

SOME TECHNIQUES FOR CALCULATING TWO-LOOP DIAGRAMS*

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A brief overview of some recent publications related to the evaluation of two-loop Feynman diagrams is given.

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We shall discuss, very briefly, recent progress in evaluating certain types of two-loop Feynman diagrams and related issues. Some emphasis will be put on the activities I was involved in, in collaboration with F.A. Berends, V.A. Smirnov, J.B. Tausk and N.I. Ussyukina. Mainly, a collection of “pointers” to the papers describing the relevant methods and algorithms is given. The lack of space makes it impossible to include references to all papers containing various applications of two-loop calculations, as well as some problems related to on-shell calculations with massive particles.

As a rule, the physical case of four dimensions is understood. In case of need, the dimensional regularization [1] is used as a regulator.

1. Two-loop self-energy diagrams with masses

For some special cases, the results were known long time ago. For example, two-loop diagrams contributing to the photon polarization operator in QED were first calculated in 1955 [2] (see also in [3]). Some other special cases were considered in Refs. [4]. In all examples, some of the internal lines were massless and there were just one or two different non-zero masses. As a rule, the results were expressible in terms of trilogarithms Li_3 , except for one of the QCD contributions to the quark selfenergy.

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Indeed, the situation becomes more complicated if one is interested in a diagram involving a three-particle threshold with all massive particles. In this case, there are arguments [5] that, for a general external momentum k , the result may not be expressible in terms of polylogarithms. The simplest example is the so-called “sunset” diagram with three massive propagators. Recently, it was considered in a number of papers [6, 7]. The results were expressed either in terms of multiple hypergeometric series or via one-dimensional integrals.

For more complicated diagrams, e.g. for the general two-loop self-energy diagram with different masses, the representations in terms of known functions are not available¹. One of the ways is to construct integral representations and then calculate the result numerically. Various approaches to this problem, including some useful integral representations, were discussed in Refs. [9, 10, 11, 12, 7]. Tensor reduction of two-loop self-energy diagrams was discussed in [13]. The problem can be reduced to evaluation of the scalar integrals.

An analytic approach to the calculation of two-loop diagrams with different masses consists in constructing the expansions of these diagrams in different regions. For example, when there is no threshold at $k^2 = 0$, the small momentum expansion is basically an ordinary Taylor expansion. For two-loop diagrams, the general algorithm for constructing the coefficients of such an expansion was presented in [14]. In general, the expansion converges when k^2 is below the first physical threshold. The coefficients of the expansion can be presented in terms of two-loop vacuum diagrams. Some results for these diagrams can also be found in [15], whereas the results for higher powers of propagators can be obtained using the integration-by-parts technique [16] (this procedure is also described in [14]). The general problem of tensor decomposition of two-loop vacuum diagrams was discussed in [17, 18]. In Ref. [19], conformal mapping and Padé approximations were used to evaluate numerical values beyond the threshold(s). In Ref. [20], a modified scheme for calculating the coefficients of the expansion was proposed.

The problem of constructing the large momentum expansion of two-loop self-energy diagrams (when k^2 is larger than all physical thresholds) was considered in [21]. To do this, the general theory of asymptotic expansions of Feynman diagrams [22] (see also in [23]) was employed. In four dimensions,

¹ In three dimensions, an essential progress has been recently made in Ref. [8]. However, the three-dimensional case is simpler, due to a very simple (exponential) form of the massive propagator in the coordinate space.

the coefficients of the large momentum expansion of the master two-loop self-energy diagram involve $\ln(k^2)$ and $\ln^2(k^2)$. The lowest term of the expansion of the master diagram is nothing but the massless integral [24] proportional to $6\zeta(3)$.

Then, the next question is what to do if one is interested in the threshold behaviour. Dealing with the thresholds may be unavoidable even in the small momentum expansion, when the lowest physical threshold vanishes. In this case, the Taylor expansion does not work, and one should use a procedure based on [22]. These algorithms for the zero-threshold expansion were described in [25] (see also in [26]) where all possible zero-threshold configurations were considered. Moreover, when there is a large mass parameter, one can also use the large mass expansion [22] to describe the non-zero threshold behaviour at the small (as compared with the large mass scale) non-zero thresholds. For two-loop selfenergies, this procedure was considered in Ref. [27]. One should not put any conditions on the relative values of k^2 and small masses. In this case, two-particle-threshold irregularities are in fact described by the one-loop two-point diagrams.

2. Massive and massless two-loop three-point functions

The problem of calculating the two-loop three-point functions is, in general, more complicated than the two-point case. In particular, there are more external invariants (p_1^2, p_2^2 and p_3^2), and the structure of singularities is more involved. Furthermore, there are two basic topologies, the planar one and the non-planar one, and there is a problem of so-called irreducible scalar numerators.

On one hand, for three-point functions one can still use approaches based on numerical integration of parametric integrals, see e.g. in [28, 10, 11, 7]. The approach used in [11] to calculate the planar vertex function was recently extended to the non-planar case [29].

On the other hand, it is also possible to extend the analytic approach based on expansion of three-point functions in different regions. For example, when all three external momenta squared are small (less than the corresponding thresholds), one can expand the three-point function in a triple Taylor series, using the algorithms similar to ones from [14]. Such an approach was developed in [19] where also conformal mapping and Padé approximations technique were used. An explicit expression for the projectors yielding the coefficients of this triple series was given in [18].

Some special cases of three-point functions involving zero thresholds were considered in [30]. As in the two-point case [25], the general theory of asymptotic expansions [22] was applied. Analogous technique can be also used in the case when all three external momenta squared are above the correspond-

ing thresholds. In particular, one needs the results for the corresponding (in general, off-shell) integrals with massless internal lines².

The exact result for the off-shell massless planar three-point function was obtained in Ref. [32]. In the derivation the “uniqueness” technique [33] was used. This approach was also generalized to the case of ladder diagrams with an arbitrary number of rungs, see in [32, 34]. For the two-loop diagram, the result was presented in terms of polylogarithms up to the fourth order, Li_4 . In Ref. [35] the non-planar diagram was also calculated³, as well as the cases when some of the propagators are shrunk. The result for the non-planar diagram was presented in terms of the square of an expression involving dilogarithms. The problem of irreducible numerators (i.e. scalar numerators which cannot be cancelled against the denominators) was also considered in [35] and some results for the integrals with such numerators were obtained. A complete solution to this problem should also include an efficient algorithm for calculating the integrals with higher integer powers of the propagators and irreducible numerators.

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² Note that the results for the (planar and non-planar) diagrams with two of the three external momenta being on shell (e.g., $p_1^2 = p_2^2 = 0$) were obtained in Refs. [31].

³ We note that in Refs. [11, 29] the results of [32, 35] were used to check the massless limit of the numerical programs.

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