  $3 \times 3$  orthogonal matrix and should be excluded. Explicitly the twelve different forms of  $R$  read as

- (1)  $R = R_{12}(\theta) R_{23}(\sigma) R_{12}(\theta')$ ,
- (2)  $R = R_{12}(\theta) R_{31}(\tau) R_{12}(\theta')$ ,
- (3)  $R = R_{23}(\sigma) R_{12}(\theta) R_{23}(\sigma')$ ,
- (4)  $R = R_{23}(\sigma) R_{31}(\tau) R_{23}(\sigma')$ ,
- (5)  $R = R_{31}(\tau) R_{12}(\theta) R_{31}(\tau')$ ,
- (6)  $R = R_{31}(\tau) R_{23}(\sigma) R_{31}(\tau')$ ,

in which a rotation in the  $(i, j)$  plane occurs twice; and

$$\begin{aligned}
 (7) \quad R &= R_{12}(\theta) R_{23}(\sigma) R_{31}(\tau) , \\
 (8) \quad R &= R_{12}(\theta) R_{31}(\tau) R_{23}(\sigma) , \\
 (9) \quad R &= R_{23}(\sigma) R_{12}(\theta) R_{31}(\tau) , \\
 (10) \quad R &= R_{23}(\sigma) R_{31}(\tau) R_{12}(\theta) , \\
 (11) \quad R &= R_{31}(\tau) R_{12}(\theta) R_{23}(\sigma) , \\
 (12) \quad R &= R_{31}(\tau) R_{23}(\sigma) R_{12}(\theta) ,
 \end{aligned}$$

where all three  $R_{ij}$  are present.

Although all the above twelve combinations represent the most general orthogonal matrices, only nine of them are structurally different. The reason is that the products  $R_{ij}R_{kl}R_{ij}$  and  $R_{ij}R_{mn}R_{ij}$  (with  $ij \neq kl \neq mn$ ) are correlated with each other, leading essentially to the same form for  $R$ . Indeed it is straightforward to see the correlation between patterns (1), (3), (5) and (2), (4), (6), respectively, as follows:

$$\begin{aligned}
 R_{12}(\theta) R_{31}(\tau) R_{12}(\theta') &= R_{12}(\theta + \pi/2) R_{23}(\sigma = \tau) R_{12}(\theta' - \pi/2) , \\
 R_{23}(\sigma) R_{31}(\tau) R_{23}(\sigma') &= R_{23}(\sigma - \pi/2) R_{12}(\theta = \tau) R_{23}(\sigma' + \pi/2) , \\
 R_{31}(\tau) R_{23}(\sigma) R_{31}(\tau') &= R_{31}(\tau + \pi/2) R_{12}(\theta = \sigma) R_{31}(\tau' - \pi/2) . \quad (3)
 \end{aligned}$$

Thus the orthogonal matrices (2), (4) and (6) need not be treated as independent choices. We then draw the conclusion that there exist *nine* different forms for the orthogonal matrix  $R$ , *i.e.*, patterns (1), (3) and (5) as well as (7)–(12).

We proceed to include the CP-violating phase, denoted by  $\varphi$ , in the above rotation matrices. The resultant matrices should be unitary such that a unitary flavor mixing matrix can be finally produced. There are several different ways for  $\varphi$  to enter  $R_{12}$ , *e.g.*,

$$R_{12}(\theta, \varphi) = \begin{pmatrix} c_\theta & s_\theta e^{+i\varphi} & 0 \\ -s_\theta e^{-i\varphi} & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad (4a)$$

or

$$R_{12}(\theta, \varphi) = \begin{pmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix} , \quad (4b)$$

or

$$R_{12}(\theta, \varphi) = \begin{pmatrix} c_\theta e^{+i\varphi} & s_\theta & 0 \\ -s_\theta & c_\theta e^{-i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (4c)$$

Similarly one may introduce a phase parameter into  $R_{23}$  or  $R_{31}$ . Then the CKM matrix  $V$  can be constructed, as a product of three rotation matrices, by use of one complex  $R_{ij}$  and two real ones. Note that the location of the CP-violating phase in  $V$  can be arranged by redefining the quark field phases, thus it does not play an essential role in classifying different parametrizations. We find that it is always possible to locate the phase parameter  $\varphi$  in a  $2 \times 2$  submatrix of  $V$ , in which each element is a sum of two terms with the relative phase  $\varphi$ . The remaining five elements of  $V$  are real in such a phase assignment. Accordingly we arrive at the nine distinctive parametrizations of the CKM matrix  $V$  listed in Table I where the complex rotation matrices  $R_{12}(\theta, \varphi)$ ,  $R_{23}(\sigma, \varphi)$  and  $R_{31}(\tau, \varphi)$  are obtained directly from the real ones in Eq. (2) with the replacement  $1 \rightarrow e^{-i\varphi}$ . These nine possibilities have been discussed recently in Ref. [9] (see also Ref. [10]).

One can see that  $P2$  and  $P3$  correspond to the Kobayashi–Maskawa [2] and Maiani [3] representations, although different notations for the CP-violating phase and three mixing angles are adopted here. The latter is indeed equivalent to the “standard” parametrization advocated by the Particle Data Group [3, 4]. This can be seen clearly if one makes three transformations of quark field phases:  $c \rightarrow c e^{-i\varphi}$ ,  $t \rightarrow t e^{-i\varphi}$ , and  $b \rightarrow b e^{-i\varphi}$ . In addition,  $P1$  is just the one discussed by Xing and me in Ref. [5].

From a mathematical point of view, all nine different parametrizations are equivalent. However this is not the case if we apply our considerations to the quarks and their mass spectrum. It is well-known that both the observed quark mass spectrum and the observed values of the flavor mixing parameters exhibit a striking hierarchical structure. The latter can be understood in a natural way as the consequence of a specific pattern of chiral symmetries whose breaking causes the towers of different masses to appear step by step [11–13]. Such a chiral evolution of the mass matrices leads, as argued in Ref. [12], to a specific way to introduce and describe the flavor mixing. In the limit  $m_u = m_d = 0$ , which is close to the real world, since  $m_u/m_t \ll 1$  and  $m_d/m_b \ll 1$ , the flavor mixing is merely a rotation between the  $t$ – $c$  and  $b$ – $s$  systems, described by one rotation angle. No complex phase is present; *i.e.*, CP violation is absent. This rotation angle is expected to change very little, once  $m_u$  and  $m_d$  are introduced as tiny perturbations. A sensible parametrization should make use of this feature. This implies that the rotation matrix  $R_{23}$  appears exactly once in the description of the CKM matrix  $V$ , eliminating  $P2$  (in which  $R_{23}$  appears twice) and  $P5$  (where  $R_{23}$  is absent). This leaves us with seven parametrizations of the flavor mixing matrix.

TABLE I

Classification of different parametrizations for the flavor mixing matrix

Parametrization	Useful relations
$P1: V = R_{12}(\theta) R_{23}(\sigma, \varphi) R_{12}^{-1}(\theta')$ $\begin{pmatrix} s_{\theta} s_{\theta'} c_{\sigma} + c_{\theta} c_{\theta'} e^{-i\varphi} & s_{\theta} c_{\theta'} c_{\sigma} - c_{\theta} s_{\theta'} e^{-i\varphi} & s_{\theta} s_{\sigma} \\ c_{\theta} s_{\theta'} c_{\sigma} - s_{\theta} c_{\theta'} e^{-i\varphi} & c_{\theta} c_{\theta'} c_{\sigma} + s_{\theta} s_{\theta'} e^{-i\varphi} & c_{\theta} s_{\sigma} \\ -s_{\theta'} s_{\sigma} & -c_{\theta'} s_{\sigma} & c_{\sigma} \end{pmatrix}$	$\mathcal{J} = s_{\theta} c_{\theta} s_{\theta'} c_{\theta'} s_{\sigma}^2 c_{\sigma} \sin \varphi$ $\tan \theta =  V_{ub}/V_{cb} $ $\tan \theta' =  V_{td}/V_{ts} $ $\cos \sigma =  V_{tb} $
$P2: V = R_{23}(\sigma) R_{12}(\theta, \varphi) R_{23}^{-1}(\sigma')$ $\begin{pmatrix} c_{\theta} & s_{\theta} c_{\sigma'} & -s_{\theta} s_{\sigma'} \\ -s_{\theta} c_{\sigma} & c_{\theta} c_{\sigma} c_{\sigma'} + s_{\theta} s_{\sigma'} e^{-i\varphi} & -c_{\theta} c_{\sigma} s_{\sigma'} + s_{\theta} c_{\sigma'} e^{-i\varphi} \\ s_{\theta} s_{\sigma} & -c_{\theta} s_{\sigma} c_{\sigma'} + c_{\sigma} s_{\sigma'} e^{-i\varphi} & c_{\theta} s_{\sigma} s_{\sigma'} + c_{\sigma} c_{\sigma'} e^{-i\varphi} \end{pmatrix}$	$\mathcal{J} = s_{\theta}^2 c_{\theta} s_{\sigma} c_{\sigma} s_{\sigma'} c_{\sigma'} \sin \varphi$ $\cos \theta =  V_{ud} $ $\tan \sigma =  V_{td}/V_{cd} $ $\tan \sigma' =  V_{ub}/V_{us} $
$P3: V = R_{23}(\sigma) R_{31}(\tau, \varphi) R_{12}(\theta)$ $\begin{pmatrix} c_{\theta} c_{\tau} & s_{\theta} c_{\tau} & s_{\tau} \\ -c_{\theta} s_{\sigma} s_{\tau} - s_{\theta} c_{\sigma} e^{-i\varphi} & -s_{\theta} s_{\sigma} s_{\tau} + c_{\theta} c_{\sigma} e^{-i\varphi} & s_{\sigma} c_{\tau} \\ -c_{\theta} c_{\sigma} s_{\tau} + s_{\theta} s_{\sigma} e^{-i\varphi} & -s_{\theta} c_{\sigma} s_{\tau} - c_{\theta} s_{\sigma} e^{-i\varphi} & c_{\sigma} c_{\tau} \end{pmatrix}$	$\mathcal{J} = s_{\theta} c_{\theta} s_{\sigma} c_{\sigma} s_{\tau} c_{\tau}^2 \sin \varphi$ $\tan \theta =  V_{us}/V_{ud} $ $\tan \sigma =  V_{cb}/V_{tb} $ $\sin \tau =  V_{ub} $
$P4: V = R_{12}(\theta) R_{31}(\tau, \varphi) R_{23}^{-1}(\sigma)$ $\begin{pmatrix} c_{\theta} c_{\tau} & c_{\theta} s_{\sigma} s_{\tau} + s_{\theta} c_{\sigma} e^{-i\varphi} & c_{\theta} c_{\sigma} s_{\tau} - s_{\theta} s_{\sigma} e^{-i\varphi} \\ -s_{\theta} c_{\tau} & -s_{\theta} s_{\sigma} s_{\tau} + c_{\theta} c_{\sigma} e^{-i\varphi} & -s_{\theta} c_{\sigma} s_{\tau} - c_{\theta} s_{\sigma} e^{-i\varphi} \\ -s_{\tau} & s_{\sigma} c_{\tau} & c_{\sigma} c_{\tau} \end{pmatrix}$	$\mathcal{J} = s_{\theta} c_{\theta} s_{\sigma} c_{\sigma} s_{\tau} c_{\tau}^2 \sin \varphi$ $\tan \theta =  V_{cd}/V_{ud} $ $\tan \sigma =  V_{ts}/V_{tb} $ $\sin \tau =  V_{td} $
$P5: V = R_{31}(\tau) R_{12}(\theta, \varphi) R_{31}^{-1}(\tau')$ $\begin{pmatrix} c_{\theta} c_{\tau} c_{\tau'} + s_{\tau} s_{\tau'} e^{-i\varphi} & s_{\theta} c_{\tau} & -c_{\theta} c_{\tau} s_{\tau'} + s_{\tau} c_{\tau'} e^{-i\varphi} \\ -s_{\theta} c_{\tau'} & c_{\theta} & s_{\theta} s_{\tau'} \\ -c_{\theta} s_{\tau} c_{\tau'} + c_{\tau} s_{\tau'} e^{-i\varphi} & -s_{\theta} s_{\tau} & c_{\theta} s_{\tau} s_{\tau'} + c_{\tau} c_{\tau'} e^{-i\varphi} \end{pmatrix}$	$\mathcal{J} = s_{\theta}^2 c_{\theta} s_{\tau} c_{\tau} s_{\tau'} c_{\tau'} \sin \varphi$ $\cos \theta =  V_{cs} $ $\tan \tau =  V_{ts}/V_{us} $ $\tan \tau' =  V_{cb}/V_{cd} $
$P6: V = R_{12}(\theta) R_{23}(\sigma, \varphi) R_{31}(\tau)$ $\begin{pmatrix} -s_{\theta} s_{\sigma} s_{\tau} + c_{\theta} c_{\tau} e^{-i\varphi} & s_{\theta} c_{\sigma} & s_{\theta} s_{\sigma} c_{\tau} + c_{\theta} s_{\tau} e^{-i\varphi} \\ -c_{\theta} s_{\sigma} s_{\tau} - s_{\theta} c_{\tau} e^{-i\varphi} & c_{\theta} c_{\sigma} & c_{\theta} s_{\sigma} c_{\tau} - s_{\theta} s_{\tau} e^{-i\varphi} \\ -c_{\sigma} s_{\tau} & -s_{\sigma} & c_{\sigma} c_{\tau} \end{pmatrix}$	$\mathcal{J} = s_{\theta} c_{\theta} s_{\sigma} c_{\sigma}^2 s_{\tau} c_{\tau} \sin \varphi$ $\tan \theta =  V_{us}/V_{cs} $ $\sin \sigma =  V_{ts} $ $\tan \tau =  V_{td}/V_{tb} $
$P7: V = R_{23}(\sigma) R_{12}(\theta, \varphi) R_{31}^{-1}(\tau)$ $\begin{pmatrix} c_{\theta} c_{\tau} & s_{\theta} & -c_{\theta} s_{\tau} \\ -s_{\theta} c_{\sigma} c_{\tau} + s_{\sigma} s_{\tau} e^{-i\varphi} & c_{\theta} c_{\sigma} & s_{\theta} c_{\sigma} s_{\tau} + s_{\sigma} c_{\tau} e^{-i\varphi} \\ s_{\theta} s_{\sigma} c_{\tau} + c_{\sigma} s_{\tau} e^{-i\varphi} & -c_{\theta} s_{\sigma} & -s_{\theta} s_{\sigma} s_{\tau} + c_{\sigma} c_{\tau} e^{-i\varphi} \end{pmatrix}$	$\mathcal{J} = s_{\theta} c_{\theta}^2 s_{\sigma} c_{\sigma} s_{\tau} c_{\tau} \sin \varphi$ $\sin \theta =  V_{us} $ $\tan \sigma =  V_{ts}/V_{cs} $ $\tan \tau =  V_{ub}/V_{ud} $
$P8: V = R_{31}(\tau) R_{12}(\theta, \varphi) R_{23}(\sigma)$ $\begin{pmatrix} c_{\theta} c_{\tau} & s_{\theta} c_{\sigma} c_{\tau} - s_{\sigma} s_{\tau} e^{-i\varphi} & s_{\theta} s_{\sigma} c_{\tau} + c_{\sigma} s_{\tau} e^{-i\varphi} \\ -s_{\theta} & c_{\theta} c_{\sigma} & c_{\theta} s_{\sigma} \\ -c_{\theta} s_{\tau} & -s_{\theta} c_{\sigma} s_{\tau} - s_{\sigma} c_{\tau} e^{-i\varphi} & -s_{\theta} s_{\sigma} s_{\tau} + c_{\sigma} c_{\tau} e^{-i\varphi} \end{pmatrix}$	$\mathcal{J} = s_{\theta} c_{\theta}^2 s_{\sigma} c_{\sigma} s_{\tau} c_{\tau} \sin \varphi$ $\sin \theta =  V_{cd} $ $\tan \sigma =  V_{cb}/V_{cs} $ $\tan \tau =  V_{td}/V_{ud} $
$P9: V = R_{31}(\tau) R_{23}(\sigma, \varphi) R_{12}^{-1}(\theta)$ $\begin{pmatrix} -s_{\theta} s_{\sigma} s_{\tau} + c_{\theta} c_{\tau} e^{-i\varphi} & -c_{\theta} s_{\sigma} s_{\tau} - s_{\theta} c_{\tau} e^{-i\varphi} & c_{\sigma} s_{\tau} \\ s_{\theta} c_{\sigma} & c_{\theta} c_{\sigma} & s_{\sigma} \\ -s_{\theta} s_{\sigma} c_{\tau} - c_{\theta} s_{\tau} e^{-i\varphi} & -c_{\theta} s_{\sigma} c_{\tau} + s_{\theta} s_{\tau} e^{-i\varphi} & c_{\sigma} c_{\tau} \end{pmatrix}$	$\mathcal{J} = s_{\theta} c_{\theta} s_{\sigma} c_{\sigma}^2 s_{\tau} c_{\tau} \sin \varphi$ $\tan \theta =  V_{cd}/V_{cs} $ $\sin \sigma =  V_{cb} $ $\tan \tau =  V_{ub}/V_{tb} $

The list can be reduced further by considering the location of the phase  $\varphi$ . In the limit  $m_u = m_d = 0$ , the phase must disappear in the weak transition elements  $V_{tb}$ ,  $V_{ts}$ ,  $V_{cb}$  and  $V_{cs}$ . In  $P7$  and  $P8$ , however,  $\varphi$  appears particularly in  $V_{tb}$ . Thus these two parametrizations should be eliminated, leaving us with five parametrizations (*i.e.*,  $P1$ ,  $P3$ ,  $P4$ ,  $P6$  and  $P9$ ). In the same limit, the phase  $\varphi$  appears in the  $V_{ts}$  element of  $P3$  and the  $V_{cb}$  element of  $P4$ . Hence these two parametrizations should also be eliminated. Then we are left with three parametrizations,  $P1$ ,  $P6$  and  $P9$ . As expected, these are the parametrizations containing the complex rotation matrix  $R_{23}(\sigma, \varphi)$ . We stress that the “standard” parametrization [4] (equivalent to  $P3$ ) does not obey the above constraints and should be dismissed.

Among the remaining three parametrizations,  $P1$  is singled out by the fact that the CP-violating phase  $\varphi$  appears only in the  $2 \times 2$  submatrix of  $V$  describing the weak transitions among the light quarks. This is precisely the phase where the phase  $\varphi$  should appear, not in any of the weak transition elements involving the heavy quarks  $t$  and  $b$ .

In the parametrization  $P6$  or  $P9$ , the complex phase  $\varphi$  appears in  $V_{cb}$  or  $V_{ts}$ , but this phase factor is multiplied by a product of  $\sin \theta$  and  $\sin \tau$ , *i.e.*, it is of second order of the weak mixing angles. Hence the imaginary parts of these elements are not exactly vanishing, but very small in magnitude.

In our view the best possibility to describe the flavor mixing in the standard model is to adopt the parametrization  $P1$ . As discussed in Ref. [5], this parametrization has a number of significant advantages in addition to that mentioned above. Especially it is well suited for specific models of quark mass matrices.

In the following part I shall show that the parametrization  $P1$  follows automatically if we impose the constraints from the chiral symmetries and the hierarchical structure of the mass eigenvalues. We take the point of view that the quark mass eigenvalues are dynamical entities, and one could change their values in order to study certain symmetry limits, as it is done in QCD. In the standard electroweak model, in which the quark mass matrices are given by the coupling of a scalar field to various quark fields, this can certainly be done by adjusting the related coupling constants. Whether it is possible in reality is an open question. It is well-known that the quark mass matrices can always be made hermitian by a suitable transformation of the right-handed fields. Without loss of generality, we shall suppose in this paper that the quark mass matrices are hermitian. In the limit where the masses of the  $u$  and  $d$  quarks are set to zero, the quark mass matrix  $\tilde{M}$  (for both charge  $+2/3$  and charge  $-1/3$  sectors) can be arranged such that its elements  $\tilde{M}_{i1}$  and  $\tilde{M}_{1i}$  ( $i = 1, 2, 3$ ) are all zero [11, 12]. Thus the quark

mass matrices have the form

$$\tilde{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{C} & \tilde{B} \\ 0 & \tilde{B}^* & \tilde{A} \end{pmatrix}. \quad (5)$$

The observed mass hierarchy is incorporated into this structure by denoting the entry which is of the order of the  $t$ -quark or  $b$ -quark mass by  $\tilde{A}$ , with  $\tilde{A} \gg \tilde{C}, |\tilde{B}|$ . It can easily be seen (see, *e.g.*, Ref. [14]) that the complex phases in the mass matrices (5) can be rotated away by subjecting both  $\tilde{M}_u$  and  $\tilde{M}_d$  to the same unitary transformation. Thus we shall take  $\tilde{B}$  to be real for both up- and down-quark sectors. As expected, CP violation cannot arise at this stage. The diagonalization of the mass matrices leads to a mixing between the second and third families, described by an angle  $\tilde{\theta}$ . The flavor mixing matrix is then given by

$$\tilde{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{c} & \tilde{s} \\ 0 & -\tilde{s} & \tilde{c} \end{pmatrix}, \quad (6)$$

where  $\tilde{s} \equiv \sin \tilde{\theta}$  and  $\tilde{c} \equiv \cos \tilde{\theta}$ . In view of the fact that the limit  $m_u = m_d = 0$  is not far from reality, the angle  $\tilde{\theta}$  is essentially given by the observed value of  $|V_{cb}|$  ( $= 0.039 \pm 0.002$  [15, 16]); *i.e.*,  $\tilde{\theta} = 2.24^\circ \pm 0.12^\circ$ .

At the next and final stage of the chiral evolution of the mass matrices, the masses of the  $u$  and  $d$  quarks are introduced. The Hermitian mass matrices have in general the form:

$$M = \begin{pmatrix} E & D & F \\ D^* & C & B \\ F^* & B^* & A \end{pmatrix} \quad (7)$$

with  $A \gg C, |B| \gg E, |D|, |F|$ . By a common unitary transformation of the up- and down-type quark fields, one can always arrange the mass matrices  $M_u$  and  $M_d$  in such a way that  $F_u = F_d = 0$ ; *i.e.*,

$$M = \begin{pmatrix} E & D & 0 \\ D^* & C & B \\ 0 & B^* & A \end{pmatrix}. \quad (8)$$

This can easily be seen as follows. If phases are neglected, the two symmetric mass matrices  $M_u$  and  $M_d$  can be transformed by an orthogonal transformation matrix  $O$ , which can be described by three angles such that they assume the form (8). The condition  $F_u = F_d = 0$  gives two constraints for the three angles of  $O$ . If complex phases are allowed in  $M_u$  and  $M_d$ , the



condition  $F_u = F_u^* = F_d = F_d^* = 0$  imposes four constraints, which can also be fulfilled, if  $M_u$  and  $M_d$  are subjected to a common unitary transformation matrix  $U$ . The latter depends on nine parameters. Three of them are not suitable for our purpose, since they are just diagonal phases; but the remaining six can be chosen such that the vanishing of  $F_u$  and  $F_d$  results.

The basis in which the mass matrices take the form (8) is a basis in the space of quark flavors, which in our view is of special interest. It is a basis in which the mass matrices exhibit two texture zeros, for both up- and down-type quark sectors. These, however, do not imply special relations among mass eigenvalues and flavor mixing parameters (as pointed out above). In this basis the mixing is of the “nearest neighbour” form, since the (1,3) and (3,1) elements of  $M_u$  and  $M_d$  vanish; no direct mixing between the heavy  $t$  (or  $b$ ) quark and the light  $u$  (or  $d$ ) quark is present. In certain models (see, *e.g.*, Refs. [6, 7]), this basis is indeed of particular interest, but we shall proceed without relying on a special texture models for the mass matrices.

A mass matrix of the type (8) can in the absence of complex phases be diagonalized by a rotation matrix, described only by two angles in the hierarchy limit of quark masses [6]. At first the off-diagonal element  $B$  is rotated away by a rotation between the second and third families (angle  $\theta_{23}$ ); at the second step the element  $D$  is rotated away by a transformation of the first and second families (angle  $\theta_{12}$ ). No rotation between the first and third families is required<sup>1</sup>. The rotation matrix for this sequence takes the form

$$\begin{aligned}
 R &= R_{12}R_{23} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \\
 &= \begin{pmatrix} c_{12} & s_{12}c_{23} & s_{12}s_{23} \\ -s_{12} & c_{12}c_{23} & c_{12}s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \tag{9}
 \end{aligned}$$

where  $c_{12} \equiv \cos \theta_{12}$ ,  $s_{12} \equiv \sin \theta_{12}$ , *etc.* The flavor mixing matrix  $V$  is the product of two such matrices, one describing the rotation among the up-type quarks, and the other describing the rotation among the down-type quarks:

$$V = R_{12}^u R_{23}^u (R_{23}^d)^{-1} (R_{12}^d)^{-1}. \tag{10}$$

The product  $R_{23}^u (R_{23}^d)^{-1}$  can be written as a rotation matrix described by a single angle  $\theta$ . In the limit  $m_u = m_d = 0$ , this is just the angle  $\tilde{\theta}$  encountered

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<sup>1</sup> This is true in the lowest order as the corresponding rotation angle is small enough, in accordance with the hierarchical structure discussed above. However, our final result of parametrizing the flavor mixing matrix will remain exact, independent of the mentioned approximation. The reason is that a rotation between the first and third families can always be absorbed by redefining the relevant rotation matrices. For a detailed discussion, see Ref. [17].

in Eq. (6). The angle which describes the  $R_{12}^u$  rotation shall be denoted by  $\theta_u$ ; the corresponding angle for the  $R_{12}^d$  rotation by  $\theta_d$ . Thus in the absence of CP-violating phases the flavor mixing matrix takes the following specific form:

$$\begin{aligned}
 V &= \begin{pmatrix} c_u & s_u & 0 \\ -s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_d & -s_d & 0 \\ s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} s_u s_d c + c_u c_d & s_u c_d c - c_u s_d & s_u s \\ c_u s_d c - s_u c_d & c_u c_d c + s_u s_d & c_u s \\ -s_d s & -c_d s & c \end{pmatrix}, \quad (11)
 \end{aligned}$$

where  $c_u \equiv \cos \theta_u$ ,  $s_u \equiv \sin \theta_u$ , etc.

We proceed by including the phase parameters of the quark mass matrices in Eq. (8). Each mass matrix has in general two complex phases. These phases can be dealt with in a similar way as described in Refs. [6, 18]. It can easily be seen that, by suitable rephasing of the quark fields, the flavor mixing matrix can finally be written in terms of only a single phase  $\varphi$  as follows [5]:

$$\begin{aligned}
 V &= \begin{pmatrix} c_u & s_u & 0 \\ -s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\varphi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_d & -s_d & 0 \\ s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} s_u s_d c + c_u c_d e^{-i\varphi} & s_u c_d c - c_u s_d e^{-i\varphi} & s_u s \\ c_u s_d c - s_u c_d e^{-i\varphi} & c_u c_d c + s_u s_d e^{-i\varphi} & c_u s \\ -s_d s & -c_d s & c \end{pmatrix}. \quad (12)
 \end{aligned}$$

Note that the three angles  $\theta_u$ ,  $\theta_d$  and  $\theta$  in Eq. (12) can all be arranged to lie in the first quadrant through a suitable redefinition of quark field phases. Consequently all  $s_u$ ,  $s_d$ ,  $s$  and  $c_u$ ,  $c_d$ ,  $c$  are positive. The phase  $\varphi$  can in general take values from 0 to  $2\pi$ ; and CP violation is present in weak interactions if  $\varphi \neq 0, \pi$  and  $2\pi$ .

This particular and exact representation of the flavor mixing matrix was first obtained by Xing and me in Ref. [5]. In comparison with all other parametrizations discussed previously [2-4], it has a number of interesting features which in our view make it very attractive and provide strong arguments for its use in future discussions of flavor mixing phenomena, in particular, those in  $B$ -meson physics. We shall discuss them below.

a) The flavor mixing matrix  $V$  in Eq. (12) follows directly from the chiral expansion of the mass matrices. Thus it naturally takes into account the hierarchical structure of the quark mass spectrum.

b) The complex phase describing CP violation ( $\varphi$ ) appears only in the (1,1), (1,2), (2,1) and (2,2) elements of  $V$ , i.e., in the elements involving only

the quarks of the first and second families. This is a natural description of CP violation since in our hierarchical approach CP violation is not directly linked to the third family, but rather to the first and second ones, and in particular to the mass terms of the  $u$  and  $d$  quarks.

It is instructive to consider the special case  $s_u = s_d = s = 0$ . Then the flavor mixing matrix  $V$  takes the form

$$V = \begin{pmatrix} e^{-i\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

This matrix describes a phase change in the weak transition between  $u$  and  $d$ , while no phase change is present in the transitions between  $c$  and  $s$  as well as  $t$  and  $b$ . Of course, this effect can be absorbed in a phase change of the  $u$ - and  $d$ -quark fields, and no CP violation is present. Once the angles  $\theta_u$ ,  $\theta_d$  and  $\theta$  are introduced, however, CP violation arises. It is due to a phase change in the weak transition between  $u'$  and  $d'$ , where  $u'$  and  $d'$  are the rotated quark fields, obtained by applying the corresponding rotation matrices given in Eq. (12) to the quark mass eigenstates ( $u'$ : mainly  $u$ , small admixture of  $c$ ;  $d'$ : mainly  $d$ , small admixture of  $s$ ).

Since the mixing matrix elements involving  $t$  or  $b$  quark are real in the representation (12), one can find that the phase parameter of  $B_q^0$ - $\bar{B}_q^0$  mixing ( $q = d$  or  $s$ ), dominated by the box-diagram contributions in the standard model [19], is essentially unity:

$$\left(\frac{q}{p}\right)_{B_q} = \frac{V_{tb}^* V_{tq}}{V_{tb} V_{tq}^*} = 1. \tag{14}$$

In most of other parametrizations of the flavor mixing matrix, however, the two rephasing-variant quantities  $(q/p)_{B_d}$  and  $(q/p)_{B_s}$  take different (maybe complex) values.

c) The dynamics of flavor mixing can easily be interpreted by considering certain limiting cases in Eq. (12). In the limit  $\theta \rightarrow 0$  (*i.e.*,  $s \rightarrow 0$  and  $c \rightarrow 1$ ), the flavor mixing is, of course, just a mixing between the first and second families, described by only one mixing angle (the Cabibbo angle  $\theta_C$  [1]). It is a special and essential feature of the representation (12) that the Cabibbo angle is *not* a basic angle, used in the parametrization. The matrix element  $V_{us}$  (or  $V_{cd}$ ) is indeed a superposition of two terms including a phase. This feature arises naturally in our hierarchical approach, but it is not new. In many models of specific textures of mass matrices, it is indeed the case that the Cabibbo-type transition  $V_{us}$  (or  $V_{cd}$ ) is a superposition of several terms. At first, it was obtained by me in discussing the two-family mixing [18].

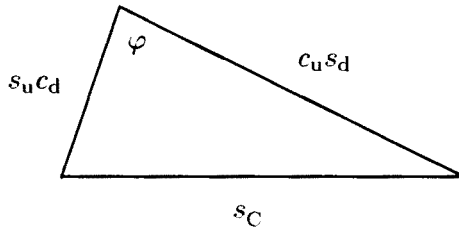


Fig. 1. The light quark (LQ) triangle in the complex plane.

In the limit  $\theta = 0$  considered here, one has  $|V_{us}| = |V_{cd}| = \sin \theta_C \equiv s_C$  and

$$s_C = \left| s_u c_d - c_u s_d e^{-i\varphi} \right|. \quad (15)$$

This relation describes a triangle in the complex plane, as illustrated in Fig. 1, which we shall denote as the “light quark (LQ) triangle”. This triangle is a feature of the mixing of the first two families (see also Ref. [18]). Explicitly one has (for  $s = 0$ ):

$$\tan \theta_C = \sqrt{\frac{\tan^2 \theta_u + \tan^2 \theta_d - 2 \tan \theta_u \tan \theta_d \cos \varphi}{1 + \tan^2 \theta_u \tan^2 \theta_d + 2 \tan \theta_u \tan \theta_d \cos \varphi}}. \quad (16)$$

Certainly the flavor mixing matrix  $V$  cannot accommodate CP violation in this limit. However, the existence of  $\varphi$  seems necessary in order to make Eq. (16) compatible with current data, as one can see below.

d) The three mixing angles  $\theta$ ,  $\theta_u$  and  $\theta_d$  have a precise physical meaning. The angle  $\theta$  describes the mixing between the second and third families, which is generated by the off-diagonal terms  $B_u$  and  $B_d$  in the up and down mass matrices of Eq. (8). We shall refer to this mixing involving  $t$  and  $b$  as the “heavy quark mixing”. The angle  $\theta_u$ , however, solely describes the  $u$ - $c$  mixing in the limit  $m_u \ll m_c \ll m_t$ , corresponding to the  $D_u$  term in  $M_u$ . We shall denote this as the “ $u$ -channel mixing”. The angle  $\theta_d$  solely describes the  $d$ - $s$  mixing in the limit  $m_d \ll m_s \ll m_b$ , corresponding to the  $D_d$  term in  $M_d$ ; this will be denoted as the “ $d$ -channel mixing”. Thus there exists an asymmetry between the mixing of the first and second families and that of the second and third families, which in our view reflects interesting details of the underlying dynamics of flavor mixing. The heavy quark mixing is a combined effect, involving both charge  $+2/3$  and charge  $-1/3$  quarks, while the  $u$ - or  $d$ -channel mixing (described by the angle  $\theta_u$  or  $\theta_d$ ) proceeds solely in the charge  $+2/3$  or charge  $-1/3$  sector. Therefore an experimental determination of these two angles would allow to draw interesting conclusions about the amount and perhaps the underlying pattern of the  $u$ - or  $d$ -channel mixing.

e) The three angles  $\theta$ ,  $\theta_u$  and  $\theta_d$  are related in a very simple way to observable quantities of  $B$ -meson physics. For example,  $\theta$  is related to the rate of the semileptonic decay  $B \rightarrow D^* l \nu_l$ ;  $\theta_u$  is associated with the ratio of the decay rate of  $B \rightarrow (\pi, \rho) l \nu_l$  to that of  $B \rightarrow D^* l \nu_l$ ; and  $\theta_d$  can be determined from the ratio of the mass difference between two  $B_d$  mass eigenstates to that between two  $B_s$  mass eigenstates. From Eq. (12) we find the following exact relations:

$$\sin \theta = |V_{cb}| \sqrt{1 + \left| \frac{V_{ub}}{V_{cb}} \right|^2}, \tag{17}$$

and

$$\begin{aligned} \tan \theta_u &= \left| \frac{V_{ub}}{V_{cb}} \right|, \\ \tan \theta_d &= \left| \frac{V_{td}}{V_{ts}} \right|. \end{aligned} \tag{18}$$

These simple results make the parametrization (12) uniquely favorable for the study of  $B$ -meson physics.

By use of current data on  $|V_{ub}|$  and  $|V_{cb}|$ , *i.e.*,  $|V_{cb}| = 0.039 \pm 0.002$  [15,16] and  $|V_{ub}/V_{cb}| = 0.08 \pm 0.02$  [19], we obtain  $\theta_u = 4.57^\circ \pm 1.14^\circ$  and  $\theta = 2.25^\circ \pm 0.12^\circ$ . Taking  $|V_{td}| = (8.6 \pm 2.1) \times 10^{-3}$  [16], which was obtained from the analysis of current data on  $B_d^0$ - $\bar{B}_d^0$  mixing, we get  $|V_{td}/V_{ts}| = 0.22 \pm 0.07$ , *i.e.*,  $\theta_d = 12.7^\circ \pm 3.8^\circ$ . Both the heavy quark mixing angle  $\theta$  and the u-channel mixing angle  $\theta_u$  are relatively small. The smallness of  $\theta$  implies that Eqs. (15) and (16) are valid to a high degree of precision (of order  $1 - c \approx 0.001$ ).

f) According to Eq. (16), as well as Eq. (15), the phase  $\varphi$  is a phase difference between the contributions to  $V_{us}$  (or  $V_{cd}$ ) from the u-channel mixing and the d-channel mixing. Therefore  $\varphi$  is given by the relative phase of  $D_d$  and  $D_u$  in the quark mass matrices (8), if the phases of  $B_u$  and  $B_d$  are absent or negligible.

The phase  $\varphi$  is not likely to be  $0^\circ$  or  $180^\circ$ , according to the experimental values given above, even though the measurement of CP violation in  $K^0$ - $\bar{K}^0$  mixing [19] is not taken into account. For  $\varphi = 0^\circ$ , one finds  $\tan \theta_C = 0.14 \pm 0.08$ ; and for  $\varphi = 180^\circ$ , one gets  $\tan \theta_C = 0.30 \pm 0.08$ . Both cases are barely consistent with the value of  $\tan \theta_C$  obtained from experiments ( $\tan \theta_C \approx |V_{us}/V_{ud}| \approx 0.226$ ).

g) The CP-violating phase  $\varphi$  in the flavor mixing matrix  $V$  can be determined from  $|V_{us}|$  ( $= 0.2205 \pm 0.0018$  [19]) through the following formula, obtained easily from Eq. (12):

$$\varphi = \arccos \left( \frac{s_u^2 c_d^2 c^2 + c_u^2 s_d^2 - |V_{us}|^2}{2s_u c_u s_d c_d c} \right). \tag{19}$$

The two-fold ambiguity associated with the value of  $\varphi$ , coming from  $\cos \varphi = \cos(2\pi - \varphi)$ , is removed if one takes  $\sin \varphi > 0$  into account (this is required by current data on CP violation in  $K^0$ - $\bar{K}^0$  mixing (*i.e.*,  $\epsilon_K$ ) [19]). More precise measurements of the angles  $\theta_u$  and  $\theta_d$  in the forthcoming experiments of  $B$  physics will remarkably reduce the uncertainty of  $\varphi$  to be determined from Eq. (19). This approach is of course complementary to the direct determination of  $\varphi$  from CP asymmetries in some weak  $B$ -meson decays into hadronic CP eigenstates.

Taking the presently known phenomenological constraints on quark mixing and CP violation into account, we find that the value of  $\varphi$  is most likely in the range  $40^\circ$  to  $120^\circ$ ; the central value is  $\varphi \approx 81^\circ$  [5]. Note that  $\varphi$  is essentially independent of the angle  $\theta$ , due to the tiny observed value of the latter. Once  $\tan \theta_d$  is precisely measured, one shall be able to fix the magnitude of  $\varphi$  to a satisfactory degree of accuracy.

h) It is well known that CP violation in the flavor mixing matrix  $V$  can be rephasing-invariantly described by a universal quantity  $\mathcal{J}$  [20]:

$$\text{Im} \left( V_{il} V_{jm} V_{im}^* V_{jl}^* \right) = \mathcal{J} \sum_{k,n=1}^3 (\epsilon_{ijk} \epsilon_{lmn}) . \quad (20)$$

In the parametrization (12),  $\mathcal{J}$  reads

$$\mathcal{J} = s_u c_u s_d c_d s^2 c \sin \varphi . \quad (21)$$

Obviously  $\varphi = 90^\circ$  leads to the maximal value of  $\mathcal{J}$ . Indeed  $\varphi = 90^\circ$ , a particularly interesting case for CP violation, is quite consistent with current data. One can see from Fig. 2 of Ref. [5] that this possibility exists if  $0.202 \leq \tan \theta_d \leq 0.222$ , or  $11.4^\circ \leq \theta_d \leq 12.5^\circ$ . In this case the mixing term  $D_d$  in Eq. (8) can be taken to be real, and the term  $D_u$  to be imaginary, if  $\text{Im}(B_u) = \text{Im}(B_d) = 0$  is assumed (see also Refs. [14, 21]). Since in our description of the flavor mixing the complex phase  $\varphi$  is related in a simple way to the phases of the quark mass terms, the case  $\varphi = 90^\circ$  is especially interesting. It can hardly be an accident, and this case should be studied further. The possibility that the phase  $\varphi$  describing CP violation in the standard model is given by the algebraic number  $\pi/2$  should be taken seriously. It may provide a useful clue towards a deeper understanding of the origin of CP violation and of the dynamical origin of the fermion masses.

In Ref. [21] the case  $\varphi = 90^\circ$  has been denoted as "maximal" CP violation. It implies in our framework that in the complex plane the u-channel and d-channel mixings are perpendicular to each other. In this special case (as well as  $\theta \rightarrow 0$ ), we have

$$\tan^2 \theta_C = \frac{\tan^2 \theta_u + \tan^2 \theta_d}{1 + \tan^2 \theta_u \tan^2 \theta_d} . \quad (22)$$

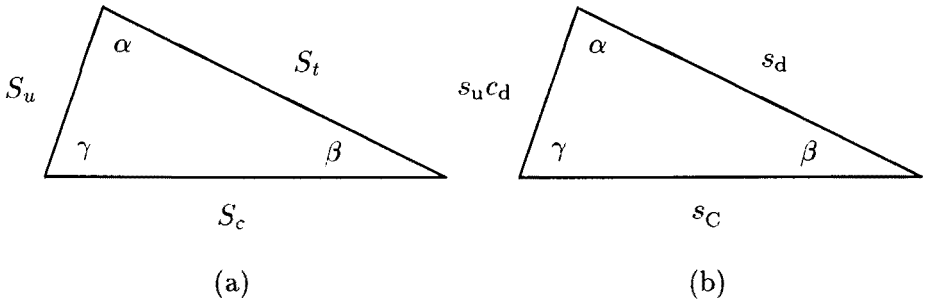


Fig. 2. The unitarity triangle (a) and its rescaled counterpart (b) in the complex plane.

To a good approximation (with the relative error  $\sim 2\%$ ), one finds  $s_c^2 \approx s_u^2 + s_d^2$ .

i) At future  $B$ -meson factories, the study of CP violation will concentrate on measurements of the unitarity triangle

$$S_u + S_c + S_t = 0, \tag{23}$$

where  $S_i \equiv V_{id}V_{ib}^*$  in the complex plane (see Fig. 2(a)). The inner angles of this triangle are denoted as

$$\begin{aligned} \alpha &\equiv \arg(-S_t S_u^*), \\ \beta &\equiv \arg(-S_c S_t^*), \\ \gamma &\equiv \arg(-S_u S_c^*). \end{aligned} \tag{24}$$

In terms of the parameters  $\theta, \theta_u, \theta_d$  and  $\varphi$ , we obtain

$$\begin{aligned} \sin(2\alpha) &= \frac{2c_u c_d \sin \varphi (s_u s_d c + c_u c_d \cos \varphi)}{s_u^2 s_d^2 c^2 + c_u^2 c_d^2 + 2s_u c_u s_d c_d c \cos \varphi}, \\ \sin(2\beta) &= \frac{2s_u c_d \sin \varphi (c_u s_d c - s_u c_d \cos \varphi)}{c_u^2 s_d^2 c^2 + s_u^2 c_d^2 - 2s_u c_u s_d c_d c \cos \varphi}. \end{aligned} \tag{25}$$

To an excellent degree of accuracy, one finds  $\alpha \approx \varphi$ . In order to illustrate how accurate this relation is, let us input the central values of  $\theta, \theta_u$  and  $\theta_d$  (i.e.,  $\theta = 2.25^\circ, \theta_u = 4.57^\circ$  and  $\theta_d = 12.7^\circ$ ) to Eq. (25). Then one arrives at  $\varphi - \alpha \approx 1^\circ$  as well as  $\sin(2\alpha) \approx 0.34$  and  $\sin(2\beta) \approx 0.65$ . It is expected that  $\sin(2\alpha)$  and  $\sin(2\beta)$  will be directly measured from the CP asymmetries in  $B_d \rightarrow \pi^+ \pi^-$  and  $B_d \rightarrow J/\psi K_S$  modes at a  $B$ -meson factory.

Note that the three sides of the unitarity triangle (23) can be rescaled by  $|V_{cb}|$ . In a very good approximation (with the relative error  $\sim 2\%$ ), one arrives at

$$|S_u| : |S_c| : |S_t| \approx s_u c_d : s_C : s_d . \tag{26}$$

Equivalently, one can obtain

$$s_\alpha : s_\beta : s_\gamma \approx s_C : s_u c_d : s_d , \tag{27}$$

where  $s_\alpha \equiv \sin \alpha$ , *etc.* The rescaled unitarity triangle is shown in Fig. 2(b). Comparing this triangle with the LQ triangle in Fig. 1, we find that they are indeed congruent with each other to a high degree of accuracy. The congruent relation between these two triangles is particularly interesting, since the LQ triangle is essentially a feature of the physics of the first two quark families, while the unitarity triangle is linked to all three families. In this connection it is of special interest to note that in models which specify the textures of the mass matrices the Cabibbo triangle and hence three inner angles of the unitarity triangle can be fixed by the spectrum of the light quark masses and the CP-violating phase  $\varphi$  (see, *e.g.*, Ref. [21]).

j) It is worth pointing out that the u-channel and d-channel mixing angles are related to the so-called Wolfenstein parameters [22] in a simple way:

$$\begin{aligned} \tan \theta_u &= \left| \frac{V_{ub}}{V_{cb}} \right| \approx \lambda \sqrt{\rho^2 + \eta^2} , \\ \tan \theta_d &= \left| \frac{V_{td}}{V_{ts}} \right| \approx \lambda \sqrt{(1 - \rho)^2 + \eta^2} , \end{aligned} \tag{28}$$

where  $\lambda \approx s_C$  measures the magnitude of  $V_{us}$ . Note that the CP-violating parameter  $\eta$  is linked to  $\varphi$  through

$$\sin \varphi \approx \frac{\eta}{\sqrt{\rho^2 + \eta^2} \sqrt{(1 - \rho)^2 + \eta^2}} \tag{29}$$

in the lowest-order approximation. Then  $\varphi = 90^\circ$  implies  $\eta^2 \approx \rho(1 - \rho)$ , on the condition  $0 < \rho < 1$ . In this interesting case, of course, the flavor mixing matrix can fully be described in terms of only three independent parameters.

k) Compared with the standard parametrization of the flavor mixing matrix  $V$  advocated in Ref. [4], the parametrization (12) has an additional advantage: the renormalization-group evolution of  $V$ , from the weak scale to an arbitrary high energy scale, is to a very good approximation associated only with the angle  $\theta$ . This can easily be seen if one keeps the  $t$  and  $b$  Yukawa couplings only and neglects possible threshold effect in the one-loop renormalization-group equations of the Yukawa matrices. Thus the parameters  $\theta_u$ ,  $\theta_d$  and  $\varphi$  are essentially independent of the energy scale, while



$\theta$  does depend on it and will change if the underlying scale is shifted, say from the weak scale ( $\sim 10^2$  GeV) to the grand unified theory scale (of order  $10^{16}$  GeV). In short, the heavy quark mixing is subject to renormalization-group effects; but the u- and d-channel mixings are not, likewise the phase  $\varphi$  describing CP violation and the LQ triangle as a whole.

We have presented a new description of the flavor mixing phenomenon, which is based on the phenomenological fact that the quark mass spectrum exhibits a clear hierarchy pattern. This leads uniquely to the interpretation of the flavor mixing in terms of a heavy quark mixing, followed by the u-channel and d-channel mixings. The complex phase  $\varphi$ , describing the relative orientation of the u-channel mixing and the d-channel mixing in the complex plane, signifies CP violation, which is a phenomenon primarily linked to the physics of the first two families. The Cabibbo angle is not a basic mixing parameter, but given by a superposition of two terms involving the complex phase  $\varphi$ . The experimental data suggest that the phase  $\varphi$ , which is directly linked to the phases of the quark mass terms, is close to  $90^\circ$ . This opens the possibility to interpret CP violation as a maximal effect, in a similar way as parity violation.

Our description of flavor mixing has many clear advantages compared with other descriptions. We propose that it should be used in the future description of flavor mixing and CP violation, in particular, for the studies of quark mass matrices and  $B$ -meson physics.

The description of the flavor mixing phenomenon given above is of special interest if for the u- and d-channel mixings the quark mass textures discussed first in Ref. [18] are applied (see also Ref. [21]). In that case one finds

$$\begin{aligned}\tan \theta_u &= \sqrt{\frac{m_u}{m_c}}, \\ \tan \theta_d &= \sqrt{\frac{m_d}{m_s}},\end{aligned}\tag{30}$$

apart from small corrections [17]. The experimental value for  $\tan \theta_u$  given by the ratio  $|V_{ub}/V_{cb}|$  is in agreement with the observed value for  $\sqrt{m_u/m_c} \approx 0.07$ , but the error bars for both  $m_u/m_c$  and  $|V_{ub}/V_{cb}|$  are almost the same (about 25%). Thus from the underlying texture no new information is obtained.

This is not true for the angle  $\theta_d$ , whose experimental value is due to a large uncertainty:  $\theta_d = 12.7^\circ \pm 3.8^\circ$ . If  $\theta_d$  is given indeed by the square root of the quark mass ratio  $m_d/m_s$ , which is known to a high degree of accuracy [23], we would know  $\theta_d$  and therefore all four parameters of the flavor mixing matrix with high precision.

As emphasized in Ref. [21], the phase angle  $\varphi$  is very close to  $90^\circ$ , implying that the LQ triangle and the unitarity triangle are essentially rectangular

triangles. In particular the angle  $\beta$ , which is likely to be measured soon from the CP asymmetry in  $B_d \rightarrow J/\psi K_S$ , is expected to be close to  $20^\circ$ .

It will be very interesting to see whether the angles  $\theta_u$  and  $\theta_d$  are indeed given by the square roots of the light quark mass ratios  $m_u/m_c$  and  $m_d/m_s$ , which imply that the phase  $\varphi$  is close to or exactly equal to  $90^\circ$ . This would mean that the light quarks play the most important role in the dynamics of flavor mixing and CP violation and that a small window has been opened allowing the first view across the physics landscape beyond the mountain chain of the Standard Model.

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