JET CROSS SECTIONS
AT NEXT-TO-LEADING ORDER *

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We briefly summarize theoretical methods for carrying out QCD calculations to next-to-leading order in perturbation theory. In particular, we describe a new general algorithm that can be used for computing arbitrary jet cross sections in arbitrary processes and can be straightforwardly implemented in general-purpose Monte Carlo programs.

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1. Motivations

During the last fifteen years many efforts have been devoted to carry out accurate QCD calculations to higher perturbative orders. These calculations are motivated by three main reasons.

First of all, the comparison between perturbative calculations and experimental data allow one to perform precision tests of QCD in the strong-interaction processes that involve a large transferred momentum Q [1, 2]. These tests are essential for measuring the strong coupling $\alpha_S(Q)$ and its running [3] as predicted by asymptotic freedom. Perturbative QCD studies are also important to evaluate the background for new physics signals. An outstanding example of that is the current investigation [1] of the discrepancy between the single-inclusive jet distribution at large p_t, as measured

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by CDF [4], and the QCD predictions. More recently, a renewed interest in perturbative calculations has been motivated by phenomenological and theoretical models of non-perturbative phenomena (see [5] and references therein). Using these models and having under control the perturbative component, one can use experimental data on high-energy cross sections to extract information on the underlying non-perturbative dynamics.

To these aims, calculations at the leading order (LO) of the perturbative expansion in the QCD coupling $\alpha_S(Q)$ are insufficient. In fact, just because of its perturbative nature, the running of the QCD coupling can be hidden in higher-order corrections. Thus at LO the value of α_S is essentially undetermined and a LO calculation predicts only the order of magnitude of a given cross section and the rough features of a certain observable. The accuracy of the perturbative QCD expansion is instead controlled by the size of the higher-order contributions. Any definite perturbative QCD prediction requires (at least) a next-to-leading order (NLO) calculation.

In general, NLO calculations are highly non-trivial. The first bottleneck one encounters in producing new NLO calculations for a certain process is the evaluation of the relevant matrix elements. In recent years new techniques [6] have been developed to compute QCD Feynman diagrams and most of the one-loop five-point amplitudes are now available [7,8]. However, even when the process-dependent matrix elements are known, there are practical difficulties in setting up a straightforward calculational procedure. The physical origin of these difficulties is in the necessity of factorizing the long- and short-distance components of the scattering processes and is reflected in the perturbative expansion by the presence of divergences. QCD theorems guarantee that these divergences eventually cancel in the evaluation of physical cross sections but do not prevent their appearance in intermediate steps. Since single intermediate expressions are usually divergent, the numerical implementation of NLO calculations forms a second bottleneck.

The main issue one has to face is thus the following. On one side many different NLO calculations (i.e. calculations for different observables) for a certain process and, possibly, for many processes are warranted. On the other side each calculation is very complicated (see also Sect. 2).

In particular, it is very important to reduce the second bottleneck by setting up efficient and simple *methods* for computing *arbitrary* quantities in a single process. It is even more important to have at our disposal simple *algorithms* for computing arbitrary quantities in *arbitrary* processes. The goal is a universal algorithm that, in principle, can be used to construct a general-purpose Monte Carlo program (not a Monte Carlo event generator) for carrying out NLO QCD calculations. Conceptually, such an algorithm could be used in the same manner as some universal Monte Carlo event generators (*e.g.* HERWIG [9]): any time one wants to compute a new quantity

or to vary the experimental cuts, one simply modifies the 'user routine' accordingly; any time one wants to study a different process, one simply enters the corresponding matrix elements.

A new general algorithm of this type was recently presented [10]. It is based on two key ingredients: the *subtraction method* for the numerical cancellation of the divergences among different contributions; and the *dipole factorization theorems* for the universal (process-independent) analytical treatment of individual divergent terms.

In this contribution, after a brief summary of general methods, we describe these two ingredients and show some numerical results for the specific cases of jets in e^+e^- annihilation and deep-inelastic lepton-hadron scattering (DIS).

2. NLO QCD calculations

The general structure of a QCD cross section in NLO is the following

$$\sigma = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} \,. \tag{1}$$

Here the LO cross section σ^{LO} is obtained by integrating the fully exclusive cross section $d\sigma^B$ in the Born approximation over the phase space for the corresponding jet quantity. Let us suppose that this LO calculation involves m partons with momenta p_k $(k = 1, \ldots, m)$ in the final state. Thus, we write

$$\sigma^{\mathrm{LO}} = \int_m d\sigma^{\mathrm{B}} \,, \tag{2}$$

where the Born-level cross section is:

$$d\sigma^{\mathrm{B}} = d\Phi^{(m)}(\{p_k\})|\mathcal{M}_m(\{p_k\})|^2 F_J^{(m)}(\{p_k\}) \,, \tag{3}$$

and $d\Phi^{(m)}$ and $\mathcal{M}_m$ respectively denote the full phase space and the tree-level QCD matrix element to produce m final state partons. These are the factors that depend on the process.

The function $F_J^{(m)}$ defines the physical quantity that we want to compute, possibly including the experimental cuts. Note that this quantity has to be a jet observable, that is, it has to be infrared and collinear safe: its actual value has to be independent of the number of soft and collinear particles in the final state. Thus, we should have (we refer to [10] for a more detailed formal definition)

$$F_J^{(m+1)} \to F_J^{(m)} \,, \tag{4}$$

in any case where the $m + 1$-parton configuration on the left-hand side is obtained from the m-parton configuration on the right-hand side by adding a soft parton or replacing a parton with a pair of collinear partons carrying the same total momentum.

Efficient techniques, based on helicity amplitudes [11] and colour sub-amplitude decomposition [12], are available for calculating tree-level matrix elements. Thus the evaluation of the LO cross section does not present any particular difficulty. Even if σ^{LO} cannot be computed analytically (because $\mathcal{M}_m$ is too cumbersome or the phase-space cuts in $F_J^{(m)}$ are very involved), one can straightforwardly use numerical integration techniques, for instance, a Monte Carlo program where the function $F_J^{(m)}$ is given as 'user routine'.

At NLO one has to consider the exclusive cross section $d\sigma^R$ with $m + 1$ partons in the final state and the one-loop correction $d\sigma^V$ to the process with m partons in the final state:

$$\sigma^{\text{NLO}} = \int\limits_{m+1} d\sigma^R + \int\limits_{m} d\sigma^V . \tag{5}$$

The exclusive cross sections $d\sigma^R$ and $d\sigma^V$ have the same structure as the Born-level cross section in Eq. (3), apart from the replacements $|\mathcal{M}_m|^2 \to |\mathcal{M}_{m+1}|^2$ and $|\mathcal{M}_m|^2 \to |\mathcal{M}_m|^2_{(1-\text{loop})}$. Here $|\mathcal{M}_m|^2_{(1-\text{loop})}$ denotes the QCD amplitude to produce m final state partons evaluated in the one-loop approximation.

The calculation of the loop integral in $|\mathcal{M}_m|^2_{(1-\text{loop})}$ leads to *ultraviolet*, *soft* and *collinear* singularities. The ultraviolet singularities can be handled in a simple way within the loop corrections by carrying out the renormalization procedure. Thus we can assume that the virtual cross section in Eq. (5) is given in terms of the renormalized matrix element and the ultraviolet divergences have been removed.

Soft and collinear singularities instead lead to the main problem. These singularities do not cancel within the sole $d\sigma^V$ and are accompanied by analogous singularities arising from the integration of the real cross section $d\sigma^R$. In the case of jet quantities, adding the real and virtual contribution, these singularities cancel and the physical NLO cross section in Eq. (5) is finite. This cancellation is guaranteed by the property in Eq. (4). However, the cancellation mechanism is not trivial because it does not take place at the integrand level.

The two integrals on the right-hand side of Eq. (5) are separately divergent so that, before any numerical calculation can be attempted, the separate pieces have to be regularized. The most widely used regularization procedure (actually, the only regularization procedure that is gauge invariant and Lorentz invariant to any order of the QCD perturbative expansion)

is obtained by means of analytic continuation in a number of space-time dimensions $d = 4 - 2\varepsilon$ different from four. Using dimensional regularization, the divergences (arising out of the integration) are replaced by double (soft and collinear) poles $1/\varepsilon^2$ and single (soft or collinear) poles $1/\varepsilon$. Thus the real and virtual contributions should be calculated independently, yielding equal-and-opposite poles in ε. These poles have to be combined and, after having achieved their cancellation, the limit $\varepsilon \to 0$ can be safely carried out.

In principle this computation procedure does not pose any problems. In practice, it is not so. On one side, analytic calculations are impossible for all but the simplest quantities because of the involved kinematics for multi-parton configurations and of the complicated phase-space cuts relative to the definition of the jet observable. On the other side, the use of numerical methods is far from trivial because real and virtual contributions have to be integrated *separately* over different phase-space regions and because of the analytic continuation in the arbitrary number d of space-time dimensions.

The most efficient solution to this practical problem consists in using a hybrid analytical/numerical procedure: one must somehow simplify and extract the singular parts of the cross section and treat them analytically; the remainder is treated numerically, independently of the full complications of the jet quantity and of the process.

2.1. General methods and algorithms

There are, broadly speaking, two general methods for doing that: the *phase-space slicing* method and the *subtraction* method. Both the slicing [13] and the subtraction [14] methods were first used in the context of NLO calculations of three-jet cross sections in e^+e^- annihilation. Then they have been applied to other cross sections, adapting the method each time to the particular process. Only recently has it become clear that both methods are generalizable in a process-independent manner. The key observation is that the singular parts of the QCD matrix elements for real emission can be singled out in a general way by using the factorization properties of soft and collinear radiation [15]. Owing to this universality, the two methods have led to general algorithms for NLO QCD calculations.

In the context of the phase-space slicing method, an algorithm has been developed for jet cross sections in lepton and hadron collisions [16, 17]. The generalization of this method to include fragmentation functions and heavy flavours is considered in Refs [18, 19].

As for the subtraction method, two approaches are available for setting up general algorithms. The 'residue approach' introduced in Ref. [20] has been further generalized in Refs [21–23]. The dipole formalism [24] has been completely worked out in Ref. [10].

The advent of these algorithms has made feasible NLO QCD calculations for multi-jet cross sections. Monte Carlo programs have been constructed for most of the physical processes that involve four particles at LO. For five-particle processes, the three-jet cross section in hadron collisions in the simplified case of pure-gluon subprocesses is available [25], as is the four-jet cross section in electron-positron annihilation in the approximation of large number of colours [26]; the full QCD results are expected to appear soon.

We refer to Sect. 12.2 of Ref. [10] for a discussion of the comparison among different general methods for NLO calculations. In the rest of this contribution we describe the approach, based on the dipole formalism.

3. The subtraction method

The general idea of the subtraction method is to use the identity

$$\sigma^{\mathrm{NLO}} = \int\limits_{m+1} \left[d\sigma^R - d\sigma^A \right] + \int\limits_{m+1} d\sigma^A + \int\limits_m d\sigma^V \,, \qquad (6)$$

which is obtained by subtracting and adding back the same quantity $d\sigma^A$. The cross section contribution $d\sigma^A$ has to fulfil two main properties.

$i)$ Firstly, it must be a proper approximation of $d\sigma^R$ such as to have the same *pointwise* singular behaviour (in d dimensions) as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a *local* counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$ under the integral sign in the first term on the right-hand side of Eq. (6). This defines a cross section contribution $\sigma^{\mathrm{NLO}\,\{m+1\}}$ with $m + 1$-parton kinematics that can be integrated numerically in four dimensions:

$$\sigma^{\mathrm{NLO}\,\{m+1\}} = \int\limits_{m+1} \left[\left(d\sigma^R \right)_{\varepsilon=0} - \left(d\sigma^A \right)_{\varepsilon=0} \right] \,. \qquad (7)$$

$ii)$ The second property of $d\sigma^A$ is its analytic integrability (in d dimensions) over the one-parton subspace leading to the soft and collinear divergences. In this case, we can rewrite the last two terms on the right-hand side of Eq. (6) as follows

$$\sigma^{\mathrm{NLO}\,\{m\}} = \int\limits_m \left[d\sigma^V + \int\limits_1 d\sigma^A \right]_{\varepsilon=0} \,. \qquad (8)$$

Performing the analytic integration $\int_1 d\sigma^A$, one obtains ε-pole contributions that can be combined with those in $d\sigma^V$, thus cancelling all the divergences. The remainder is finite in the limit $\varepsilon \to 0$ and thus defines the integrand of

a cross section contribution $\sigma^{\mathrm{NLO}\,\{m\}}$ with m-parton kinematics that can be integrated numerically in four dimensions.

The final structure of the NLO calculation is as follows

$$\sigma^{\mathrm{NLO}} = \sigma^{\mathrm{NLO}\,\{m+1\}} + \sigma^{\mathrm{NLO}\,\{m\}}\,, \tag{9}$$

and can be easily implemented in a 'partonic Monte Carlo' program, which generates appropriately weighted events with $m+1$ and m final state partons.

Note that, using the subtraction method, no approximation is actually performed in the evaluation of the NLO cross section. Rather than approximating the cross section, the subtracted contribution $d\sigma^A$ defines a fake cross section that has the same dynamical singularities as the real one and whose kinematics are sufficiently simple to permit its analytic integration.

The real cross section contribution $d\sigma^R$ has the following general structure

$$d\sigma^R = d\Phi^{(m+1)}|\mathcal{M}_{m+1}(\{p_k\})|^2 F_J^{(m+1)}(\{p_k\})\,, \tag{10}$$

where $d\Phi^{(m+1)}$ and $|\mathcal{M}_{m+1}|^2$ depend on the process and $F_J^{(m+1)}$ depends on the quantity we want to compute. Obviously, for any given $d\sigma^R$ one can try to construct a corresponding $d\sigma^A$ by properly approximating $d\Phi^{(m+1)}$, $|\mathcal{M}_{m+1}|^2$ and $F_J^{(m+1)}$. It is less obvious that one can use the subtraction method to compute *arbitrary* quantities in a given process, because one needs a fake cross section $d\sigma^A$ that depends only on the process and, hence, is independent of the actual definition of the jet function $F_J^{(m+1)}$. It is still less obvious that one can use the subtraction method to construct a universal algorithm for computing arbitrary quantities in *arbitrary* processes. To this purpose the fake cross section $d\sigma^A$ also has to be somehow independent of $\mathcal{M}_{m+1}$.

Our method to achieve this generality is based on the dipole formalism.

4. Dipole formalism and universal subtraction term

4.1. Soft and collinear limits

The starting point of the dipole formalism are the soft and collinear factorization theorems for the QCD matrix elements. According to these theorems, the singular behaviour in d dimensions of a generic tree-level matrix element $\mathcal{M}_{m+1}(p_1,\ldots,p_{m+1})$ with $m+1$ final state partons can be obtained by means of factorized limiting formulae that, respectively in the soft (when the parton momentum p_j vanishes) and collinear (when the parton momenta p_i and p_j become parallel) regions, have the following structure

$$|\mathcal{M}_{m+1}(p_1,\ldots,p_j,\ldots,p_{m+1})|^2 \to |\mathcal{M}_m(p_1,\ldots,p_{m+1})|^2 \otimes_c J^2(p_j)\,, \tag{11}$$

$$|\mathcal{M}_{m+1}(p_1,\ldots,p_j,p_i,\ldots,p_{m+1})|^2 \to |\mathcal{M}_m(p_1,\ldots,p_j+p_i,\ldots,p_{m+1})|^2 \otimes_h P_{ij}\,.$$
$$(12)$$

The notation in Eqs (11), (12) is symbolic (see Ref. [10] for more details) but sufficient to recall their main features.

The contributions $\mathcal{M}_m$ on the right-hand sides are the tree-level matrix elements to produce m partons and are respectively obtained from the original $m+1$-parton matrix element by removing the soft parton p_j or combining the two collinear partons p_j and p_i into a single-parton momentum.

The other contributions on the right-hand sides are responsible for the soft and collinear divergences. The factor $\boldsymbol{J}^2(p_j)$ in Eq. (11) is the eikonal current for the emission of the soft gluon p_j, and P_{ij} is the Altarelli–Parisi splitting function. These factors are *universal*: they do not depend on the process but only on the momenta and quantum numbers of the QCD partons in $\mathcal{M}_m$. In particular, $\boldsymbol{J}^2(p_j)$ depends on the colour charges of the partons in $\mathcal{M}_m$, and P_{ij} depends on their helicities. Because of these colour and helicity correlations (symbolically denoted by $\otimes_c$ and $\otimes_h$), Eqs (11), (12) are not real factorized expressions. Moreover, there is another important reason, due to kinematics, why Eqs (11), (12) cannot be regarded as true factorization formulae but rather as limiting formulae. Indeed, the tree-level matrix elements in Eqs (11), (12) are unambiguously defined only when momentum conservation is fulfilled exactly. Since, in general, the $m+1$- parton phase space does not factorize into an m-parton times a single-parton phase space, the right-hand sides of these equations are unequivocally defined only in the strict soft and collinear limits.

Owing to their universality, the limiting formulae (11), (12) can be used to approximate the matrix element $|\mathcal{M}_{m+1}|^2$ in Eq. (10) and thus to find a fake cross section $d\sigma^A$ that matches the real cross section $d\sigma^R$ in all the singular regions of phase space. However, the implementation of Eqs (11), (12) in the calculation of QCD cross sections requires a careful treatment of momentum conservation away from the soft and collinear limits. Care also has to be taken to avoid double counting the soft and collinear divergences in their overlapping region (*e.g.* when p_j is both soft and collinear to p_i). The use of the dipole factorization theorem introduced in Ref. [24] allows one to overcome these difficulties in a straightforward way.

4.2. Dipole formulae

The dipole factorization formulae have the following symbolic structure

$$|\mathcal{M}_{m+1}(p_1,\ldots,p_{m+1})|^2 = |\mathcal{M}_m(\tilde{p}_1,\ldots,\tilde{p}_m)|^2 \otimes \boldsymbol{V}_{ij} + \ldots. \qquad (13)$$

The dots on the right-hand side stand for contributions that are not singular when $p_i \cdot p_j \to 0$. The dipole splitting functions $\boldsymbol{V}_{ij}$ are universal (process-

independent) singular factors that depend on the momenta and quantum numbers of the m partons in the tree-level matrix element $|\mathcal{M}_m|^2$. Colour and helicity correlations are denoted by the symbol $\otimes$. The set $\tilde{p}_1, \ldots, \tilde{p}_m$ of modified momenta on the right-hand side of Eq. (13) is defined starting from the original $m + 1$ parton momenta in such a way that the m partons in $|\mathcal{M}_m|^2$ are physical, that is, they are on-shell and energy-momentum conservation is implemented exactly:

$$\tilde{p}_i^2 = 0, \quad \tilde{p}_1 + \ldots + \tilde{p}_m = p_1 + \ldots + p_{m+1} \, . \tag{14}$$

The detailed expressions for these parton momenta and for the dipole splitting functions are given in Ref. [10].

Apart from the presence of colour and helicity correlations, Eq. (13) can be considered as a true factorization formula because its left-hand and right-hand sides live on the same phase-space manifold. Equation (14) indeed guarantees that exact kinematics are retained in the definition of the m-parton configuration $\{\tilde{p}_1, \ldots, \tilde{p}_m\}$. These m parton momenta depend on p_i and p_j in such a way that in the soft and collinear regions the m-parton configuration become indistinguishable from the original $m + 1$-parton configuration. Correspondingly, the dipole splitting function $\boldsymbol{V}_{ij}$ is defined in order to coincide with the eikonal current and with the Altarelli–Parisi splitting function respectively in the soft and collinear limits.

It follows that Eq. (13) provides a *single* formula that approximates the real matrix element $|\mathcal{M}_{m+1}|^2$ for an arbitrary process, in *all* of its singular limits. These limits are approached smoothly, thus avoiding double counting of overlapping soft and collinear singularities. The exact implementation of momentum conservation makes possible this smooth transition and the extrapolation of the limiting formulae (11), (12) away from the soft and collinear regions.

4.3. Universal subtraction term

These main features of the dipole formulae allow us to construct a universal subtraction term with the following form

$$d\sigma^A = d\Phi^{(m+1)} \sum_{ij} |\mathcal{M}_m(\{\tilde{p}_k\})|^2 \otimes \boldsymbol{V}_{ij} F_J^{(m)}(\{\tilde{p}_k\}) \, . \tag{15}$$

Note that the only dependence on the jet observable is in the jet-defining function $F_J^{(m)}$ and the only dependence on the process is in the tree-level matrix element $|\mathcal{M}_m|^2$. These are the same m-parton functions as enter in the calculation of the Born-level cross section of Eq. (3). The only other ingredients needed to construct $d\sigma^A$ are the dipole splitting functions, which

are completely process-independent and given once and for all [10]. This specifies the universal character of Eq. (15): the fake cross section $d\sigma^A$ used for the NLO calculation is straightforwardly obtained in terms of the sole (process-dependent) information that is necessary for the corresponding LO calculation.

Having the subtraction term in the explicit form (15), we can discuss how it fulfils the properties i) and ii) listed in Sect. 3. As for the property i), we note that there are several dipole terms on the right-hand side of Eq. (15). Each of them mimics one of the $m + 1$-parton configurations in $d\sigma^R$ that are kinematically degenerate with a given m-parton state. Any time the $m + 1$-parton state in $d\sigma^R$ approaches a soft and/or collinear region, there is a corresponding dipole factor in $d\sigma^A$ that approaches the same region with exactly the same probability as in $d\sigma^R$. The equality of the two probabilities directly follows from (15) and from the limiting behaviour in Eqs (4), (11), (12) of the cross section factors on the right-hand side of Eq. (10). In this manner $d\sigma^A$ acts as a local counterterm for $d\sigma^R$. Note, in particular, that the cancellation mechanism is completely independent of the actual form of the jet-defining function and works for any jet observable (i.e. for any quantity that fulfils Eq. (4)).

As for the property ii), we start by noting that $d\sigma^A$ (likewise $d\sigma^R$) depends on the $m + 1$ parton momenta $p_1, \ldots, p_{m+1}$. However, having introduced the modified momenta $\widetilde{p}_1, \ldots, \widetilde{p}_m$, for each dipole term in Eq. (15) we can define a *one-to-one* mapping

$$\{p_1, \ldots, p_{m+1}\} \leftrightarrow \{\widetilde{p}_1, \ldots, \widetilde{p}_m, p_i + p_j\}. \tag{16}$$

The key feature of this mapping is that the m modified momenta can be chosen in such a way that they obey *exact* phase-space factorization as follows

$$d\Phi^{(m+1)}(p_1, \ldots, p_{m+1}) = d\Phi^{(m)}(\widetilde{p}_1, \ldots, \widetilde{p}_m) d\varphi_{(\{\widetilde{p}_k\})}(p_i + p_j), \tag{17}$$

where $d\varphi$ is a single-particle subspace that, for fixed $\widetilde{p}_1, \ldots, \widetilde{p}_m$, depends only on the dipole momenta p_i and p_j [10]. Owing to the exact phase-space factorization and to the fact that the fake cross section in Eq. (15) is proportional to the jet quantity calculated from the modified m-parton configuration, the integration of the singular dipole contributions can be completely factorized (modulo colour and helicity correlations) with respect to a term that exactly reproduces the Born-level cross section:

$$\int\limits_{m+1} d\sigma^A = \int\limits_m d\Phi^{(m)}(\{\widetilde{p}_k\}) |\mathcal{M}_m(\{\widetilde{p}_k\})|^2 F_J^{(m)}(\{\widetilde{p}_k\})$$

$$\otimes \sum_{ij} \int\limits_1 d\varphi_{(\{\widetilde{p}_k\})}(p_i + p_j) \boldsymbol{V}_{ij} = \int\limits_m d\sigma^B \otimes \boldsymbol{I}(\{\widetilde{p}_k\}). \tag{18}$$

The last factor on the right-hand side of Eq. (18) is defined by

$$I(\{\widetilde{p}_k\}) \equiv \sum_{ij} \int_1 d\varphi_{(\{\widetilde{p}_k\})}(p_i + p_j)\mathbf{V}_{ij}. \tag{19}$$

and contains all the soft and collinear singularities that are necessary to compensate those in the virtual cross section $d\sigma^V$. Owing to the convenient definition of the dipole splitting function $\mathbf{V}_{ij}$, it is possible to carry out analytically the integration in Eq. (19).over the dipole phase space in d dimensions. This leads to an explicit and universal expression [10] for the factor I, whose ε-poles cancel those in the one-loop matrix element.

5. Final results and numerical implementation

The discussion in the previous section shows that, by using the subtraction method and the dipole formulae, one can extract and treat analytically the singular parts of any NLO cross section in a way that is independent of the exact details of the observable and of the process. This leaves a remainder that depends on the full complications of the jet quantity, but which is finite so that it can be treated either numerically or analytically (whenever possible).

In general, the use of numerical integration techniques (typically, Monte Carlo methods) is certainly more convenient. First of all, the numerical approach allows one to calculate any number and any type of observable simultaneously by simply histogramming the appropriate quantities, rather than having to make a separate analytic calculation for each observable. Furthermore, using the numerical approach, it is easy to implement different experimental conditions, for example detector acceptances and experimental cuts.

In order to summarize the final results of our algorithm and to describe their numerical implementation, we start by recalling how the LO cross section in Eq. (2) is evaluated by using a Monte Carlo program. One first generates an m-parton event in the phase-space region $d\Phi^{(m)}$ and gives it the weight $|\mathcal{M}_m|^2$. Then this weighted event is analysed by a user routine according to the actual definition of the phase-space function $F_J^{(m)}$ and inserted into a corresponding histogram bin.

Following the decomposition in Eq. (9), the NLO cross section is obtained by adding two contributions (which are not necessarily positive definite) with m-parton (as in the LO calculation) and $m + 1$-parton kinematics, respectively. Unlike the original real and virtual contributions, these two terms are separately finite and can directly be integrated in four space-time dimensions.

5.1. The term with m-parton kinematics

The first contribution is obtained by inserting Eq. (18) into Eq. (8) and can be written as follows

$$\sigma^{\mathrm{NLO}\,\{m\}} = \int_m d\Phi^{(m)} F_J^{(m)}(\{p_k\}) \mathcal{F}_m(\{p_k\})\,, \tag{20}$$

where the *master function* $\mathcal{F}_m(\{p_k\})$ is explicitly given by

$$\mathcal{F}_m(\{p_k\}) = \left\{ |\mathcal{M}_m(\{p_k\})|^2_{(1-\mathrm{loop})} + |\mathcal{M}_m(\{p_k\})|^2 \otimes \boldsymbol{I}(\{p_k\}) \right\}_{\varepsilon=0}. \tag{21}$$

The first term in the curly bracket is the one-loop *renormalized* matrix element for producing m final state partons. The second term is obtained by combining the tree-level matrix element to produce m partons and the universal factor $\boldsymbol{I}$ in Eq. (19). These two terms are defined in $d = 4 - 2\varepsilon$ dimensions. Owing to the progress made in recent years in the analytical techniques for evaluating one-loop amplitudes [6], many of them have been calculated. The explicit expression of the universal factor $\boldsymbol{I}$ is provided by our algorithm. Thus, one has to carry out the expansion in ε-poles of the two terms in the curly bracket, cancel analytically (by trivial addition) the poles and perform the limit $\varepsilon \to 0$. This simple algebraic manipulation is sufficient to construct an effective m-parton weight, the master function $\mathcal{F}$, that is finite. As a result, Eq. (20) can be handled by the Monte Carlo program exactly in the same way as the LO cross section.

Note that the two terms on the right-hand side of Eq. (21) separately depend on the regularization prescription of the soft and collinear divergences, namely dimensional regularization. Since different versions of dimensional regularization can be used to compute the one-loop matrix element, the second term in the curly bracket has to be evaluated accordingly. Alternatively, one can fix the latter and use the transition rules derived in Ref. [27] to relate the one-loop amplitudes in different dimensional-regularization schemes.

The necessity to consistently regularize the separately divergent components of the cross section is a common feature of any NLO calculation, independently of the method that is actually used in the computation. Failure in the consistent implementation of the regularization procedure leads to violation of unitarity and, ultimately, to an incorrect (although possibly finite) final result. The dipole formalism is extremely efficient to guarantee unitarity because all the divergences are isolated in the right-hand side of Eq. (21). As explained in Ref. [28], for any regularization prescription that is unambiguously defined at the level of one-loop matrix elements, one can compute in a simple and consistent way the universal factor $\boldsymbol{I}$ that provides the finite and unitary master function $\mathcal{F}$.

5.2. The term with $m + 1$-parton kinematics

The NLO contribution with $m + 1$-parton kinematics, which is obtained by subtracting the fake cross section in Eq. (15) from the real cross section in Eq. (10), has the following explicit expression:

$$
\sigma^{NLO\{m+1\}} = \int\limits_{m+1} d\Phi^{(m+1)}
$$

$$
\times \left\{ |\mathcal{M}_{m+1}(\{p_k\})|^2 F_J^{(m+1)}(\{p_k\}) - \sum_{ij} |\mathcal{M}_m(\{\tilde{p}_k\})|^2 \otimes \boldsymbol{V}_{ij} F_J^{(m)}(\{\tilde{p}_k\}) \right\}.
$$

$$
(22)
$$

The terms in the curly bracket define an effective matrix element that is integrable in four space-time dimensions. It follows that the NLO matrix element $\mathcal{M}_{m+1}$, with $m + 1$ final state partons, can be directly evaluated in $d = 4$ dimensions, thus leading to an extreme simplification of the Lorentz algebra. Knowing the tree-level matrix elements and the dipole splitting functions, the Monte Carlo integration of Eq. (22) is straightforward. One simply generates an $m+1$-parton configuration and uses it to define an event with positive weight $+|\mathcal{M}_{m+1}|^2$ and several counter-events, each of them with the negative weight $-|\mathcal{M}_m|^2 \otimes \boldsymbol{V}_{ij}$. Then these event and counter-events are analysed by the user routine. The role of the two different jet functions $F_J^{(m+1)}$ and $F_J^{(m)}$ is that of binning the weighted event and counter-events into different bins of the jet observable. Any time that the generated $m + 1$-parton configuration approaches a singular region, the event and one counter-event fall into the same bin and the cancellation of the large positive and negative weights takes place.

6. Monte Carlo programs

Generalizing the procedure for constructing NLO Monte Carlo programs for arbitrary quantities has several advantages. These are principally due to the reduction in the number and complexity of ingredients that have to be calculated for each new process, and because the d-dimensional integrals only need be done once and can be easily checked independently, rather than being buried inside a specific calculation.

Using the general algorithm described in this contribution, we have already constructed two Monte Carlo programs (they can be obtained from http://surya11.cern.ch/users/seymour/nlo/), EVENT2 and DISENT.

EVENT2 [24] computes three-jet observables in e^+e^- annihilation. In the case of un-oriented three-jet events, this program is comparable and in

agreement with the program EVENT [29], which is based on the subtraction procedure of Ref. [14] and has been used for most of the QCD analyses at LEP and SLC [2,3]. As an example we show the NLO coefficients for the thrust and C-parameter distributions in Fig. 1. We find that, in general, the numerical convergence of EVENT2 is similar to the program of Ref. [29], except close to the two-jet region in which ours becomes progressively better. In the case of oriented events [30], EVENT2 should be compared with a corresponding program, EERAD [16], based on the phase-space slicing method.
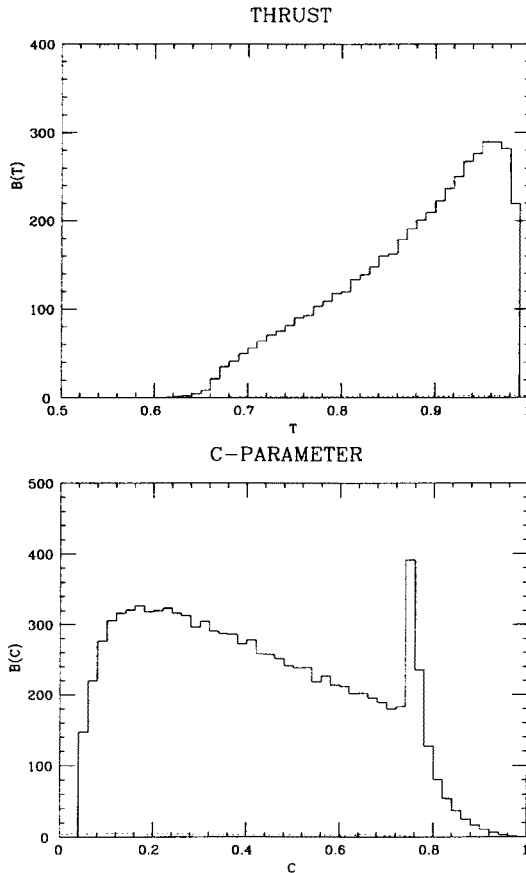
THRUST

C—PARAMETER

Fig. 1. Coefficient of $(\alpha_S/2\pi)^2$ for the thrust and C-parameter distributions. The dotted histograms show the size of the statistical errors.

DISENT [10,31] is a NLO program for $2+1$-jet observables in DIS. The program uses the matrix elements evaluated by the Leiden group [32]. In Fig. 2a we show as an example the differential jet rate as a function of jet resolution parameter f_{cut}, using the $k_\perp$ jet algorithm [33] at HERA ener-

gies [34]. We see that the NLO corrections are generally small and positive, except at very small f_{cut} (where large logarithmic terms, $-\alpha_s \log^2 f_{cut}$, arise at each higher order). In Fig. 2b, we show the variation of the jet rate at a fixed f_{cut} with factorization and renormalization scales. The scale dependence is considerably smaller at NLO. DISENT can be compared with the Monte Carlo MEPJET [35] that uses the phase-space slicing algorithm of Ref. [17].
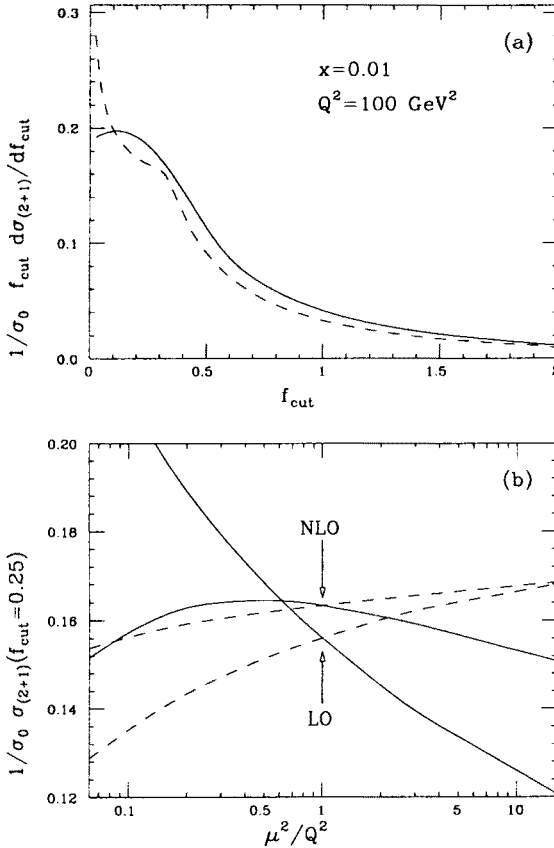
Fig. 2. Jet cross sections in ep collisions at HERA energies ($\sqrt{s} = 300$ GeV). (a) The distribution of resolution parameter f_{cut} at which DIS events are resolved into $(2 + 1)$ jets according to the $k_\perp$ jet algorithm. Curves are LO (dashed) and NLO (solid) using factorization and renormalization scales equal to Q^2, and the MRSD$'_-$ distribution functions. Both curves are normalized to the LO cross section. (b) The rate of events with exactly $(2 + 1)$ jets at $f_{cut} = 0.25$ with variation of renormalization (solid) and factorization (dashed) scales. Normalization is again the LO cross section with fixed factorization scale.

The results of the algorithm based on the dipole formalism have also been implemented in a program [36] for the calculation of NLO QCD corrections to four-fermion final states in e^+e^- annihilation.

7. Summary and outlook

The calculation of jet cross sections in perturbative QCD requires the integration of multiparton matrix elements over complicated phase-space regions that depend on the actual definition of the jet observables and on the experimental cuts. In general, these phase-space integrations can be carried out only by using numerical methods. Beyond LO, however, numerical techniques cannot straightforwardly be applied because real-emission contributions and virtual contributions are separately divergent. These divergences have to be first regularized, then evaluated analytically, combined together and cancelled before any numerical calculation can be attempted.

General methods are now available to overcome all the analytical difficulties related to the treatment of soft and collinear divergences in NLO calculations. In this contribution we have mainly described one of these general formalisms, which has been used to set up an explicit algorithm to compute NLO jet cross sections.

The algorithm combines the subtraction method and the dipole formulae to carry out all the analytical work that is necessary to evaluate and cancel the singularities. The final output of the algorithm is given in terms of effective matrix elements that can be automatically constructed starting from the original (process-dependent) matrix elements and universal (process-independent) dipole factors. The effective matrix elements can be integrated numerically or analytically (whenever possible) over the available phase space in four dimensions to compute the actual value of the NLO cross section. If the numerical approach is chosen, Monte Carlo integration techniques can be easily implemented to provide a general-purpose Monte Carlo program for carrying out NLO QCD calculations in any given process.

The simplified discussion of the algorithm presented in this contribution directly applies to processes, like $e^+e^- \to n$ jets, in which there are neither initial-state hadrons nor identified hadrons in the final state. However, the formalism and the algorithm are completely general in the sense that they apply to *any* jet observable in a given scattering process as well as to *any* hard-scattering process. Full details and explicit results for lepton-hadron and hadron-hadron collisions and for fragmentation processes are given in Ref. [10].

At present, next-to-next-to-leading order (NNLO) QCD calculations are feasible only for some fully inclusive quantities [37]. In these cases one considers all possible final states and integrates the QCD matrix elements over

the whole final state phase space. Thus one can add real and virtual contributions before performing the relevant momentum integrations in such a way that only ultraviolet singularities appear at the intermediate steps of the calculation. In the case of less inclusive jet observables, one cannot take advantage of the cancellation of soft and collinear divergences at the integrand level and, at present, no systematic method is available to handle these divergences at NNLO. Even once the necessary two-loop matrix elements for several processes are calculated, the amount of work needed to provide a numerical implementation will be enormous. The main features of the dipole formalism, which permit a universal treatment of soft and collinear singularities at NLO, seem particularly suited to set up a general method for carrying out NNLO QCD calculations.

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