QUANTUM FIELD DEGREES OF FREEDOM AND ASYMPTOTICS OF THE SKYRMION FIELD*

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A brief introduction to the Skyrme model is given. An approximate method is proposed for constructing stationary soliton solutions of the Skyrme model with nonzero spin and isospin values. The vicinity of the skyrmion center is represented as a rigidly rotating field configuration and is quantized in terms of collective coordinates. Nonzero modes are taken into account at the periphery by means of perturbation theory. In this approach, the asymptotic behavior of the soliton field is consistent with the Yukawa law, and the transition to the chiral limit is smooth.

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Quantum chromodynamics, the fundamental theory of strong interactions, runs into serious difficulties in describing the low-energy physics of hadrons. The effective theories of mesons and baryons are more suitable for this energy region [1, 2]. One of the simplest models of this type is the Skyrme model [3, 4] (see also [5] and references therein) in which pion fields are fundamental, while baryons arise as topologically nontrivial soliton solutions (skyrmions).

The Lagrangian of the Skyrme model has the form

$$\tilde{\mathcal{L}} = -\frac{f_\pi^2}{4}\,\mathrm{Tr}\,(L_\alpha L^\alpha) + \frac{1}{32e_S^2}\,\mathrm{Tr}\left([L_\alpha, L_\beta]\left[L^\alpha, L^\beta\right]\right) + \frac{1}{4} f_\pi^2 m_\pi^2\,\mathrm{Tr}(U + U^+ - 2), \tag{1}$$

where $L_\alpha = U^\dagger \frac{\partial U}{\partial x^\alpha}$; $U \in \mathrm{SU}(2)$ is a chiral-field matrix, parametrized by the isovector of the pion fields $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$

$$U(x^0, \vec{x}) = \exp\left(i\frac{\vec{\tau}\vec{\pi}(x^0, \vec{x})}{f_\pi}\right); \tag{2}$$

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$\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ is the isovector of the Pauli matrices; f_π and m_π are the pion decay constant and mass, respectively; and e_S is the dimensionless Skyrme constant.

Going over to the dimensionless variable $z^\alpha = (z^0, \vec{z}) = f_\pi x^\alpha$, $\mu_\pi = \frac{m_\pi}{f_\pi}$ and $\mathcal{L} = \frac{\tilde{\mathcal{L}}}{f_\pi^4}$ and introducing the following notation

$$\Phi(z^0, \vec{z}) = \begin{bmatrix} \Phi_1(z^0, \vec{z}) \\ \Phi_2(z^0, \vec{z}) \\ \Phi_3(z^0, \vec{z}) \end{bmatrix} = \frac{1}{f_\pi} \begin{bmatrix} \pi_1(z^0, \vec{z}) \\ \pi_2(z^0, \vec{z}) \\ \pi_3(z^0, \vec{z}) \end{bmatrix} \tag{3}$$

one can rewrite the Skyrme Lagrangian in the form

$$\mathcal{L} = \frac{1}{2}\dot{\Phi}^T G(\Phi, \nabla\Phi)\dot{\Phi} - \mathcal{M}(\Phi, \nabla\Phi), \tag{4}$$

where $G(\Phi, \nabla\Phi)$ is a symmetrical 3×3 matrix, which specifies the metric in the isotopic space.

The Lagrangian is symmetrical under the Poincare group as well as under rotations in the isotopic space. The latter symmetry yields an integral of motion that is known as the isotopic spin

$$T_a = \int d^3 z \dot{\Phi}^T G X_a \Phi \,. \tag{5}$$

Here X_a, $a = 1, 2, 3$ are the generators of the SO(3) group, which posses the following properties:

$$[X_a, X_b] = \varepsilon_{abc} X_c \,, \qquad (X_a)_{bc} = -\varepsilon_{abc} \,. \tag{6}$$

It should be emphasized that the Skyrme Lagrangian contains only the pion fields as the dynamic variables and at first sight this is a purely mesonic theory, nevertheless it does describe baryons as well. The distinctive property of this model is that it has, in parallel with Noether integrals of motion, the integral of motion of another type, which is not associated with the symmetries of the Lagrangian but rather with topological properties of the field manifold. This exactly conserved number

$$\begin{aligned} B = & -\frac{1}{12\pi^2} \int d^3 x \varepsilon_{lmn} \nabla_l \Phi_a \nabla_m \Phi_b \nabla_n \Phi_c \\ & \left(\varepsilon_{abc} \frac{\sin^3 F \cos^3 F}{F^3} + 3\Phi_a \Phi_d \varepsilon_{dbc} \frac{\sin^2 F}{F^4} \left(1 - \frac{\sin F \cos^3 F}{F} \right) \right) \end{aligned} \tag{7}$$

is called "topological index' or "topological charge" and can be interpreted as baryon charge. In other words, the field configuration with nonzero topological index can be considered as baryons, nuclei *etc.*

The simplest solution of the Skyrme model with nonzero topological index is known as the hedgehog skyrmion. It satisfies the variational equation applied to the static energy

$$\frac{\delta}{\delta\Phi}\int d^3z\mathcal{M}(\Phi,\nabla\Phi)=0 \tag{8}$$

and has the form

$$\Phi_s(\vec{z})=\frac{\vec{z}}{r}F_s(r),\ \ r=|\vec{z}|. \tag{9}$$

The chiral angle $F_s(r)$ can be found by integrating numerically the ordinary differential equation

$$\begin{aligned}&(r^2+\frac{2}{e_S^2}\sin^2 F_s)\frac{d^2F_s}{dr^2}+2r\frac{dF_s}{dr}\\ &-\left(1+\frac{1}{e_S^2}\left(\frac{\sin^2 F_s}{r^2}-\left(\frac{dF_s}{dr}\right)^2\right)\right)\sin 2F_s-\mu_\pi^2r^2\sin F_s=0,\end{aligned} \tag{10}$$

supplemented by the boundary conditions

$$F_s(r)|_{r=0}=\pi,\quad F_s(r)|_{r\to\infty}=0, \tag{11}$$

which correspond to the topological charge $B=1$. In topological sectors $B=\pm1$ the hedgehog solution is stable and represents the absolute static energy minimum [5].

It should be note that the asymptotic behavior of the field of the static skyrmion corresponds to the Yukawa law

$$F(r)\sim\left(\frac{\mu_\pi}{r}+\frac{1}{r^2}\right)e^{-\mu_\pi r}. \tag{12}$$

Unfortunately, the static solution possesses zero spin and zero isospin; hence, it cannot describe the properties of physical baryons.

To solve this problem Adkins, Nappi and Witten [6] proposed a semi-classical approach, in which a baryonlike soliton is constructed as a rigidly rotating field configuration;

$$\Phi(z^0,\vec{z})=R(z^0)\Phi_s(\vec{z}), \tag{13}$$

($R(z^0)\in SO(3)$ is the matrix of isotopic rotations). That is, only rotational modes were quantized, while other degrees of freedom were assumed to be frozen. Moreover it was assumed that the shape of the rotating soliton coincides with the static solution. This approach allows to reproduce the static properties of lightest baryons (nucleon and Δ-isobar) to within 30%.

Nonetheless, there is no reason to expect that a rotating field configuration is not substantially affected by centrifugal forces and is close to the static solution. The attempts of many authors [7–9] to take centrifugal effects into account without going beyond the semiclassical approach led to considerable difficulties. The chiral angle minimizing the total energy of the rotating skyrmion

$$\frac{\delta E_{\rm tot}}{\delta F_j} = 0 \tag{14}$$

decreases at infinity more slowly than what is required by the Yukawa law:

$$F_j(r) \approx \left(\frac{\mu_j}{r} + \frac{1}{r^2}\right) e^{-\mu_j r}, \tag{15}$$

$$\mu_j^2 = \mu_\pi^2 - \frac{2j(j+1)}{3\Lambda^2} \tag{16}$$

(j and Λ are the skyrmion spin and moment of inertia, respectively). Moreover there is no solution in the chiral limit ($m_\pi \to 0$).

In my opinion the reason behind the above difficulties is that the semiclassical approach is inapplicable in the case of small pion mass and is therefore incompatible with the chiral limit.

Indeed, the semiclassical approach can be applied when the probability of exciting the frozen degrees of freedom is small. In the case under study, the minimum energy required for exciting the vibrational degrees of freedom is determined by the pion mass μ_π, it is to be compared with the kinetic energy of rotation $\frac{j(j+1)}{2\Lambda}$. Hence the condition that ensures the applicability of the adiabatic approximation has the form

$$m_\pi \gg \frac{j(j+1)}{2\Lambda}. \tag{17}$$

It is clear that this inequality cannot be satisfied in the chiral limit. Calculations show that, even for the pion mass equal to its experimental value, the two sides of inequality are on the same order of magnitude.

Another shortcoming of the semiclassical approach consists in its contradiction to the special theory of relativity. In the case of the rigidly rotating skyrmion the speed of segments that are remote from the rotation center exceeds the speed of light.

The obvious way of solving this problem would be the treatment of all (both zero and nonzero) modes from the point of view of the relativistic quantum theory. However an implementing this way in practice is quite difficult. Therefore, in the present talk I propose to take into account only those nonzero modes that are characterized by their excitation energy below (or on the order of) the kinetic energy of rotation. The essential feature of the approach is that the field degrees of freedom are unfrozen only at the asymptotic region of the skyrmion,

$$\Phi(z^0, \vec{z}) = R(z^0)\Phi_0(\vec{z}) + \phi(z^0, \vec{z}), \text{ at } |\vec{z}| > r_0, \tag{18}$$

while its core — that is the region near its center within a sphere of radius r_0 — is treated as a rigidly rotating field configuration:

$$\Phi(z^0, \vec{z}) = R(z^0)\Phi_0(\vec{z}), \text{ at } |\vec{z}| \leq r_0 \tag{19}$$

(r_0 is a parameter of the model). The accuracy of this approximation depends on the degree to which the production of quanta with wavelength less or equal r_0 ($\lambda \leq r_0$) is small. To be more specific, it is applicable provided that the energy of a pion with wavelength equal to r_0 is much greater than the rotating energy:

$$\sqrt{m_\pi^2 + \left(\frac{2\pi}{r_0}\right)^2} \gg \frac{j(j+1)}{2\Lambda}. \tag{20}$$

It is obvious that this condition is less stringent than previous one; for sufficiently small r_0, the former can be satisfied even in the chiral limit.

Since we assume that the soliton core rotates as a rigid body, the proposed approach is nonrelativistic; however, the impact of the contradiction to the special theory of relativity is less profound than in previous approach: if r_0 is sufficiently small, the speed of segments on the surface of the rigidly rotating core can be much smaller then the speed of light.

In the framework of the proposed approach the baryon-like state vector and its energy can be found by perturbative methods. For a small parameter one can choose the value of the chiral angle F in the point r_0.

The stationary baryon-like state vector with spin and isospin j_B, third spin projection s_B and isospin projection t_B was found in the first order of the perturbation theory

$$\begin{aligned} |B\rangle &= |j_B s_B t_B\rangle \\ &+ \sum_{j=0}^{\infty}{}' \sum_{s=-j}^{j} \sum_{t=-j}^{j} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{q=-1}^{1} \int_0^\infty dk \\ &\times C^{(1)}_{jstlmq} |jst\rangle b^{(+)}_{lmq}(k)|0\rangle. \end{aligned}$$

Here $b^{(+)}_{lmq}(k)$ is the creation operator for the pion with angular momentum l, third angular momentum projection m, isospin projection q and energy

$\omega(k) = \sqrt{k^2+\mu_\pi^2}$; $|jst\rangle$ is the state of the hedgehog core with spin and isospin j, third spin projection s and isospin projection t,

$$\begin{aligned} C^{(1)}_{jst1mq}(k) &= (-1)^{j-j_B+1+m}\sqrt{\frac{2j+1}{2j_B+1}} \\ &\times (j,1,t,q|j_B t_B)(j,1,s,m|j_B s_B)\sqrt{\frac{4\pi}{3}}\frac{1}{\sqrt{2\omega(k)}} \\ &\times \left(\frac{r_0^2 F_0(r_0)\left.\frac{dR_1(r,k)}{dr}\right|_{r=r_0}}{\frac{(j-j_B)(j+j_B+1)}{2\Lambda_0}+\omega(k)} - \omega(k)\int\limits_{r_0+0}^{\infty} dr r^2 F_0(r) R_1(r,k)\right), \quad (21) \\ C^{(1)}_{jstlmq}(k) &= 0 \text{ for } l \neq 1\,, \end{aligned}$$

where $R_1(r,k)$ is a special function which can be expressed in terms of spherical Bessel functions:

$$R_l(k,r) = \sqrt{\frac{2}{\pi}}k\frac{n_l(kr_0)j_l(kr) - j_l(kr_0)n_l(kr)}{\sqrt{(j_l(kr_0))^2+(n_l(kr_0))^2}}. \quad (22)$$

The expectation value of the nucleon field has the proper dipole structure

$$\langle B|\Phi_{1,2}|B\rangle = 0, \quad (23)$$

$$\langle B|\Phi_3|B\rangle = \begin{cases} -\frac{1}{3}\cos\theta F(r) & \text{at} \quad r \le r_0, \\ -\frac{1}{3}\cos\theta f(r) & \text{at} \quad r > r_0, \end{cases} \quad (24)$$

and its asymptotic behavior corresponds to the Yukawa law:

$$f(r) = F(r_0)\frac{e^{-\mu_\pi r}\left(\frac{\mu_\pi}{r}+\frac{1}{r^2}\right)}{e^{-\mu_\pi r_0}\left(\frac{\mu_\pi}{r_0}+\frac{1}{r_0^2}\right)}. \quad (25)$$

The nucleon energy was found in the second order of perturbation theory

$$\begin{aligned} E_N^\star &= \int d^3z \mathcal{M}(\Phi,\nabla\Phi) + \frac{3}{8\Lambda_0} \\ &- \frac{2f_\pi^2}{\Lambda_0^2}(r_0^2F(r_0))^2 \\ &\times \int\limits_0^\infty \frac{dkk^4}{(1+k^2r_0^2)\omega^2(k)\left(\omega(k)+\frac{1}{\Lambda_0}\right)\left(\omega(k)+\frac{3}{2\Lambda_0}\right)}. \quad (26) \end{aligned}$$

The behavior of the chiral angle in the region $r < r_0$ can be determined from the variational principle applied to the total skyrmion energy

$$\frac{\delta E_N^\star}{\delta F_0(r)} = 0. \tag{27}$$

So in contrast to the adiabatic approximation, the proposed method for constructing approximate soliton solutions with nonzero spin and isospin values enables us to take into account centrifugal effects without running into problems peculiar to the semiclassical approach.

At infinity, the asymptotic behavior of the resulting soliton solutions is consistent with the Yukawa law. The asymptotic behavior and the energy of the soliton both have a well-defined limit for $m_\pi \to 0$; that is, restoration of chiral symmetry does not lead to abrupt changes in the properties of solutions of the Skyrme model. This conclusion fits well into the generally accepted view on the chiral SU(2) symmetry of hadronic matter.

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