DUAL VORTICES, SPONTANEOUS GAUGE SYMMETRY BREAKING AND CONFINEMENT*

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We apply lattice methods to the physics of quark confinement. We exploit the close correspondence between the confinement of color flux and vortices in superconductivity.

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1. Introduction

In these lectures I would like to describe the lattice approach to the study of quark confinement. These techniques have opened up investigations of the seminal ideas that were proposed long ago by Nielsen and Olesen [1], Kogut and Susskind [2], Nambu [3], Parisi [4], Mandelstam [5] and 't Hooft [6].

A lattice simulation showing a linearly rising static quark potential gives ample evidence that quarks are confined [7, 8]. But a global calculation such as this can not reveal whether there is a simple underlying mechanism governing the physics of confinement.

Calculations of the distribution of energy and action density surrounding a quark-antiquark pair have shown that these densities are enhanced between quarks forming a string as the linear potential implies [9, 8]. This is a calculation of local properties which begins to reveal details of the confinement configurations.

The present effort attempts to go a step further by looking at local relationships among field strengths and currents pointing toward the existence of an effective theory of confinement.

Abrikosov vortices in type II superconductors are an example of a configuration with confined flux. In Sec. 2 I will place this in the context of

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particle physics and show how one can verify these properties using lattice simulations. In Sec. 3 I would like to describe the generalization to non-Abelian theories based on the notion of an Abelian projection [10, 11] after a partial gauge fixing that preserves a U(1) invariance. In Sec. 4 I report on recent work by DiCecio, Hart and myself [12] in which we use an Ehrenfest theorem to tighten up the definitions of field strengths and currents on the lattice in an Abelian projected theory, which has been lacking in earlier work.

2. U(1) gauge theories and superconductivity

The onset of superconductivity is governed by the spontaneous breaking of the U(1) gauge symmetry via a non-zero vacuum expectation value of a charged field [13]. An immediate consequence of this is the generation of a photon mass and, for type II superconductors, the formation of vortices which confine magnetic flux to narrow tubes [14] as revealed by the Ginzburg–Landau effective theory. Lattice studies of dual superconductivity in SU(N) gauge theories seek to exploit this connection in establishing the underlying principle governing color confinement [15].

On a four dimensional lattice, the effective Euclidean lattice Higgs theory is the appropriate generalization of the Ginzburg–Landau effective theory. The Higgs field is a 0-form living on the sites and the gauge field is a 1-form living on the links. Classical solutions to this theory exhibit the connection between the non-zero vacuum expectation value of the Higgs field and vortex formation.

In U(1) lattice pure gauge theory (no Higgs field), this same connection is seen to be present, not in the defining variables, but rather in the dual variables. More specifically:

1. A field with non-zero magnetic monopole charge, Φ, has been constructed [16]. It is a composite 4-form living on hypercubes constructed from gauge fields. There are also monopole current 3-forms.

   On the dual lattice this monopole operator is a 0-form living on dual sites. The monopole currents are 1-forms living on dual links. These currents either form closed loops or end at monopole operators.

   The monopole operator has a non-zero vacuum expectation value in the dual superconducting phase, $\langle\Phi\rangle \neq 0$, thereby signaling the spontaneous breaking of the U(1) gauge symmetry.

2. Dual Abrikosov vortices have been seen in simulations [17]. They are identified by the signature relationship between the electric field and the curl of the monopole current in the transverse profile of the vortex. The dual coherence length, ξ_d measures the characteristic dis-

tance from a dual-normal-superconducting boundary over which the dual-superconductivity turns on. The dual London penetration length, λ_d measures the attenuation length of an external field penetrating the dual-superconductor. The dual photon mass $\sim 1/\lambda_d$ and the dual Higgs mass $\sim 1/\xi_d$.

A signal $\langle \Phi \rangle \neq 0$ without the consequent signal of a dual photon mass does not imply confinement. An observation of a dual photon mass, *i.e.* vortex formation, without $\langle \Phi \rangle \neq 0$ does not reveal the underlying principle governing the phenomenon.

2.1. Higgs effective theory

The Higgs theory, treated as an effective theory, *i.e.* limited to classical solutions, and considered in the dual sense, provides a model for interpreting simulations of the pure gauge theory that can reveal these important connections. Recalling the Higgs' current

$$J_\mu = -\frac{ie}{2}\left(\phi^*\left(\partial_\mu - ieA_\mu\right)\phi - \phi\left(\partial_\mu + ieA_\mu\right)\phi^*\right), \tag{1}$$

and spontaneous gauge symmetry breaking through a constraint Higgs potential

$$\theta = ve^{ie\omega(x)}, \qquad v = \text{constant}, \tag{2}$$

leads to the London theory of a type II superconductor. Using Eqns.(1) and (2) we obtain

$$\begin{aligned} J_\mu &= -e^2v^2\left(A_\mu - \partial_\mu\omega\right), \\ &\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right) + m_\gamma^2\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right) = 0, \\ &\nabla \times \boldsymbol{J} + \frac{1}{\lambda^2}\boldsymbol{B} = 0, \end{aligned} \tag{3}$$

where

$$e^2v^2 = m_\gamma^2 = \frac{1}{\lambda^2}.$$

Using Ampere's law $\nabla \times \boldsymbol{B} = \boldsymbol{J}$, we obtain

$$\nabla^2\boldsymbol{B} = \frac{\boldsymbol{B}}{\lambda^2},$$

identifying λ as the London penetration depth.

If the manifold is multiply connected, then the gauge term in Eq. (3) can contribute, as long as $e\omega(x)$ is periodic, with period 2π on paths that surround a hole.

$$\begin{aligned}\int_S (\boldsymbol{B}+\lambda^2\boldsymbol{\nabla}\times\boldsymbol{J})\cdot\boldsymbol{n}da &= \oint_C (\boldsymbol{A}+\lambda^2\boldsymbol{J})\cdot d\boldsymbol{l}\,,\\ &= \oint_C \boldsymbol{\nabla}\omega\cdot d\boldsymbol{l}\,,\\ &= N\frac{2\pi}{e} = Ne_m = \Phi_m\,.\end{aligned}$$

where N quanta of magnetic flux penetrates the hole in the manifold. In real superconductors, the hole is a consequence of the large magnetic field at the center which drives the material normal.

A cylindrically symmetric vortex solution is given by

$$\begin{aligned}\boldsymbol{B}+\lambda^2\boldsymbol{\nabla}\times\boldsymbol{J} &= \Phi_m\delta^2(\boldsymbol{r}_\perp)\boldsymbol{n_z}\,,\\ (1-\lambda^2\boldsymbol{\nabla}_\perp^2)B_z(r_\perp) &= \Phi_m\delta^2(\boldsymbol{r}_\perp)\,,\\ B_z(r_\perp) &= \frac{\Phi_m}{2\pi\lambda^2}K_0(r_\perp/\lambda)\,.\end{aligned}$$

The delta function core of this vortex is normal, *i.e.* no spontaneous symmetry breaking, and the exponential tail of the vortex is a penetration depth effect at the superconducting-normal boundary. The key point is that the modulus of the Higgs field must be independent of position to get these idealized vortices. For a "Mexican hat" Higgs potential, there is a coherence length setting the length scale from a normal-superconducting boundary over which the vacuum expectation value of the Higgs field changes from zero to its asymptotic value.

On the lattice, the same phenomenon occurs but there is more. We can generate vortices from finite configurations. In the continuum these objects are singular. Since the lattice formulation is based on group elements, rather than the Lie algebra the periodic behavior of the compact manifold is manifest. This gives the $2\pi N$ ambiquity in the group angle leading to N units of quantized flux. To see how this works, consider the lattice Higgs action

$$\begin{aligned}S = \beta &\sum_{x,\mu>\nu}(1-\cos\,\theta_{\mu\nu}(x))\\ &-\kappa\sum_{x,\mu}(\phi^*(x)e^{i\theta_\mu(x)}\phi(x+\mu)+H.c.)+\sum_x V_{\text{Higgs}}(|\phi(x)|^2)\,,\end{aligned}$$

where $\theta_{\mu\nu}(x)$ is the curl of the gauge field,

$$\theta_{\mu\nu}(x) = \Delta_\mu^+\theta_\nu(x) - \Delta_\nu^+\theta_\mu(x)\,,$$

and where $\phi(x)$ is the Higgs field and $\phi(x+\mu)$ refers to the Higgs field at the neighboring site in the μ direction and Δ_ν^+ is the forward difference operator. The electric current is given by

$$\frac{a^3}{e\kappa}J_\mu^e(x) = \text{Im}\,(\phi^*(x)e^{i\theta_\mu(x)}\phi(x+\mu))\,,$$

where a is the lattice spacing. Let us choose a Higgs potential that constrains the Higgs field $|\phi(x)| = 1$. Then if

$$\sin[\theta + 2N\pi] \approx \theta\,,$$

we find a relation between the field strength tensor and the curl of the current

$$\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} - \frac{a^2}{e^2\kappa}\frac{(\Delta_\mu^+ J_\nu^e(x) - \Delta_\nu^+ J_\mu^e(x))}{a} = \frac{2\pi N}{e}\frac{1}{a^2} = N e_m \frac{1}{a^2}\,,$$

where

$$ea^2 F_{\mu\nu} = \sin[\theta_\mu(x) + \theta_\nu(x+\mu) - \theta_\mu(x+\nu) - \theta_\nu(x)]\,.$$

If $N = 0$ then this is a London relation which implies a Meissner effect. For $N \neq 0$ there are N units of quantized flux penetrating that plaquette, indicating the presence of an Abrikosov vortex.

2.2. Pure U(1) gauge theory

In a pure U(1) lattice gauge simulation (without Higgs field), lattice averages over many configurations exhibit superconductivity in the dual variables. The superconducting current carriers are monopoles. They can be defined in a natural way on the lattice using the DeGrand Toussaint [18] construction. They arise in non-singular configurations again because of the $2\pi N$ ambiguity in group elements. These are the magnetic charge carriers for dual superconductivity. For a review of the monopole construction and vortex operators in U(1) gauge theory see *e.g.* the 1995 Varenna Proceedings [14].

As a brief summary, consider the unit 3-volume on the lattice at fixed x_4. The link angle is compact, $-\pi < \theta_\mu \leq \pi$. The plaquette angle is also compact, $-4\pi < \theta_{\mu\nu} \leq 4\pi$ and defined

$$ea^2 F_{\mu\nu}(x) = \theta_{\mu\nu}(x) = \Delta_\mu^+\theta_\nu(x) - \Delta_\nu^+\theta_\mu(x)\,,$$

where a is the lattice spacing. This measures the electromagnetic flux through the face. Consider a configuration in which the absolute value of the link angles, $|\theta_\mu|$, making up the cube are all small compared to $\pi/4$. Gauss' theorem applied to this cube then clearly gives zero total flux. Because of the 2π periodicity of the action we decompose the plaquette angle into two parts

$$\theta_{\mu\nu}(x) = \bar{\theta}_{\mu\nu}(x) + 2\pi n_{\mu\nu}(x)\,. \tag{4}$$

where $-\pi < \bar{\theta}_{\mu\nu} \leq \pi$. If the four angles making up one of the six plaquette are adjusted so that *e.g.* $\theta_{\mu\nu} > \pi$ then there is a discontinuous change in $\bar{\theta}_{\mu\nu}$ by -2π and a compensating change in $n_{\mu\nu}$. We can clearly choose the configuration that leaves the plaquette angles on all the other faces safely away from a discontinuity. We then define a Dirac string $n_{\mu\nu}$ passing through this face (or better a Dirac sheet since the lattice is 4D). $\bar{\theta}_{\mu\nu}$ measures the electromagnetic flux through the face.

This construction gives the following definition of the magnetic monopole current.

$$\frac{a^3}{e_m} J^m_\mu(x) = \varepsilon_{\mu\nu\sigma\tau} \Delta^+_\nu \bar{\theta}_{\sigma\tau}(x)\,. \tag{5}$$

This lives on dual links on the dual lattice. Although Eq. (4) is not gauge invariant, Eq. (5) is. Further it is a conserved current, satisfying the conservation law $\Delta^+_\mu J^m_\mu(x) = 0$.

In simulations of a pure U(1) gauge theory we find that lattice averages give a relation similar to Eq. (4), but in the dual variables

$$\langle {}^*\mathcal{F}_{\mu\nu} \rangle \equiv \langle {}^*F_{\mu\nu} \rangle - \lambda_d^2 \frac{\left\langle \Delta^-_\mu J^m_\nu(x) - \Delta^-_\nu J^m_\mu(x) \right\rangle}{a} = Ne\frac{1}{a^2}\,,$$

where ${}^*F_{\mu\nu}$ is dual of $F_{\mu\nu}$. This is the signal for the detection of dual vortices [14].

3. Non-Abelian theory

The link of these considerations to confinement in non-Abelian gauge theory is through the Abelian projection [10, 11]. One first fixes the gauge while preserving U(1) gauge invariance. The non-Abelian gauge fields can be parametrized in terms of a U(1) gauge field and charged coset fields. The working hypothesis is that operators constructed from the U(1) gauge field alone, *i.e.* Abelian plaquettes, Abelian Wilson loops, Abelian Polyakov lines and monopole currents, will exhibit the correct large distance correlations relevant for confinement. But there is as yet no definitive way to choose the gauge which defines the Abelian projection and hence no unique way to define Abelian links and coset fields from the SU(2) links.

Baker, Ball, Zachariasen [19] and co-workers have postulated a dual form of QCD in the continuum. Their formulation describes a non-Abelian dual superconductor and hence it confines color. They have calculated flux tubes, static quark potentials, temperature dependent effects and many other quantities in the tree approximation. Although their formulation does not focus on monopoles, there is a large potential overlap between their work and the lattice description developed here.

We are seeking a judicious choice of dynamical variables — defined by a particular Abelian projection — which can account for confinement via an effective theory involving these dynamical variables. This should be our first goal. If this is solved satisfactorily, then we can investigate how the picture changes if we go to an alternative set of variables — a different Abelian projection — since the phenomenon we are describing is of course gauge invariant. We do not expect that the same mechanism would describe confinement in two different Abelian projections.

There is a very nice illustration of this point in a paper by Chernodub, Polikarpov and Veselov [20]. They compare two Abelian projections.

- The first is the maximal Abelian gauge [11] which is the most widely studied and perhaps the most promising candidate. Monopoles are the magnetic charge carriers of the persistent currents.
- Second they exhibit an Abelian projection in which confinement is due to objects other than monopoles. They choose the "minimal Abelian projection" and show that confinement is due to topological objects which are denoted "minopoles".

Here we have two projections, two sets of dynamical variables, and two different descriptions of confinement, both viable candidate mechanisms.

There is an intimate connection between vortex formation and a non-zero vacuum expectation of the monopole field in a Higgs theory and hence it is a strong test of the idea for the dual theory. Two candidate projections are the Polyakov gauge and the Maximal Abelian gauge.

3.1. Polyakov gauge

Del Debbio, Di Giacomo, Paffuti and Pieri [21] have constructed a lattice monopole field operator that shows a very strong signal with a sharp discontinuity in the vacuum expectation value of the monopole operator as a function of temperature in the neighborhood of T_c, the deconfining temperature. The critical indices for the deconfining phase transition have been measured in simulations and they agree with the critical indices measured in other operators sensitive to the phase transition [22]. Their calculation is manifestly gauge invariant.

The monopole field operator creates the Abelian magnetic charge and the accompanying Abelian field strength. The definition of gauge invariant field strength is that defined by 't Hooft [23] in the Georgi Glashow model,

$$\begin{aligned} F_{\mu\nu} &= \mathrm{Tr}\Big[\phi\Big(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\Big)\Big] \\ &+\frac{1}{2}\ \mathrm{Tr}\Big[\phi\Big(\partial_\mu\phi - ig[A_\mu,\phi]\Big)\Big(\partial_\nu\phi - ig[A_\nu,\phi]\Big)\Big]\,, \end{aligned}$$

where ϕ transforms under the adjoint representation of SU(2). However there is no adjoint scalar field in QCD. Instead one constructs a composite field from a Wilson line, which begins and ends at the same site x

$$W(x) = \cos\theta_W(x) + i\phi(x)\sin\theta_W(x)\,, \quad \phi = \boldsymbol{\phi}\cdot\boldsymbol{\sigma}\,, \quad \boldsymbol{\phi}\cdot\boldsymbol{\phi} = 1\,, \quad \theta_{\mathrm{W}} \in [0,\pi]\,.$$

Del Debbio *et al.* chose $W(x) = P(x)$, a Polyakov line which closes through the time dimension. The Polyakov gauge is obtained by rotating ϕ(x) into the the 3–direction at every site. Then the gauge invariant field strength reduces to the curl of the the 3rd component of the SU(2) gauge field. Even though the construction is gauge invariant we associate their calculation with the Polyakov gauge.

In Ref. [24] we established vortex formation in the Polyakov gauge. We anticipated that the monopoles in the Polyakov projection would give a picture very similar to the maximal Abelian gauge. However we found that the coset fields greatly suppress the static sources. It could be that we are much farther from the continuum limit than we thought. But that seems unlikely since other quantities are close to scaling values. It could be that the dynamical variables arising from the Polyakov Abelian projection do not adequately separate the short distance and long distance physics. This leaves the maximal Abelian gauge as the prime candidate to define an effective theory of confinement in this scenario.

In Sec. 4 of these lectures, we will offer another possible solution to this discrepancy.

3.2. Maximal Abelian gauge

The maximal Abelian gauge is defined by a gauge configuration that maximizes $\mathcal{R}$, where

$$\mathcal{R}[U] \equiv \sum_{x,\mu}\frac{1}{2}\mathrm{Tr}\left(\sigma_3 U_\mu(x)\sigma_3 U_\mu^\dagger(x)\right)\,,$$

and where $U_\mu(x)$ is the link starting a site x and extending in the μ direction. In the continuum limit this becomes the gauge condition

$$\left(\partial_\mu \pm gA_\mu^3(x)\right)A_\mu^\pm(x) = 0\,.$$

The vortices are well established [25, 24] in this gauge. The typical behavior is shown in Fig. 1. The London relation is seen in the confining case for transverse distances larger than about two lattice spacing. The dual coherence length $\xi_d \approx 2$, *i.e.* the onset of the violation of the London relation. The unconfined case is also shown where curl $\boldsymbol{J}$is almost zero, and the electric field falls more gradually than in the confining case.

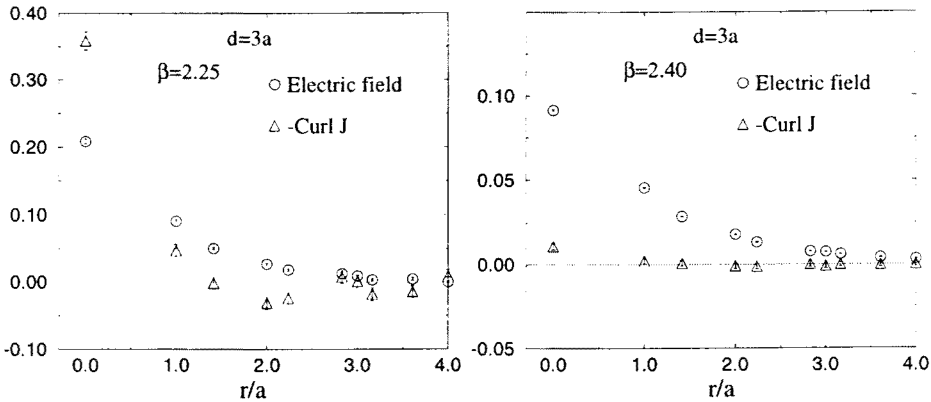

Fig. 1. Transverse profile of the electric field and curl of the monopole current in the mid plane between a static $q\bar{q}$ pair on the maximal Abelian gauge at finite temperature for a confining (left) and unconfining (right) phases Ref. [24].

The static potential constructed from Abelian links gives as definitive a signal of confinement as the gauge invariant static potential as found by Suzuki *et al.* [15, 26], Stack *et al.* [27] and Bali *et al.* [28]. Bali *et al.* find the Abelian string tension calculation gives 0.92(4) times the full string tension for $\beta = 2.5115$. Whether this approaches 1.0 in the continuum limit remains to be seen.

The calculation of Del Debbio et.al [21] of the monopole field operator as a practical matter is not adaptable to this gauge. It would require hundreds of gauge fixing sweeps of the whole lattice in order to accept or reject a single link update. Chernodub, Polikarpov and Veselov [29] have more recently calculated the constraint effective potential as a function of the monopole field in this gauge and found a symmetry breaking minimum. However they reported a problem of obtaining statistics and instead calculated an approximation to the effective potential.

More recently Nakamura *et al.* [30] studied an alternative monopole operator defined in terms of the variables which occur in the monopole form of the action. We have not yet addressed the issue of establishing vortex formation in that framework.

4. Ehrenfest theorem for field strength and electric current

After applying Abelian projection to non–Abelian gauge theories, the result is mathematically identical to a set of charged fields (vector–like in the maximally Abelian gauge) coupled to an electromagnetic field, governed by a complicated U(1) gauge invariant action. The charged fields have traditionally been ignored, but numerical work suggests that we should re-examine their role which can be quite significant in, *e.g.*, the Polyakov gauge.

A comparison between the two different projections [24] shows that in the Polyakov gauge the peak values of the electric field and of the curl of the magnetic currents are more than an order of magnitude smaller than the correspondent values in the maximal Abelian gauge. Suzuki *et al.* [33] reported similarly that the string tension was suppressed in the Polyakov gauge.

To show the different role of the coset fields in the two projections, consider the divergence of the electric field giving the spatial distribution of the electric charge. The divergence at site x is defined as

$$\left\langle \mathrm{div}\,\vec{E}(x)\right\rangle = \sum_{i=1}^{3} \frac{\langle \sin\theta_{PP\dagger}(\sin\theta_{i4}(x) - \sin\theta_{i4}(x-i))\rangle}{a^3 e\,\langle\cos\theta_{PP\dagger}\rangle}\,,$$

where $\theta_{PP\dagger}$ is the angle corresponding to a product of Polyakov line sources. All six plaquettes have in common the link starting at x and extending in the time direction. In Fig. 2 we show the results for two Polyakov lines separated by three lattice spacings. The figure clearly shows that the effective charge of the static source is much smaller in the Polyakov gauge and that the coset fields in the two prescriptions respond in opposite ways to the presence of an electric charge with the fields in the Polyakov gauge shielding the static charge. This led us to look for a definition of charge density and electric flux on the lattice.

We are presently completing a numerical check of an exact relation for SU(2) lattice averages after Abelian projection [12] that will accomplish both of these ends:

$$\langle \Delta_\mu F_{\mu\nu} - J_\nu \rangle_{\text{static Abelian source}} = 0\,. \tag{6}$$

J_ν, the electric current in the remaining U(1), contains terms from the external source, the "off-diagonal" gauge potentials, the gauge fixing condition and the Faddeev–Popov determinant:

$$J_\nu = J_\nu^{(\text{source})} + J_\nu^{(\text{dynamical})} + J_\nu^{(\text{gauge fixing})} + J_\nu^{(\text{Faddeev–Popov})}\,.$$

Eq. (6) resembles the Euler–Lagrange–Maxwell equation, of course, satisfied at the extremum of the action. With a suitable choice of lattice operators,

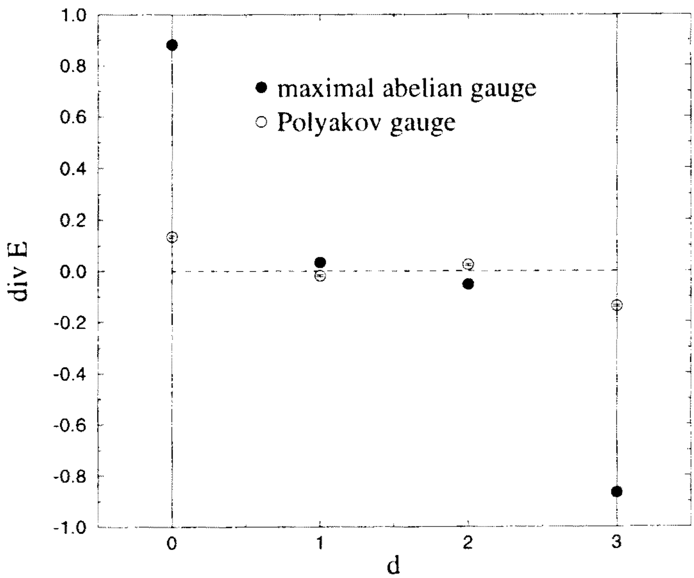

Fig. 2. Divergence of the electric field along the axis connecting two static charges separated by three lattice spacings. Ref. [24].

however, lattice averages also satisfy this relation. The corresponding relations for U(1) and for SU(3) without gauge fixing are given in Ref. [31]. The term "Ehrenfest theorem" is taken from the context of quantum mechanics, where a classical equation is exactly satisfied for expectation values, *e.g.* $\frac{d^2}{dt^2}\langle \boldsymbol{r}\rangle = -\langle \boldsymbol{\nabla} V(\boldsymbol{r})\rangle$.

4.1. Motivation

Equation (6) defines the electric current density on the lattice. Unlike pure U(1), an electric current density occurs in the Abelian projected theory, and is capable of screening sources and affecting the string tension; Bali *et al.* [32] found that the string tension after Abelian projection to the maximal Abelian gauge is only 92% of the full SU(2) result. The contributions from gauge fixing and the Faddeev–Popov determinant contribution to the current can be measured by making use of this relation. The latter is the contribution from the ghost fields. One uses the Ginzburg–Landau theory interpreted as a dual effective theory to model the simulations. One needs to modify this model, however, to include the effects of a dynamical charge density.

Equation (6) also defines the Abelian field strength on the lattice. For regions where $J_\nu = 0$, this defines exactly the conservation of flux. It then gives precise meaning to the intuitive picture of the vacuum squeezing the field lines. Crucial to this mechanism for confinement is the connection between the spontaneous breaking of a gauge symmetry as indicated by a non-zero vacuum expectation value of the monopole creation operator and the formation of vortices. Both the monopole operator and the vortex operators [25] rely on a definition of electric field strength. Therefore a tightening of these definitions could enhance our understanding of this crucial connection. One can compare this definition of Abelian field strength with the lattice implementation [24] of the 't Hooft expression [23] which would lead to an exact Ehrenfest theorem only to leading order in the lattice spacing, a.

4.2. U(1) example

Consider the partition function for the U(1) plaquette action, $S_a = \sum_{n,\mu>\nu}(1 - \cos\theta_{\mu\nu}(x))$, including a Wilson loop:

$$\mathcal{Z}_W = \int [d\theta_\mu] e^{i\theta_W} \exp\left(-\beta S_a\right).$$

It is invariant under shifting any link angle, $\theta_\mu(x) \to \theta_\mu(x) + \varepsilon$. Using this invariance Zach *et al.* [31] derived the relation:

$$\frac{\int[d\theta_\mu]\sin\theta_W \left(\frac{1}{a}\Delta_\mu \frac{\sin\theta_{\mu\nu}}{ea^2}\right) e^{-\beta S_a}}{\int[d\theta_\mu]\cos\theta_W e^{-\beta S_a}} = e\frac{\delta_W}{a^3},$$

where $\delta_W = 0$ unless the shifted link lies on the Wilson loop external source, when $\delta_W = \pm 1$. By identifying $\sin\theta_{\mu\nu}/(ea^2)$ as the field strength we then obtain an Ehrenfest theorem of the form:

$$\left\langle \frac{1}{a}\Delta_\mu F_{\mu\nu} - J_\nu^{(\mathrm{static})} \right\rangle_{\mathrm{source}} = 0.$$

The choice of $\theta_{\mu\nu}/(ea^2)$ as the field strength would not lead to an Ehrenfest theorem for finite lattice spacing.

4.3. Generalization to SU(2)

The SU(2) Wilson plaquette action, S, gives

$$\mathcal{Z}_W = \int [dU_\mu] e^{i\theta_W} \exp\left(-\beta S\right),$$

where W now indicates an Abelian Wilson loop. Ignoring gauge invariance for the moment, we exploit the invariance of the measure under a right (or left) multiplication of a link variable by a constant SU(2) matrix, $U_\mu \to U_\mu\left(1 - \frac{i}{2}\varepsilon\sigma_3\right)$. The derivative of S with respect to ε, which we denote as S_μ, inserts a σ_3 in the six plaquettes contiguous to the shifted link. Similarly the Abelian Wilson loop has a σ_3 insertion if it contains the shifted link. This gives the relation

$$\left\langle \frac{2}{ga^3} S_\nu(U) - J_\nu^{(\text{static})} \right\rangle_{\text{Abelian source}} = 0\,, \tag{7}$$

This can be cast into the form Eq. (6) using the parametrization of the link matrix:

$$U_\mu = \begin{pmatrix} \cos(\phi_\mu)e^{i\theta_\mu} & \sin(\phi_\mu)e^{i\chi_\mu} \\ -\sin(\phi_\mu)e^{-i\chi_\mu} & \cos(\phi_\mu)e^{-i\theta_\mu} \end{pmatrix}, \tag{8}$$

where $\phi_\mu \in [0, \frac{\pi}{2})$ and θ_μ, $\chi_\mu \in [-\pi, \pi)$. Separate this into the diagonal, $\mathcal{D}_\mu$, and off-diagonal, $\mathcal{O}_\mu$, parts; $U_\mu = \mathcal{D}_\mu + \mathcal{O}_\mu$. Applying this to Eq. (7) gives

$$\left\langle \frac{1}{a}\Delta_\mu F_{\mu\nu} - J_\nu^{(\text{dynamical})} - J_\nu^{(\text{static})} \right\rangle_{\text{Abelian source}} = 0\,, \tag{9}$$

where $\frac{1}{a}\Delta_\mu F_{\mu\nu}$ contains only $\mathcal{D}_\mu$ contributions to the links and $J_\nu^{(\text{dynamical})}$ the rest. The first term in Eq. (9) becomes an ordinary divergence because the inserted σ_3 in the loop commutes with the links and can take any position in the loop.

4.4. Gauge fixing

Gauge fixing complicates the issue, although the essence of the argument goes through as before. Prior to Abelian projection we gauge fix to satisfy $F_i(U;n) = 0$ for $i = 1, 2$. When shifting a link $U_\mu \to U_\mu\left(1 - \frac{i}{2}\varepsilon\sigma_3\right)$ we must in general perform a simultaneous gauge transformation, $g(x) = 1 - \frac{i}{2}\boldsymbol{\eta}\cdot\boldsymbol{\sigma}$, where $\eta_i(x) \propto \varepsilon$, to avoid leaving the gauge condition. S_μ is gauge invariant, but we obtain extra terms from the Wilson loop source and the Faddeev–Popov determinant when we differentiate with respect to ε.

The partition function now reads:

$$\mathcal{Z}_W = \int [dU_\mu]\, e^{i\theta_W}\, \Delta_{FP} \prod_{n,i} \delta(F_i(U;n))\, e^{-\beta S}\,.$$

We are primarily interested in the maximal Abelian gauge,

$$F_i(U;n) = \frac{1}{2}\sum_\mu \left\{ \mathrm{Tr}\left(\sigma_i U_\mu^\dagger(x)\sigma_3 U_\mu(x)\right) + \mathrm{Tr}\left(\sigma_i U_\mu(x-\hat\mu)\sigma_3 U_\mu^\dagger(x-\hat\mu)\right)\right\},$$

and integrating out the ghost fields gives

$$\Delta_{FP} = \det \left(\frac{\partial F_i(U;n)}{\partial \eta_j(m)} \right)_{F_i(U;n)=0} .$$

The Ehrenfest theorem is now given by

$$\int [dU_\mu]\, e^{i\theta_W}\, \Delta_{FP} \prod_{n,i} \delta(F_i(U;n)) \left(-\beta S_\mu + \frac{(\Delta_{FP})_\mu}{\Delta_{FP}} + i(\theta_W)_\mu \right) e^{-\beta S} = 0 .$$

which is recast as Eq. (6).

4.5. Status

When shifting the link does not conflict with the gauge condition, *e.g.* when no gauge condition is imposed, no extra gauge transformation is required and we find Eq. (9) is satisfied exactly. Ignoring this conflict, in the maximal Abelian gauge, the sum of the first two terms in Eq. (9) is 1.128 (5) for the case of a shift of the source link. (This is in a normalization where $\langle J_\nu^{(\mathrm{static})} \rangle = 1$.) We have used a Abelian plaquette source ($\beta = 2.3$ on a 12^4 lattice). Hence there is a 13% violation of the identity. On the same lattice, in the numerically simpler gauge that diagonalizes all plaquettes in a particular plane, the violation is −23%. In both cases there is a rapidly decreasing, but non-zero signal for the summed terms that extends away from the source where $\langle J_\nu^{(\mathrm{static})} \rangle = 0$.

The corrective gauge transformation at every site that accompanies the shift of a single link introduces a non-locality. The Wilson loop derivative, $(\theta_W)_\mu$, is increased, and picks up a contribution even when the shifted link is not one of those making up the loop. The magnitude of the gauge transformation falls off exponentially with distance from the shifted link, however, and is a small effect. In other words, the derivative of the source is no longer a delta function of position, but is slightly smeared by the gauge fixing.

Inclusion of this reduces the violation of the maximal Abelian gauge Ehrenfest identity to 3% on a 6^4 lattice at $\beta = 2.3$ (since this is a lattice identity, finite volume and scaling considerations are irrelevant). On such a lattice, the plaquette gauge relation is improved to −15%.

The calculation of the Faddeev–Popov term is incomplete, but preliminary results indicate that $\langle J_\nu^{(\mathrm{Faddeev-Popov})} \rangle$ is relatively small and of the correct sign and magnitude to cancel the remaining violation and satisfy Eq. (6) both in the maximal Abelian and plaquette gauges.

The lattice definition of Abelian field strength that follows from this approach is

$$F_{\mu\nu}(x) = \frac{1}{ga^2}\mathrm{Tr}\left[\sigma_3 \mathcal{D}_\mu(x)\mathcal{D}_\nu(x+\hat{\mu})\mathcal{D}_\mu^\dagger(x+\hat{\nu})\mathcal{D}_\nu^\dagger(x)\right]. \quad (10)$$

This differs from the lattice version [24] of the 't Hooft Abelian field strength [23]. Fig. 3 gives the lattice implementation of this. This is SU(2) gauge invariant because the adjoint fields transform under a gauge transformation along with the links. We choose $\phi = \sigma_3$ to compare with Eq. (10). Both

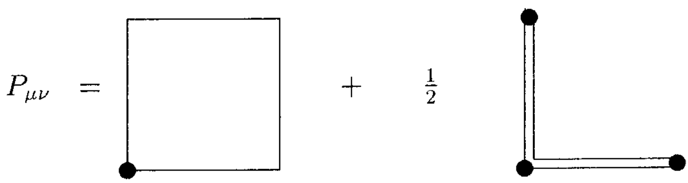

Fig. 3. Lattice generalization of the 't Hooft gauge invariant Abelian field strength. The lines represent single links, U. The adjoint field, indicated by the circle, is normalized: $\phi^2 = I$, $\phi = \boldsymbol{\phi} \cdot \boldsymbol{\sigma}$. $P_{\mu\nu} = -iga^2 F_{\mu\nu}$.

agree in the continuum limit, but the latter would satisfy the Ehrenfest theorem only to leading order in a.

In terms of the link parametrization, Eq. (8), the field strength is:

$$F_{\mu\nu}(x) = \frac{\cos\theta_{\mu\nu}(x)}{ga^2}\{\cos\phi_\mu(x)\cos\phi_\nu(x+\hat{\mu})\cos\phi_\mu(x+\hat{\nu})\cos\phi_\nu(x)\}. \quad (11)$$

The factors $\cos\phi_\mu(x) \to 1$ in the continuum limit.

5. Summary and conclusions

The upshot of the previous section is that flux should be defined by Eq. (11). We need to recalculate our London relation and G-L coherence length calculations with this new definition applied to electric and magnetic flux. The previous calculations were done without the factor $\cos\phi_\mu(x)\cos\phi_\nu(x+\mu)\cos\phi_\mu(x+\nu)\cos\phi_\nu(x)$. We anticipate only a small correction to the previous calculations in the maximal Abelian gauge. The reason follows from a fact pointed out by Cernodub *et al.* [20] and Poulis [34]. In this gauge this factor has very small fluctuations and is essentially a constant. If it were exactly constant, it could cancel out of the London relation.

In other gauges, the London relation calculations suffered problems and was very hard to interpret. We think now that some of the problems can be traced to the definitions of flux and we will revisit these calculations.

We now know how to define charge density on the lattice after Abelian projection. Hence there are new issues to address such as charge screening. One is whether we have interpreted the charge of the source correctly.

The goal of this lattice work is to identify a confinement mechanism. Currently these directions lead us to an effective Higgs theory in which spontaneous gauge symmetry breaking provides the order parameter for the confining phase transition. However many issues with this approach remain unresolved. We look forward to addressing some of them with the newly derived Ehrenfest theorem at our disposal.

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REFERENCES

[1] H.B. Nielsen, P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).

[2] J. Kogut, L. Susskind, *Phys. Rev.* **D9**, 3501 (1974).

[3] Y. Nambu, *Phys. Rev.* **D10**, 4262 (1974).

[4] G. Parisi, *Phys. Rev.* **D11**, 970 (1975),

[5] S. Mandelstam, *Phys. Rep.* **23** 245,(1976).

[6] G. 't Hooft, High energy physics, ed. A. Zichichi, Editrice Compositori, Bologna, 1976.

[7] M. Creutz *Phys. Rev. Lett.* **45** 313 (1980).

[8] G. S. Bali, C. Schlichter, K. Schilling, *Phys. Rev.* bf D51, 5165 (1995).

[9] R.W. Haymaker, J. Wosiek, *Phys. Rev.* **D43**, 2676 (1991); R. W. Haymaker, J. Wosiek, V. Singh, D. Browne, Proceedings of the Workshop on QCD Vacuum Structure, The American university of Paris, June, 1992, p184, World Scientific, 1993; R. W. Haymaker, V. Singh, Y. Peng, J. Wosiek, *Phys. Rev.* **D53**, 389 (1996).

[10] G. 't Hooft, *Nucl. Phys.* **B190**, 455 (1981).

[11] A.S. Kronfeld, G. Schierholz, U.J. Wiese, *Nucl. Phys.* **B293**, 461 (1987); A.S. Kronfeld, M.L. Laursen, G. Schierholz, U.J. Wiese, *Phys. Lett.* **B198**, 516 (1987); T. Suzuki, I. Yotsuyanagi, *Phys. Rev.* **D42**, 4257 (1990).

[12] G. DiCecio, A. Hart, R.W. Haymaker, LSU preprint LSUHE No. 266 - 1997, hep-lat 9709084.

[13] S. Weinberg, *The Quantum Theory of Fields* Vol. II, Cambridge University Press, Cambridge 1996, See chapter 21.6.

[14] For a review and further references to superconductivity on the lattice see *e.g.* R.W. Haymaker, *Dual Abrikosov Vortices in U(1) and SU(2) Lattice Gauge Theories*, Proceedings of the international School of Physics "Enrico Fermi", Course CXXX, A. Di Giacomo and D. Diakonov (Eds.) IOS Press, Amsterdam 1996, hep-lat 9510035.

[15] For a review of monopoles and confinement, see *e.g.* T. Suzuki, *Nucl. Phys.* **B** Proc. Suppl. **30**, 176 (1993).

[16] J. Frölich, P. A. Marchetti, *Euro. Phys. Lett.* **2**, 933 (1986); *Commun. Math. Phys.* **112**, 343 (1987); **116**, 127 (1988); **121**, 177 (1989); *Lett. Math. Phys.* **16**, 347 (1988); L. Polley, U.-J. Wiese, *Nucl. Phys.* **B356**, 621 (1991); M. I. Polikarpov, L. Polley, U.-J. Weise, *Phys. Lett.* **B253**, 212 (1991);. L. Del Debbio, A. Di Giacomo, G. Paffuti, *Phys. Lett.* **B349**, 513, (1995); *Nucl. Phys.* **B** Proc. Suppl. **42**, 231 (1995).

[17] V. Singh, R.W. Haymaker, D.A. Browne, *Phys. Rev.* **D47**, 1715 (1993).

[18] T. A. DeGrand, D. Toussaint, *Phys. Rev.* **D22**, 2478 (1980).

[19] M. Baker, J.S. Ball, F. Zachariasen, *Phys. Rep.* **209**, 73 (1991); *Phys. Rev.* **51**, 1968 (1995).

[20] M.N. Chernodub, M.I. Polikarpov, A.I. Veselov, *Phys. Lett.* **B342**, 303 (1995).

[21] L. Del Debbio, A. Di Giacomo, G. Paffuti, P. Pieri, *Phys. Lett.* **B355**, 255 (1995).

[22] A. Di Giacomo, G. Paffuti, hep-lat/9707003.

[23] G. 't Hooft, *Nucl. Phys.* **B79**, 276 (1974).

[24] K. Bernstein, G. DiCecio, R.W. Haymaker, *Phys. Rev.* **D55**, 6730 (1997).

[25] V. Singh, D.A. Browne, R.W. Haymaker, *Nucl. Phys.* **B** Proc. Suppl. **30**, 658 (1993); *Phys. Lett.* **B306**, 115 (1993); Y. Matsubara, S. Ejiri, T. Suzuki, *Nucl. Phys.* **B** Proc. Suppl. **34** 176 (19940; Y. Peng, R.W. Haymaker, *Phys. Rev.* **D52** 3030, (1995).

[26] H. Shiba, T. Suzuki, *Phys. Lett.* **B333**, 461 (1994).

[27] J. Stack, R. Wensley, *Nucl. Phys.* **D22**, 597 (1992); J. Stack, S. Neiman, R. Wensley, *Phys. Rev.* **D50**, 3399 (1994).

[28] G.S. Bali, V. Bornyakov, M. Müller-Preussker, K. Schilling, *Phys. Rev.* **D54**, 2863 (1996).

[29] M.N. Chernodub, M.I. Polikarpov, A.I. Veselov, *Nucl. Phys.* **B** Proc. Suppl. **49**, 307 (1996).

[30] N. Nakamura, V. Bornyakov, S. Ejiri, S. Kitahara, Y. Matsubara, T. Suzuki, *Nucl. Phys.* **B** Proc. Suppl. **53**, 512 (1997).

[31] M. Zach, M. Faber, W. Kainz, P. Skala, *Phys. Lett.* **B358**, 325 (1995); P. Skala, M. Faber, M. Zach, *Nucl. Phys.* **B494**, 293 (1997).

[32] G.S. Bali, V. Bornyakov, M. Müller-Preussker, K. Schilling, *Phys. Rev.* **D54**, 2863 (1996).

[33] T. Suzuki, S. Ilyar, Y. Matsubara, T. Okude, K. Yotsuji, *Phys. Lett.* **B347**, 375 (1995); ERRATUM *Phys. Lett.* **B351**, 603 (1995).

[34] G. Poulis, *Phys. Rev.* **D54**, 6974 (1996).