A UNIFIED BFKL AND DGLAP EVOLUTION EQUATIONS FOR QUARKS AND GLUONS *

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We discuss the partonic description of the HERA data for F_2, paying particular attention to a recent k_T factorization approach which unifies DGLAP and BFKL effects.

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1. Introduction

One of the most striking features of the measurements of deep inelastic scattering at HERA is the strong rise of the proton structure function F_2 as x decreases from 10^{-2} to below 10^{-4}. At first sight it appeared that the rise was due to the (BFKL) resummation of leading $\ln(1/x)$ contributions. In the small x domain the basic dynamical quantity is the gluon distribution $f(x, k_T^2)$ unintegrated over its transverse momentum k_T. Observables are computed in terms of f via the k_T factorization theorem. For example

$$F_2(x, Q^2) = \int \frac{dk_T^2}{k_T^2} \int_x^1 \frac{dx'}{x'} F_2^{\gamma g}(\frac{x}{x'}, k_T^2, Q^2) f(x', k_T^2) \qquad (1)$$

where $F_2^{\gamma g}$ is the off-shell gluon structure function which at lowest order is given by the "quark box and crossed-box" contributions, $\gamma g \to q\bar{q}$, see Fig. 1(a). In principle the $\ln(1/x)$ resummation predicts the small x behaviour of f and hence, via (1), the behaviour of F_2. At leading order, for

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fixed α_S, the BFKL equation gives $f \sim x^{-\lambda}$ as $1/x \to \infty$ with $\lambda = \overline{\alpha}_S 4 \ln 2$ (with $\overline{\alpha}_S = 3\alpha_S/\pi$). However, even allowing for the running of α_S, the leading order prediction is found to be too steep to describe the recent, much more precise, measurements of F_2. Of course, at subasymptotic values of $1/x$ the prediction is subject to considerable correction. We return to this in a moment.

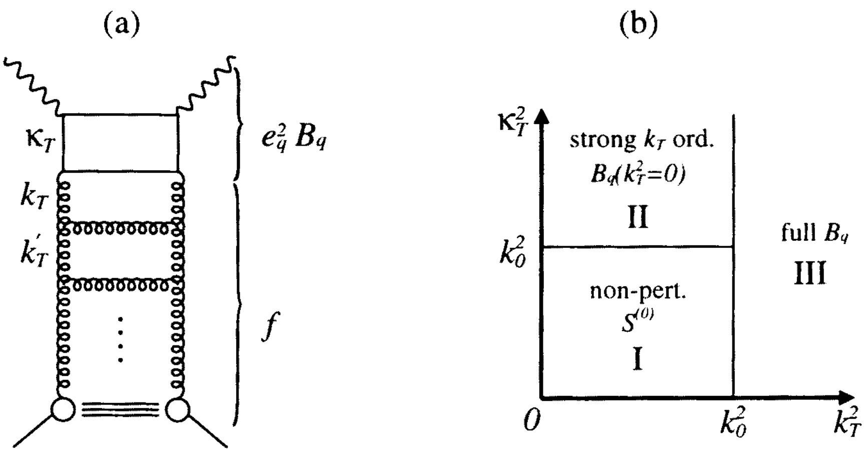

Fig. 1. (a) The diagrammatic representation of the $k_{\rm T}$ factorization formula $F_2 = F_2^{\gamma g} \otimes f$ of equation (1). At lowest order $F_2^{\gamma g}$ is given by the quark box shown (together with the crossed box) summed over all types of quark, $F_2^{\gamma g} = \Sigma e_q^2 B_q$. (b) The different regions of integration used to evaluate the sea quark distribution. In regions II and III the sea is driven by the gluon, $S_q = B_q \otimes f$.

On the other hand the conventional DGLAP approach is able to describe the F_2 data at the smallest x values observed (even for $Q^2 \sim 1{\rm GeV}^2$) with an appropriate choice of parton distributions at some starting scale Q_0^2 for the evolution. DGLAP evolution effectively resumes the leading $\ln Q^2$ contributions, which arise from strongly ordered transverse momenta $Q^2 \gg k_{\rm T}^2 \gg k_{\rm T}'^2 \gg \ldots$ along the gluon chain in Fig. 1(a). In this approximation (1) can be expressed in terms of the familiar integrated gluon distribution,

$$xg(x,Q^2) = \int^{Q^2} \frac{dk_{\rm T}^2}{k_{\rm T}^2} f(x,k_{\rm T}^2), \tag{2}$$

and F_2 is given by the DGLAP collinear factorization formula. The free choice of non-perturbative input shapes in x for the parton distributions $g(x,Q_0^2)$, $\overline{q}(x,Q_0^2) \ldots$ at the starting scale for DGLAP evolution means that there is more freedom than in the BFKL approach. In practice this distinction is not so clear cut since non-perturbative and subleading $\ln(1/x)$ effects modify the BFKL prediction. Of course, the issue is not whether DGLAP or BFKL give a better description of F_2 at small x, since both $\ln Q^2$ and $\ln(1/x)$ resummations are necessary. Rather we seek a unified description of F_2 which embodies both of these perturbative QCD effects.

2. Unification of BFKL and DGLAP with collinear factorization

A unified BFKL/DGLAP description can be obtained from a DGLAP analysis, but with the $\ln(1/x)$ summations included in the anomalous dimensions γ_{ij} and coefficient functions [1]. Then the splitting functions P_{ij} take the form

$$xP_{gg} = \sum_{n=1} A_n \frac{[\alpha_S \ln(1/x)]^{n-1}}{(n-1)!}, \tag{3}$$

$$xP_{qg} = \frac{\alpha_S}{2\pi} xP_{qg}^{(0)}(x) + \alpha_S^2 \sum_{n=1} B_n \frac{[\alpha_S \ln(1/x)]^{n-1}}{(n-1)!}. \tag{4}$$

We see that P_{qg} is formally non-leading at small x in contrast to P_{gg}. However, for moderately small values of x, which are relevant for the HERA data, it is found [2] that the resummation effects in P_{qg} have a stronger impact on F_2 than those in P_{gg}. This is due to the fact that the coefficients $A_2 = A_3 = A_5 = 0$ while all the B_n coefficients are non-zero. Moreover, the BFKL resummation effects in P_{qg} can significantly affect the extraction of the gluon distribution from the observed scaling violations of F_2, since $\partial F_2/\partial \ln Q^2 \sim P_{qg}\overline{\otimes} g$, where $\overline{\otimes}$ denotes a convolution in longitudinal momentum only.

We are interested in the case when the input parton distributions are flat in x at small x, so that the rise of F_2 is due to perturbative QCD. Unfortunately, for flat input distributions the small x behaviour of F_2 is found to be sensitive to the renormalization scheme that is adopted.

An interesting development is the possibility of writing evolution equations for observables, for example

$$\frac{\partial F_2}{\partial \ln Q^2} = \Gamma_{22}\overline{\otimes}F_2 + \Gamma_{2L}\overline{\otimes}F_L, \tag{5}$$

where Γ_{ij} are independent of the scheme [3]. It is encouraging that an analysis [4] based on such an approach finds that the $\ln(1/x)$ resummations are not excluded by the data, but rather that their presence improves the description of F_2.

3. Unification of BFKL and DGLAP in the k_T factorization approach

As we mentioned above, the unintegrated gluon and the k_T factorization theorem provide the natural framework for describing small x physics (see, for example, (1)). The $\ln(1/x)$ resummations, which have to be performed explicitly in the collinear approach, are automatically implicitly included in the integration over the entire k_T^2 phase space of the gluon ladder and in the k_T factorization integrals.

The early attempts [5] to obtain the unintegrated gluon distribution $f(x, k_T^2)$ by numerically solving the BFKL equation were plagued by the dependence on the treatment of the infrared region, $k_T^2 < k_0^2$. Here we describe an improved treatment [6] in which the BFKL equation is arranged so that we only need to solve it in the perturbative domain $k_T^2 > k_0^2$ and in which the residual DGLAP contributions are now included. To be precise we solve the unified BFKL/DGLAP equation

$$f(x,k_T^2) = \frac{\alpha_S}{2\pi}\int_x^1 dz P_{gg}(z)\frac{x}{z}g(\frac{x}{z},k_0^2) + \frac{\alpha_S}{2\pi}\int_x^1 dz P_{gq}(z)\Sigma\left(\frac{x}{z},k_T^2\right)$$
$$+\overline{\alpha}_S k_T^2 \int_x^1 \frac{dz}{z}\int_{k_0^2} \frac{dk_T'^2}{k_T'^2}\left[\frac{f(\frac{x}{z},k_T'^2)\theta\left(\frac{k_T^2}{z}-k_T'^2\right) - f(\frac{x}{z},k_T^2)}{|k_T'^2-k_T^2|}\right.$$
$$\left.+\frac{f(\frac{x}{z},k_T^2)}{(4k_T'^4+k_T^4)^{\frac{1}{2}}}\right] + \overline{\alpha}_S\int_x^1 \frac{dz}{z}\left(\frac{z}{6}P_{gg}(z)-1\right)\int_{k_0^2}^{k_T^2}\frac{dk_T'^2}{k_T'^2} f\left(\frac{x}{z},k_T'^2\right). \tag{6}$$

Symbolically this equation for the gluon has the structure

$$f = \text{non-pert.input} + f \leftarrow f(\text{BFKL}) + f \leftarrow f(\text{DGLAP}-1) + f \leftarrow q\text{singlet}$$

where -1 is taken from DGLAP because it is already included in BFKL. The input term comes from the $k_T^2 < k_0^2$ parts of BFKL and of (DGLAP $-$ 1). We specify the input in terms of a simple two parameter form

$$xg(x,k_0^2) = N(1-x)^{\beta}. \tag{7}$$

In addition to restricting the solution of the BFKL equation to the perturbative region $k_T^2 > k_0^2$ and to including the DGLAP terms, we have also introduced a θ function which imposes the constraint $k_T'^2 < k_T^2/z$ on the real gluon emissions. The origin of this constraint[1] is the requirement that the virtuality of the exchanged gluon is dominated by its transverse momentum $|k'|^2 \simeq k_T'^2$. We take a running coupling $\alpha_S(k_T^2)$, which is consistent with the results of the next-to-leading order $\ln(1/x)$ analyses of Fadin and Lipatov, and Camici and Ciafaloni [9].

The final term in (6) depends on the quark singlet momentum distribution Σ. At small x the sea quark components S_q of Σ dominate. They are driven by the gluon via the $g \to q\bar{q}$ transitions, that is

$$S_q = B_q \otimes f \tag{8}$$

where at lowest order B_q is the box (and crossed box) contribution indicated in Fig. 1(a). Besides the z and k_T^2 integrations symbolically denoted by $\otimes$ the box contribution implicitly includes an integration over the transverse momentum κ_T of the exchanged quark. The evolution equation for Σ may be written in the form

$$\Sigma = S^{(0)} + \sum_q B_q(k_T^2 = 0)\bar{\otimes} zg(z, k_0^2) + \sum_q B_q \otimes f + P_{qq} \otimes S_q + V \tag{9}$$

where $\bar{\otimes}$ is simply a convolution over longitudinal momentum. The first three terms on the right hand side are the "$B_q \otimes f$" contributions of (8) coming from the regions I, II, III of the k_T^2 and κ_T^2 integrations that are shown in Fig. 1(b). First, in the non-perturbative domain, region I, the u, d, s sea quark contribution is parametrized in the form

$$S^{(0)} = C_{\boldsymbol{P}} x^{-0.08}(1-x)^8 \tag{10}$$

consistent with soft pomeron and counting rule expectations, where $C_{\boldsymbol{P}}$ is independent of Q^2. The constant $C_{\boldsymbol{P}}$ is fixed in terms of the two parameters, N and β, of (7) by the momentum sum rule. In region II we apply the strong k_T ordering approximation with $B_q \approx B_q(k_T^2 = 0)$ so that the k_T^2 integration can be carried out to give a contribution proportional to $g(x/z, k_0^2)$ [10]. Finally in region III we evaluate the full box contribution; this gives the main contribution and is responsible for the rise of F_2 with decreasing x. The last two terms in (9) give the sea $\to$ sea evolution contribution, and the valence contribution $V(x, Q^2)$ which is taken directly from a recent parton set. The charm quark component of the sea is given totally by *perturbative*

[1] A more general treatment of the gluon ladder which incorporates both the BFKL equation and DGLAP evolution is given by the CCFM equation [7], which is based on angular ordering of the gluon emissions. The angular ordered and kinematic constraints lead to similar subleading $\ln(1/x)$ effects, but the kinematic constraint overrides the angular ordered constraint, except when $Q^2 < k_T^2$ in the large x domain [8].

QCD, since for $k_T^2 < k_0^2$ the box $B(k_T^2 = 0)$ is finite as $\kappa_T^2 \to 0$ due to $m_c \neq 0$.

The coupled integral equations (6) and (9) for the gluon f and the quark singlet Σ are solved in the perturbative domain, $k_T^2 > k_0^2$. The only input is the gluon $g(x, k_0^2)$ of (7), where $k_0^2 = 1\text{GeV}^2$. The values of the two input parameters are determined by fitting to the available data for F_2 with $x < 0.05$ and $Q^2 > 1.5\text{GeV}^2$. The continuous curves in Fig. 2(a) show the description of a sample of the data. Overall the fit is excellent; at least as good as that achieved in the recent global analyses. When we repeat the analysis with the kinematic constraint omitted we see that the description (given by the dashed curves in Fig. 2(a)) is not so good and, moreover, the extrapolation of the gluon to $x \approx 0.4$ no longer describes the WA70 prompt photon data.

How important are the $\ln(1/x)$ effects? First we replace the BFKL kernel in (6) by the standard DGLAP splitting function, P_{gg}. We find that the gluon is not changed much in the HERA domain, compare the dashed and dotted curves in Fig. 2(b), as anticipated from the earlier observation that the coefficients $A_2 = A_3 = A_5 = 0$ in (3). On the other hand, when we use pure DGLAP evolution for the quark singlet, as well as the gluon, the difference is pronounced; compare the dashed and dot-dashed curves in Fig. 2(b), indicating the importance of $\ln(1/x)$ effects in P_{qg}.

4. Discussion

We argue that it is more advantageous to describe small x observables in terms of a universal unintegrated gluon together with the k_T factorization theorem, rather than to reduce the BFKL effects to collinear form. We list some points to consider. In the unintegrated approach it is straightforward to identify the perturbative contributions which contribute at all Q^2. We can therefore avoid subsuming them in the input distributions. Second, there is a natural way to introduce running α_S, which for sufficiently small x goes beyond the Renormalization Group behaviour. Thirdly the kinematic constraint along the BFKL ladder is easy to implement in the k_T factorization approach — the constraint is a subleading $\ln(1/x)$ effect which appears to embody a major part of the next-to-leading contribution (compare the α_S^2 term in the exponents of [8] and [9]). Another point is that BFKL contains all twists, whereas only the leading twist is retained in the reduction to collinear form. Finally the BFKL kernel and the off-shell gluon structure function $F_2^{\gamma g}$ are calculable perturbatively. We simply use expressions to leading order in α_S. The $\alpha_S \ln(1/x)$ summations are implicit in the integration over the *entire* k_T^2 phase space of the gluon ladder and in the k_T factorization integrals.

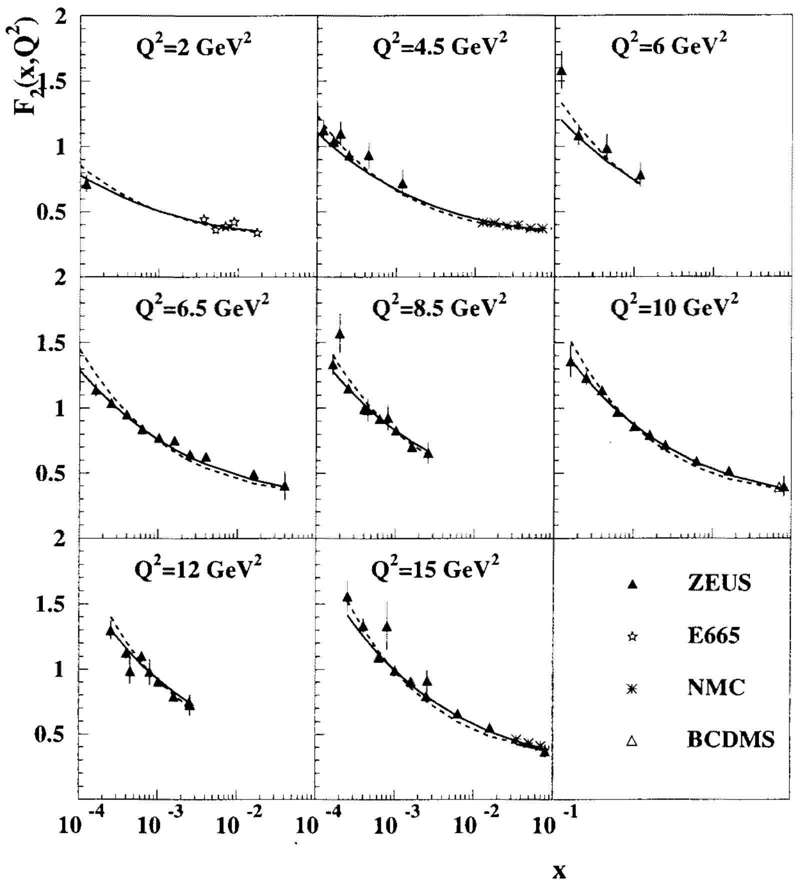

Fig. 2. (a) The two-parameter description of some of the F_2 data at small x using $f(x, k_T^2)$ evaluated with (continuous curves) and without (dashed curves) the kinematic constraint $k_T'^2 < k_T^2/z$.

In conclusion, the fact that we achieve an excellent two-parameter fit of the F_2 data at small x is not, in itself, remarkable. Other equally good phenomenological fits have been obtained. What is encouraging is that we have a theoretically well-grounded and consistent formalism which, with the minimum of non-perturbative input, is able to give a good *perturbative* description of the observed structure of F_2. Moreover the BFKL/DGLAP components of F_2 are decided by dynamics. In this way we have made a determination of the *universal* gluon distribution $f(x, k_T^2)$ which can be used, via the k_T factorization theorem, to predict the behaviour of other observables at small x. The predictions for F_2 (charm) and F_L can be found in [6].

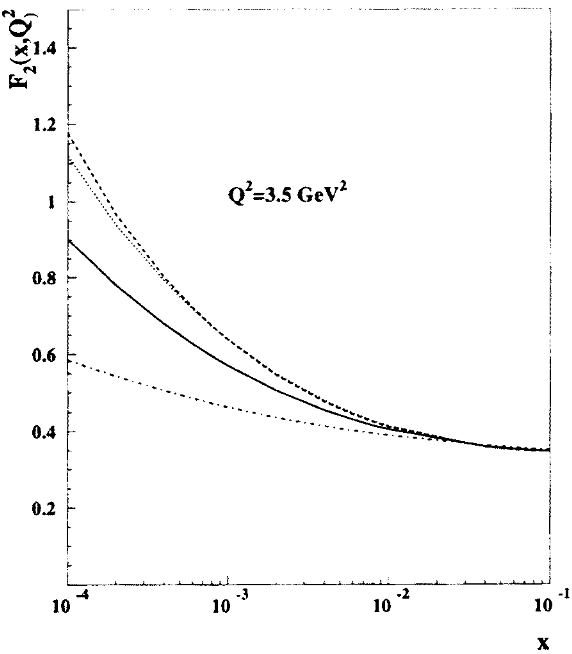

Fig. 2. (b) The continuous and dashed curves come from solving (6) and (9) with and without the kinematic constraint. The dotted and dot-dashed curves are obtained using DGLAP in the gluon sector and in both the gluon and quark sectors respectively. The same input is used for all curves.

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REFERENCES

[1] S. Catani, M. Ciafaloni, F. Hautmann, *Phys. Lett.* **B242**, 97 (1990); *Nucl. Phys.* **366**, 657 (1991); J.C. Collins, R.K. Ellis, *Nucl. Phys.* **B360**, 3 (1991); S. Catani, F. Hautmann, *Nucl. Phys.* **B427**, 475 (1994).

[2] R.K. Ellis, Z. Kunszt, E.M. Levin, *Nucl. Phys.* **B420**, 517 (1994); R.K. Ellis, F. Hautmann, B.R. Webber, *Phys. Lett.* **B348**, 582 (1995); R.D. Ball, S. Forte, *Phys. Lett.* **B351**, 313 (1995); J.R. Forshaw, R.G. Roberts, R.S. Thorne, *Phys. Lett.* **B356**, 79 (1995).

[3] S. Catani, *Proc. of Workshop on Deep Inelastic Scattering and Related Phenomena (DIS 96), Rome, 1996,* eds. G. D'Agostini and A. Nigro, World Scientific, Singapore 1996, p.165.

[4] R.S. Thorne, Rutherford preprint, RAL-96-065, January 1997.

[5] A.J. Askew, J. Kwiecinski, A.D. Martin, P.J. Sutton, *Phys. Rev.* **D47**, 3775 (1993); *Mod. Phys. Lett.* **A8**, 3813 (1993); *Phys. Rev.* **D49**, 4402 (1994); see also, I. Bojak, M. Ernst, Dortmund preprint DO-TH 96/25, hep-ph/9702282.

[6] J. Kwiecinski, A.D. Martin, A.M. Stasto, *Phys. Rev.* **D56**, 3991 (1997).

[7] M. Ciafaloni, *Nucl. Phys.* **B296**, 49 (1988); S. Catani, F. Fiorani, G. Marchesini, *Phys. Lett.* **B234**, 339 (1990); *Nucl. Phys.* **B336**, 18 (1990).

[8] J. Kwiecinski, A.D. Martin, P.J. Sutton, *Z. Phys.* **C71**, 585 (1996); B. Anderson, G. Gustafson, H. Kharraziha, J. Samuelson, *Z. Phys.* **C71**, 613 (1996).

[9] G. Camici, M. Ciafaloni, hep-ph/9707390 and references therein.

[10] J. Blumlein, *J. Phys. G* **19**, 1623 (1993).