FOUR AND THREE LOOP CALCULATIONS IN QCD: THEORY AND APPLICATIONS *

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The talk briefly reviews the current state of the art in doing multiloop QCD calculations in a completely analytic way. In particular, we discuss recent analytical calculations of 4-loop $\mathcal{O}(\alpha_s^3)$ gluino contribution to $R(s)$ and its implications to $\Gamma_{\rm tot}(Z \to \text{hadrons})$ and $\Gamma(\tau^- \to \nu_\tau + \text{hadrons})$;

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1. Introduction

The firm knowledge of radiative corrections (r.c.) is a must in confronting the Standard Model with experimental data. While the electroweak r.c. are usually required to be known in the leading and next-to-leading approximations, this is certainly not always the case for the strong interaction ones. The current (relatively large) value of the strong coupling constant $\alpha_s(M_Z) = 0.1202 \pm 0.0033$ [1] and the remarkably high precision achieved in high energy e^+e^- experiments make the latter sensitive to really higher order QCD r.c. such as $\mathcal{O}(\alpha_s^3)$ contributions to $R(s)$ and $\Gamma_{\rm tot}(Z \to \text{hadrons})$.

The talk briefly reviews the current state of the art in doing multiloop QCD calculations in a completely analytic way. It is *not* intended as a sort of exhaustive review of the vast subject of the QCD r.c. We deal with analytical calculations of 3- and 4-loop r.c. to 2-point correlators of bilinear quark currents. Even more, only massless correlators are considered (high energy limit). The choice is dictated in part by my personal taste and in part by the fact that most advanced results (at least as for the number of

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loops involved) have been obtained in this particular field. The reader who is looking for a broader exposition including leading and next-to-leading r.c. should consult numerous reviews [2].

2. Feynman integrals up to 3 loops

Here we will discuss briefly the tools now available to analytically compute massless propagators in higher orders. We limit ourselves to these rather restricted class of Feynman integrals due to the following reasons:

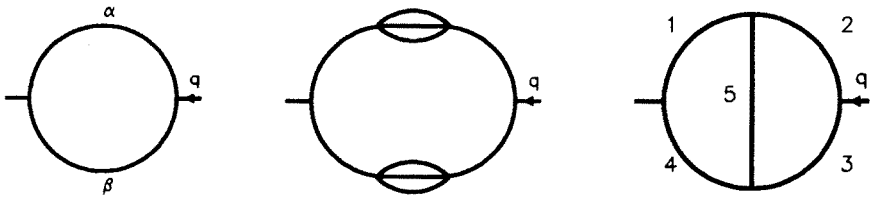
1. Practice shows that in many cases the methods of asymptotic expansions of Feynman integrals (for a recent review see [3]) do produce results numerically very well approximating the exact results when the latter are available. These methods reduce initial multi-scale Feynman amplitudes to combinations of massless propagators and massive tadpoles.
2. A number of problems can be eventually reduced to evaluation of massless propagators. A important example is the evaluation of so-called RG-functions (that is beta-functions and anomalous dimensions).
3. Thanks to the intrinsic simplicity of the integrals under discussion, — they depend on only one nontrivial scale: an external momentum — their analytical evaluation proves to be feasible in quite high orders in the coupling constant. The same simplicity provides the possibility of constructing *regular* algorithms for evaluating these integrals as well as dedicated computer programs allowing to perform the calculations in a convenient and automatic way.

For brevity massless Feynman integrals depending on exactly one external momentum will be denoted by *p-integrals*. At the moment there are tools to analytically compute arbitrary one- two- and three-loop p-integrals (see below). Fortunately, in many important cases one is interested only in the absorptive part of massless two-point correlators. In this case available theoretical tools are enough to guarantee at least *in principle* the analytical calculability of absorptive part of an arbitrary 4-loop p-integral (see below).

One-loop p-integrals

We start from a well-known elementary formula for a generic 1-loop p-integral (see Fig. 1a; we shall consider Feynman integrals in the Euclidean momentum space throughout this section)

$$\int \frac{\mathrm{d}^{\mathrm{D}}\ell}{(2\pi)^{\mathrm{D}}} \frac{1}{(q^2)^{\alpha}(q-l)^{2\beta}} = \frac{(q^2)^{2-\varepsilon-\alpha-\beta}}{(4\pi)^{2-\varepsilon}} G(\alpha,\beta)\,,$$
$$G(\alpha,\beta) \equiv \frac{\Gamma(\alpha+\beta-2+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(2-\alpha-\varepsilon)\Gamma(2-\beta-\varepsilon)}{\Gamma(4-\alpha-\beta-2\varepsilon)}\,. \qquad (1)$$



(a) (b) (c)

Fig. 1. Some p-integrals: (a) the generic one-loop p-integral and (b) an example of primitive five-loop p-integral; (c) the master two-loop p-integral.

It is of importance to note that any p-integral depends homogeneously on its external momentum. This facts allows the immediate analytic evaluation of the whole class of *primitive* p-integrals which, by definition, may be performed by repeated application of the one-loop integration formula. For example, the five-loop scalar integral of Fig. 1b is performed by (1) with the result

$$\left((q^2)^{-\varepsilon}(4\pi^2)^{2-\varepsilon}\right)^5 (q^2)^{-\varepsilon}(G(1,1)G(1,\varepsilon))^2 G(1+2\varepsilon,1+2\varepsilon)\,. \qquad (2)$$

Two-loop p-integrals

Not all p-integrals are primitive ones. One first encounters nontrivial p-integrals already at the two loop level. While one-loop integrals are performed with ease the evaluation of the master two-loop diagram (see Fig. 1c) is not trivial. The corresponding Feynman integral reads

$$\frac{(4\pi)^{4-2\varepsilon}F(\alpha_1,\ldots,\alpha_5)}{(q^2)^{-4+2\varepsilon+\sum_i \alpha_i}} \equiv \int \frac{\mathrm{d}^D\ell_1 \mathrm{d}^D\ell_2}{(2\pi)^{2D}} \frac{1}{p_1^{2\alpha_1}p_2^{2\alpha_2}p_3^{2\alpha_3}p_4^{2\alpha_4}p_5^{2\alpha_5}}\,, \qquad (3)$$

with the loop momenta

$$p_1 = \ell_1,\quad p_2 = \ell_2,\quad p_3 = q-\ell_2,\quad p_4 = q-\ell_1,\quad p_5 = \ell_2-\ell_1\,.$$

A closed expression for the function $F(\alpha_1,\ldots\alpha_5)$ for generic values of the arguments is not known. However, results do exist for particular cases. The first one, valid for a generic space-time dimension D, was obtained with the help of the so-called Gegenbauer polynomial technique in x-space (GPTX) [4]. It reads

$$\begin{aligned} F(\alpha,1,1,\beta,1) &= \frac{G(1,1)}{D-2-\alpha-\beta} \\ &\quad \times \{\alpha[G(\alpha+1,\beta)-G(\alpha+1,\beta+\varepsilon)]+(\alpha\leftrightarrow\beta)\} \end{aligned} \qquad (4)$$

It has been also shown in Ref. [4] that similar results may be obtained for the case when the indices α_2, α_3 and α_5 are integers while α_1 and α_4 are arbitrary.

In practice one often needs only a few first terms of the expansion of $F(\alpha_1 \ldots \alpha_5)$ in the Laurent series in ε. This expansion is known for generic values of the $\alpha_1 \ldots \alpha_5$ up to a fixed (quite high) order (see Refs [7, 8] and references therein).

Three-loop p-integrals

In principle GTPX is also applicable to compute some nontrivial three-loop p-integrals. For example, the basic scalar non-planar three-loop diagram of Fig. 3a was first calculated via GPTX in Ref. [4]. However, calculations quickly get clumsy, especially for diagrams with numerators.

The main breakthrough at the three-loop level happened with elaborating the method of integration by parts of dimensionally regularized integrals Refs [5, 6]. The key identity for the method is[1]

$$\int \mathrm{d}^{\mathrm{D}}\ell \frac{\partial}{\partial \ell_\mu} I(\ell, \ldots) \equiv 0\,, \tag{5}$$

where $I(\ell, \ldots)$ is a Feynman *integrand* and ℓ is one of its loop momenta. The identity reflects the possibility of neglecting the surface terms, which holds true in dimensional regularization [10]. The use of (5) along with tricks like completing momentum squares and cancelling similar factors in the nominator against those in the denominator constitutes the essence of the approach. The identity depicted in Fig. 2 is a typical example of relations obtainable with the help of the integration by parts method.

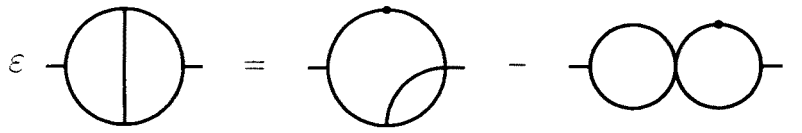


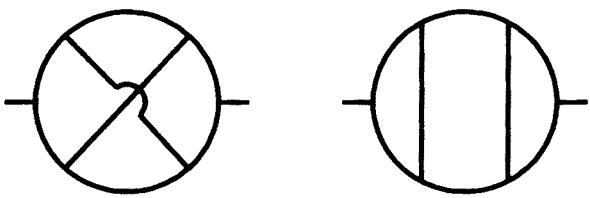
Fig. 2. The exact relation expressing a nonprimitive two-loop scalar p-integral through primitive integrals; a dot on a line means a squared scalar propagator.

It should be, however, heavily stressed that, the validity of such operations for *divergent* dimensionally regulated integrals with deeply intermixed UV and IR (sub)divergences is *not* obvious any more. It had to be rigorously justified within a proper generalization of the dimensional regularization itself. It has been done in Ref. [10] (see also a monograph [11]).

The general scheme of the use of the integration by parts method is based on the exploitation the identities of type (5) in the form of recurrence

[1] For two-loop massive integrals a similar identity was used in the classical work by 't Hooft and Veltman Ref. [9].

relations with the aim to express a complicated diagram through the simpler ones. All (about a dozen) topologically different three-loop p-integrals were neatly analyzed in Ref. [6] and a concrete calculational algorithm was suggested for every topology. As a result the algorithm of integration by parts for three-loop p-integrals was developed. The algorithm constitutes a series of involved identities which are used to identically transform any three-loop p-integral into a sum of primitive one-loop p-integrals and two basic three-loop p-integrals pictured in Fig. 3.



(a) (b)

Fig. 3. (a), (b) the master three-loop non-planar and planar scalar diagrams.

It goes without saying that the calculation of higher order corrections in gauge theories is almost impossible without intensive use of computer algebra methods. In addition to the old problem of taking long traces of Dirac γ matrixes, the algorithm of integration by parts, when applied even to a single 3-loop p-integral, generically produces dozens or even hundreds of terms. At the moment there exist essentially three different packages which implement the algorithm. For p-integrals they are written in SCHOONSCHIP [12] (see Refs [13, 14] and in FORM [15] (see Refs [16, 17]).

3. Infrared rearrangement and R^*-operation

At the moment there is no any general algorithm allowing to analytically compute arbitrary 4-loop p-integrals. The problem of creating such an algorithm seems to be hopelessly difficult. See Ref. [18] where the point is dealt with in some detail and also a discussion in the conclusion of the present talk.

Nevertheless, a large group of important 4-loop problems — calculation of RG-functions or, equivalently, UV renormalization constants — proves to be reducible to the 3-loop case and, thus, be doable in a completely analytic way. This is because in MS scheme any UV counterterm is polynomial in

momenta and masses [19]. This observation was effectively employed in Ref. [20] to simplify considerably the calculation of UV counterterms. The method was further developed and named Infrared Rearrangement (IRR) in Ref. [4]. It essentially amounts to an appropriate transformation of the IR structure of FI's by setting zero some external momenta and masses (in some cases after some differentiation is performed with respect to the latter). As a result the calculation of UV counterterms is much simplified by reducing the problem to evaluating massless p-integrals. The method of IRR was ultimately refined and freed from unessential qualifications in Ref. [21]. The following Theorem has been proven there by the explicit construction of the corresponding algorithm:

Any UV counterterm for any $(h+1)$-loop Feynman integral can be expressed in terms of pole and finite parts of some appropriately constructed (h)-loop p-integrals.

In many important cases one is interested only in the absorptive part of massless two-point correlators. In this case available theoretical tools are enough to guarantee at least *in principle* the analytical calculability of absorptive part of an arbitrary 4-loop p-integral. This is eventually due to the fact that the absorptive part of a four-loop p-integral is fully expressible in terms of the corresponding four-loop UV counterterm along with some three-loop p-integrals. As a typical example we will consider in next section the ratio $R(s) = \sigma_{\rm tot}(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ which is essentially given by the absorptive part of the vacuum polarization.

The Theorem coupled with the integration by parts method solves *at least in principle* the task of analytical evaluation of RG functions and absorptive parts of 4-loop p-integrals. It should be noted that the R^*-operation is essential to prove the Theorem, though in most (but not in all) practical cases one could proceed without it. However, such a practice, in fact, forces diagram-wise renormalization mode, what, in turns, brings down a heavy penalty of manual treatment of hundreds of diagrams.

Indeed, in genuine four-loop calculations the reduction to three-loop p-integrals is far from being trivial and includes a lot of manipulations. Typical steps here are

a to reduce the initial Feynman integral to logarithmically divergent ones via a proper differentiation with respect masses and external momenta;

b to identify UV and IR divergent subgraphs of the resulting integral;

c to remove in a recursive way the corresponding UV and IR divergences;

d to compute resulting p-integrals.

Among these steps only the calculation of p-integrals can be at present completely performed by a computer. All others, especially **b** and **c** are difficult

to computerize. As a result, in spite of the fact that the necessary theoretical tools have been around for more than a decade, yet until very recently there existed just one QCD four-loop result: the $\mathcal{O}(a_s^3)$ contribution to $R(s)$. It was done in Refs [22, 23] within the diagram-wise renormalization approach. The core of the problem is the calculation of the four-loop contribution to the photon anomalous dimension entering into the RG equation for the photon polarization operator. The initial 98 four-loop diagrams contributing to the photon polarization operator proliferate to about 250 after the IRR procedure is applied. In addition these diagrams contain about 600 various subdiagrams which should be computed separately in order to subtract UV subdivergences. No wonder that both calculations were done in a particular gauge — the Feynman one — and, thus, a valuable possibility of using the gauge invariance as a strong test of the correctness of the results had not been used at all.

The calculation of Ref. [22] did not use the R^*-operation at all while that of Ref. [23] employed it only for a few diagrams. We will see in the next section that a proper use of power of the R^*-operation allows one not only to repeat with ease these calculations in the general covariant gauge but also to extend them considerably.

4. 4-loop case: photon vacuum polarization

The photon vacuum polarization function $\Pi(-q^2)$is defined through the correlator of the hadronic EM current $j_\mu^{\rm em} = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f$ as follows

$$\Pi_{\mu\mu}(q) = (4\pi)^2 i \int {\rm d}x e^{iqx} \langle 0|T[\ j_\mu^{\rm em}(x) j_\nu^{\rm em}(0)\ ]|0\rangle = -3q^2 \Pi(-q^2)\,. \qquad (6)$$

The optical theorem relates the inclusive cross-section and thus the function $R(s)$ to the discontinuity of Π in the complex plane

$$R(s) = \frac{3}{4\pi}\,{\rm Im}\,\Pi(-s - i\delta)\,. \qquad (7)$$

The renormalization mode of the polarization operator $\Pi(Q^2)$ reads

$$\Pi(Q^2/\mu^2, \alpha_s) = Z^{\rm em} + \Pi_0(Q^2, \alpha_s^0),, \qquad (8)$$

where $\alpha_s = g^2/(4\pi)$ is the strong coupling constant. Within the $\overline{\rm MS}$ scheme (here and below we are using a convenient combination $a_s = \alpha_s/\pi$)

$$Z^{\rm em} = \sum_{1\le j\le i} (Z^{\rm em})_{ij}\,\frac{a_s^{i-1}}{\varepsilon^j}\,, \qquad (9)$$

with the coefficients $(Z_{\rm em})_{ij}$ being pure numbers and $D = 4 - 2\varepsilon$ standing for the space-time dimension.

As a result we arrive at the following renormalization group (RG) equation for the polarization operator ($L_Q = \ln \frac{\mu^2}{Q^2}$)

$$\frac{\partial}{\partial L_Q}\Pi = Q^2\gamma_{\rm em}(a_s) - \left(\beta(a_s)a_s\frac{\partial}{\partial a_s}\right)\Pi. \tag{10}$$

Here the photon anomalous dimension and the $\beta(a_s)$-function are defined in the usual way

$$\gamma_{\rm em} = \mu^2\frac{d}{d\mu^2}(Z_{\rm em}) - \varepsilon Z_{\rm em} = -\sum_{i\geq 0}(i+1)(Z_{\rm em})_{i1}a_s^i, \tag{11}$$

$$\mu^2\frac{d}{d\mu^2}a_s = \alpha_s\beta(a_s) \equiv -\sum_{i\geq 0}\beta_i a_s^{i+2}. \tag{12}$$

The relation (10) explicitly demonstrates the main computational advantage of finding first the polarization function $\Pi(Q^2)$ against a direct calculation of $R(s)$ in the case of massless QCD Indeed, in order a_s^n the derivative $(\partial/\partial L_Q)\Pi$ and, consequently, $R(s)$ depends on the very function Π which is multiplied by at least one factor of a_s. This means that one needs to know Π up to order a_s^{n-1} only to unambiguously reconstruct all Q-dependent terms in Π to order a_s^n, provided, of course, the beta function and anomalous dimension $\gamma_{\rm em}$ is known to order a_s^n. On the other hand, as we have discussed before the calculation of an anomalous dimension or a beta-function is known to be much easier than computing a correlator of the same order in the coupling constant.

Thus, in order to check the results of Refs [22, 23] by an independent calculation one should compute Π in order a_s^2 (a dozen three-loop diagrams) and $\gamma_{\rm em}$ in order a_s^3 (more than a hundred of four-loop diagrams). The first task is almost trivial nowadays (see the previous two sections) so let us concentrate on the second one. As we have already discussed, the problem is doable via IRR and the integration by parts method but the amount of various nontrivial manipulations with separate diagrams to be performed by hand is really terrific.

In a recent work [24] it has been demonstrated how the formalism of the R^*-operation can be applied to derive an explicit formula for the renormalization constant $Z^{\rm em}$ and, consequently for $\gamma_{\rm em}$. The formula reads:

$$Z^{\rm em} = -K_\varepsilon\left\{\frac{1}{Z_2}\frac{1}{4D}Tr[\delta\Gamma^0_{\tilde\alpha}(0,0,a_s^0)\gamma_\alpha]Z^{\rm em} - \frac{\delta Z_2}{Z_2}Z^{\rm em}\right.$$
$$\left. - \frac{1}{2D(D-1)}\Box_q Tr[\gamma_{\tilde\alpha}G^0(p+q,a_s^0)\Gamma^0_\alpha(p,q,a_s^0)G^0(p,a_s^0)]\Big|_{q=0}\right\}. \tag{13}$$

Here G^0 and Γ^0_α are the full quark propagator and the EM current vertex function respectively; the integration with respect to the loop momentum p with the weight function $(4\pi)^2/(2\pi)^D$ is not explicitly displayed. Z_2 is the quark wave function renormalization constant.

Several extra comments are in order regarding this formula. First, Eq. (13) is, rigorously speaking, applicable as it stands only to the so-called non-singlet diagrams, that is to those where both EM currents belong to one and the same quark loop. The four singlet diagrams, violating this requirement, appear first in order a_s^3 and should be treated separately.

Second, by $\delta\Gamma^0_{\tilde\alpha}(p,q,a_s^0)$ we denote the vertex function of the electromagnetic current with the tree contribution removed. The "tilde" atop the index α again that in every diagram the quark propagator entering to the vertex j_α is softened at small momenta by means of the auxiliary mass m_0 according to the rule

$$\gamma_\alpha \to \gamma_{\tilde\alpha} = \gamma_\alpha \, p^2/(p^2 - m_0^2). \tag{14}$$

The bare coupling constant a_s^0 is to be understood as $a_s = Z_a a_s$, with Z_a being the coupling constant renormalization constant.

An inspection of (13) immediately shows that, in order to find the $(n+1)$-loop correction to $Z^{\rm em}$, one needs only to know the renormalization constants Z_2 and $Z^{\rm em}$ to order a_s^n as well as the bare Green functions

$$G^0(p,a_s^0), \quad \frac{\partial}{\partial q_\beta}[\Gamma^0_\alpha(p,q,a_s^0)]\big|_{q=0}, \quad \Box_q[\Gamma^0_\alpha(p,q,a_s^0)]\big|_{q=0}, \quad \delta\Gamma^0_{\tilde\alpha}(0,0,a_s^0) \tag{15}$$

up to (and including) n-loops, that is to order $(a_s^0)^n$.

We have computed with the program MINCER [16] the unrenormalized three-loop Green functions (15) as well as the quark wave function renormalization constant Z_2 to order a_s^3. The calculations have been performed in the general covariant gauge with the gluon propagator $(g_{\mu\nu} - \xi\frac{q_\mu q_\nu}{q^2})/q^2$. We have also taken into account the singlet diagrams as well as extra diagrams with some of virtual quarks replaced by colour octet neutral fermions. In the minimal supersymmetric standard model such a fermion known as gluino appears as the superpartner of the gluon [29].

Our result for $R(s)$ with $\mu^2 = s$ reads [24]

$$R(s) = 3\sum_f Q_f^2 \left\{1 + a_s + a_s^2 r_2 + a_s^3 r_3\right\} + a_s^3 \left(\sum_f Q_f\right)^2 r_3^S, \tag{16}$$

where $r_3^S = \left(\frac{55}{72} - \frac{5}{3}\zeta(3)\right)$ and

$$r_2 = \frac{365}{24} - 11\,\zeta(3) - \frac{11}{12}\,n_f + \frac{2}{3}\,\zeta(3)\,n_f - \frac{11}{4}\,n_{\tilde g} + 2\,\zeta(3)\,n_{\tilde g},$$

$$
\begin{aligned}
r_3 = {} & \frac{87029}{288} - \frac{121}{48}\pi^2 - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5) - \frac{7847}{216}n_f + \frac{11}{36}\pi^2 n_f \\
& + \frac{262}{9}\zeta(3)\,n_f - \frac{25}{9}\zeta(5)\,n_f + \frac{151}{162}n_f^2 - \frac{1}{108}\pi^2 n_f^2 - \frac{19}{27}\zeta(3)\,n_f^2 \\
& - \frac{32903}{288}n_{\tilde{g}} + \frac{11}{12}\pi^2 n_{\tilde{g}} + \frac{277}{3}\zeta(3)\,n_{\tilde{g}} - \frac{25}{3}\zeta(5)\,n_{\tilde{g}} + \frac{151}{27}n_f\,n_{\tilde{g}} \\
& - \frac{1}{18}\pi^2 n_f\,n_{\tilde{g}} - \frac{38}{9}\zeta(3)\,n_f\,n_{\tilde{g}} + \frac{151}{18}n_{\tilde{g}}^2 - \frac{1}{12}\pi^2 n_{\tilde{g}}^2 - \frac{19}{3}\zeta(3)n_{\tilde{g}}^2.
\end{aligned}
$$

Here $n_{\tilde{g}}$ is the number of neutral gluino multiplets which we take either zero or one. We observe that neither $\gamma_{\rm em}$ no $R(s)$ depend on the gauge fixing parameter ξ as it must be. If $n_{\tilde{g}}$ is set to zero then $R(s)$ is in complete agreement with the results of Refs [22, 23]. In the numerical form,

$$
\begin{aligned}
R(s) = {} & 3\sum_f Q_f^2 \Big\{1 + a_s + a_s^2\,(1.98571 - 0.115295 n_f - 0.345886 n_{\tilde{g}}) \\
& + a_s^3 \Big(-6.63694 - 1.20013 n_f - 0.00518 n_f^2 - 2.85053 n_{\tilde{g}} \\
& - 0.03107 n_f n_{\tilde{g}} - 0.04661 n_{\tilde{g}}^2\Big)\Big\} - \alpha_s^3 \left(\sum_f Q_f\right)^2 1.2395.
\end{aligned}
$$

5. $\mathcal{O}(\alpha_s^3)$ gluino contribution to $\Gamma_{\rm tot}(Z \to \text{h's})$ and $\Gamma(\tau^- \to \nu_\tau + \text{h's})$

The result (16) can be straightforwardly applied to find the $\mathcal{O}(\alpha_s^3)$ gluino contribution to the Z-boson decay rate into hadrons (Γ_Z^h). As is well-known this decay rate may be viewed as an incoherent sum of vector (Γ_Z^V) and axial (Γ_Z^A) contributions. For massless u, d, s, c and b quarks the QCD corrections to Γ_Z^V are in one-to-one correspondence to those for $R(s)$. If one neglects as we do[2] the power suppressed terms of order $M_Z^2/(4M_t^2)$ and higher the same statement is valid also for Γ_Z^A except for a specific subset of so-called singlet diagrams. As a result the total decay rate is naturally presented as a sum of three terms

$$
\Gamma_Z^{A,S} = \Gamma_Z^V + \Gamma_Z^{A,NS} + \Gamma_Z^{A,S}. \tag{17}
$$

Here $\Gamma_Z^{A,S}$ and $\Gamma_Z^{A,NS}$ stand for the contributions to Γ_Z^A due to singlet and the rest (non-singlet) diagrams respectively. The first two terms in (17) are

[2] These terms have been found to be extremely small in Ref. [25]

directly expressed through the coefficients $r_1 - r_3$ and r_3^S as follows:

$$\frac{\Gamma_Z^V + \Gamma_Z^{A,NS}}{\Gamma_0} = \left\{ 3 \sum_f (v_f^2 + a_f^2) \left[1 + a_s + a_s^2 r_2 + a_s^3 r_3 \right] + \left(\sum_f v_f \right)^2 r_3^S \right\},$$

where $\Gamma_0 = G_F M_Z^3 / 24\sqrt{2}\pi = 82.94$ MeV, $v_f = 2I_3^f - 4Q_f \sin^2\theta_w$, $a_f = 2I_3^f$, $a_s = a_s(M_Z)$ and we have set $\mu = M_Z$.

The singlet diagrams are sensitive to the huge mass splitting in the top-bottom doublet and should be computed afresh. To order α_s^2 this was done in Ref. [26] (even without neglecting the power suppressed terms) while the $\mathcal{O}(\alpha_s^3)$ corrections were computed in [27]. In fact, no extra calculations are necessary to get the gluino contribution. A simple inspection of relevant diagrams immediately reveals that gluinos appear exclusively through the one-loop fermion correction to a gluon propagator This means that the result in QCD with gluinos can be obtained from the one in pure QCD by the replacement $n_f \to n_f + 3n_{\tilde{g}}$. It reads

$$\begin{aligned} \Gamma_Z^{A,S} &= \Gamma_0 \left\{ a_s^2 \left[-\frac{37}{4} + 3 \ln \frac{M_Z^2}{M_t^2} \right] \right. \\ &+ a_s^3 \left[-\frac{5825}{72} + \frac{11\pi^2}{4} + 3\zeta(3) + \frac{19}{2} \ln \frac{M_Z^2}{M_t^2} + \frac{33}{4} \ln^2 \frac{M_Z^2}{M_t^2} \right. \\ &\left. \left. + (n_f + 3n_{\tilde{g}}) \left(\frac{25}{12} - \frac{\pi^2}{6} + \frac{1}{3} \ln \frac{M_Z^2}{M_t^2} - \frac{1}{2} \ln^2 \frac{M_Z^2}{M_t^2} \right) \right] \right\}, \end{aligned} \tag{18}$$

where M_t is the pole mass of the top quark and $n_f = 5$ is the number of (light) quarks flavours. Numerically, the gluino term is tiny: the above equation with $n_f = 5$, and $M_Z^2/M_t^2 = (91.187/180)^2$ is evaluated to

$$\Gamma_Z^{A,S} = \Gamma_0 \left\{ -13.33 a_s^2 + a_s^2 \left[-52.514 - 2.8197 n_{\tilde{g}} \right] \right\}. \tag{19}$$

Another obvious application of Eq. (16) is the gluino contribution to the semihadronic decay rate of the τ-lepton. As is well-known in the massless limit the perturbative contributions to the ratio $R_\tau = \dfrac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau e^- \bar{\nu}_e)}$ can be written as follows (for a review including non-perturbative and other effects see Ref. [28])

$$R_\tau = 2 \int_0^{M_\tau^2} \frac{\mathrm{d}s}{M_\tau^2} (1 - s/M_\tau^2)^2 (1 + 2s/M_\tau^2) \tilde{R}(s), \tag{20}$$

where M_τ^2 is the τ-lepton mass. Here $\tilde{R}(s)$ is given by the expression (16) for $R(s)$ with $3\sum Q_f^2$ replaced by $3(|V_{ud}|^2 + |V_{us}|^2)$, $n_f = 3$ and $(\sum Q_f)^2$ set to zero. After a straightforward integration with respect to s in (20) one arrives at (for gluino in the octet representation)

$$R_\tau = 3(|V_{ud}|^2 + |V_{us}|^2)\left\{1 + a_s + a_s^2[5.2023 - 1.13755n_{\tilde{g}}] + a_s^3\left[26.366 - 21.0358n_{\tilde{g}} + 1.42119n_{\tilde{g}}^2\right]\right\}, \qquad (21)$$

with $a_s = a_s(M_\tau)$.

Note that the result (21) is applicable only if the gluino mass is significantly lower than that of the τ-lepton. Taken at its face value it could spoil the current agreement between the values of $\Lambda_{\overline{\rm MS}}$ extracted from the R_τ and $\Gamma(Z \to \text{hadrons})$. Such a conclusion seems to be premature as for a meaningful phenomenological discussion of the $\mathcal{O}(\alpha_s^3)$ gluino contributions one should also take into account the running of the coupling constant in the next-next-to-leading order. This requires the knowledge of the gluino contribution to the three-loop coefficient β_2 which is not yet available in the literature.

6. Conclusion

The skillful use of the R^*-operation leads to significant simplifications of analytical 4-loop calculations if we are interested in the divergent part of 4-loop diagrams. Thus, 4-loop calculations in QCD are gradually becoming practically feasible.

It is only natural to ask now about next level, that is about possibility to analytically compute *finite* parts of 4-loop p-integrals and consequently (according to the Theorem from Section 3) divergent parts of 5-loop p-integrals. Even a superficial glance at the problem shows that it hardly could be done completely "analytically". Still, there is, to my opinion, a chance that in a sense the problem could be solved. I mean that a better understanding of all kinds of identical relations connecting various p-integrals[3] could eventually result to the reduction of an arbitrary 5-loop p-integral to a combination of some limited number (a few dozens?) of master p-integrals. Once it is done there should be not very difficult to evaluate the latter analytically or numerically with sufficiently high accuracy. A fresh example of such approach has been recently demonstrated in a work by Laporta and Remiddi who achieved a computer-algebraic reduction and eventual analytical calculation of all 3-loop diagrams contributing to the electron's anomalous magnetic moment [31].

[3] For recent developments in this direction see [30].

The present work has been submitted to the HEP-PH e-print archive today, that is 29.10.1996. A related article [32] was posted on the same archive yesterday. Our results for the $\mathcal{O}(\alpha_s^3)$ gluino contributions to $R(s)$ and to related quantities have been confirmed there by an independent calculation.

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