THE ELECTRON $(g_e - 2)$ AND THE VALUE OF α: A CHECK OF QED AT 1ppb *

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The current experimental error in the electron (g_e-2), 4 ppb, is now bigger than corresponding theoretical error, which is 1 ppb. We review shortly the latest theoretical results, concentrating mainly in the analytical calculation of the QED 3-loop contributions, and then we discuss the possibility of a 1% numerical calculation of the 4-loop contributions, aiming at a final theoretical error less than 1ppb.

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1. The current best experimental value of the electron (g_e-2) is [1]

$$a_e(\exp) = 1\,159\,652\,188.4\ (4.3)\ \times 10^{-12}\ (4\text{ppb}). \tag{1}$$

Two new experiments are in progress and wishing the best success to our experimentalist colleagues we expect that new and very accurate values, with an absolute error below 10^{-12}, *i.e.* a relative error below 1 ppb (1 part in a billion), will be soon available.
As it will be shown in a moment, for a meaningful check of the QED theoretical prediction for the electron (g_e-2) one needs also an equally accurate value of the fine structure constant α. The best current solid-state, 'non QED' value is [2]

$$\alpha^{-1}(\text{NIST}) = 137.036\,0037(33)\ (24\text{ppb})\ , \tag{2}$$

with an error which is still more than an order of magnitude higher than the aimed precision. Other 'non solid-state' measurements are in progress

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[3], we can just hope that in due time they will also provide us with a 1ppb value of α.

Section 2. Let us write the theoretical value of the electron anomaly as

$$a_e(\mathrm{th}) = c_1\left(\frac{\alpha}{\pi}\right) + c_2\left(\frac{\alpha}{\pi}\right)^2 + c_3\left(\frac{\alpha}{\pi}\right)^3 + c_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots + \Delta a_e(\mathrm{th}) \ , \tag{3}$$

where the first line is the perturbative expansion in α/π of the QED contributions involving only the electron in the intermediate states, all the other contributions being contained in $\Delta a_e(\mathrm{th})$. We write in turn $\Delta a_e(\mathrm{th})$ in the form

$$\Delta a_e(\mathrm{th}) = a_e(\mu) + a_e(\tau) + a_e(\mathrm{w}) + a_e(\mathrm{h}) \ , \tag{4}$$

where $a_e(\mu), a_e(\tau)$ are the QED contributions due to internal μ and τ-lepton loops, $a_e(\mathrm{w})$ is the contribution of the weak interactions and $a_e(\mathrm{h})$ the hadronic contribution. $a_e(\mu)$ is small; even at 1ppb level it is sufficient to consider only the leading term in α and m_e^2/m_μ^2 [4],

$$a_e(\mu) = \frac{1}{45}\frac{m_e^2}{m_\mu^2}\left(\frac{\alpha}{\pi}\right)^2 \simeq 2.80 \times 10^{-12} \ . \tag{5}$$

As, in general, any contribution due to a large internal mass behaves as the ratio of the squares of the electron mass and of the large mass, $a_e(\tau)$ and $a_e(\mathrm{w})$, which can be both evaluated within the standard electroweak model, are negligibly small at this level. $a_e(\mathrm{h})$, on the contrary, can be evaluated only phenomenologically, by using the experimental e^+e^- annihilation data; at present, its best value is (from Ref.[5]; systematical error only)

$$a_e(\mathrm{h}) = (1.8847 \pm 0.0375) \ \times \ 10^{-12} \ , \tag{6}$$

again a small value, but significant at 1ppb level. It may be important to stress that the error in the prediction of the hadronic contribution will be the ultimate error in the theoretical prediction of the electron (g_e-2); that error (less than 0.04 ppb!) is however much smaller than the experimental and theoretical precision attainable in the foreseeable future. (The situation is different in the case of the (g-2) of the μ, were the hadronic error is the main source of error in the theoretical prediction.)

Summarizing, we can say that $\Delta a_e(\mathrm{th})$ is known,

$$\Delta a_e(\mathrm{th}) = 4.7 \ \times \ 10^{-12} \ , \tag{7}$$

with an error less than 0.1×10^{-12}, *i.e.* safely below the aimed 1ppb level.

Coming back to "electron-only-QED" part, the leading contribution in α is the Schwinger [6] term

$$\frac{1}{2}\left(\frac{\alpha}{\pi}\right) . \tag{8}$$

The leading term induces in the final value of $a_e(\text{th})$ the error corresponding to the experimental error in α (as α is small, $\alpha \simeq 1/137$, the experimental error in α does not show up in the higher powers of α); hence the already mentioned need for an independent, non-QED experimental value of α with a 1ppb (or less) relative error.

Also c_2 is known exactly since a long time [7]

$$\begin{aligned} c_2 &= \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) \\ &= -0.328\ 478\ 965\ 579\ \ldots , \end{aligned} \tag{9}$$

where $\zeta(3) = 1.202\ 056\ 903\ 159\ 59\ldots$ is the $p = 3$ value of the Riemann ζ-function

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}. \tag{10}$$

(For even p the constants $\zeta(p)$ can be expressed as powers of π, such as $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$.)

We have recently completed [8] (for previous results, see [9–11])the analytical calculation of c_3, obtaining

$$\begin{aligned} c_3 = &\ \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) \\ &+ \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right] \\ &- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) \\ &- \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} \\ = &\ 1.181\ 241\ 456\ \ldots , \end{aligned} \tag{11}$$

where $\zeta(5) = 1.036\ 927\ 755\ 143\ 37\ldots$ and $a_4 = 0.517\ 479\ 061\ 673\ 899\ \ldots$ is the $p = 4$ value of

$$a_p = \sum_{n=1}^{\infty} \frac{1}{2^n n^p} . \tag{12}$$

Thanks to Eq. (11) the first 3 coefficients of the expansion in α of Eq. (3) are by now all exactly known; the theoretical error is due to the next coefficient, c_4, of which only an approximate numerical value exists, thanks to the tireless work of Kinoshita. The latest value for the 4-loop contribution is [12]

$$c_4 = -1.4092(384) ,$$

with a twofold precision improvement with the respect to the previous value [13]

$$c_4 = -1.557\ (70) . \tag{13}$$

By inserting the above results and the value of α given by Eq. (2) into Eq. (3) one finds

$$a_e(\mathrm{th}) = 1\ 159\ 652\ 156.7\ (1.1)\ (28)\ \times\ 10^{-12} , \tag{14}$$

where the first error, (1.1), is due to the error in c_4 Eq. (13), while the second (and by far dominant) error comes from the experimental error in α, Eq. (2). The difference between the theoretical and experimental values is

$$a_e(\mathrm{th}) - a_e(\mathrm{exp}) = 32\ (28) \times\ 10^{-12} ; \tag{15}$$

the agreement is reasonably good — but is absolutely obvious that a more precise value of α would give a much more stringent test.

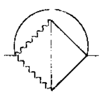
Let us (optimistically) assume that the experiments will soon reach — and overcome — the 1 ppb wall; how much work will then be needed on the theoretical side to catch up? As (fortunately) the 5-loop $\left(\frac{\alpha}{\pi}\right)^5$ contribution is still negligible, even at that level, one is confronted with the problem of reducing by a factor 5-10 the error in c_4, down to, say, an 1% relative error. A fully analytical calculation is at the moment out of question; the numerical approach will surely be continued to give better and better results, but the difficulties cannot be underestimated (one has to deal with ill-behaved integrals in many dimensions, the proper numerical treatment of spikes and end-point singularities of the integrands is very delicate), and in any case the problem of an independent calculation, to provide with a cross check, would remain.

In the next section we will shortly review the techniques which lead to the completion of the 3-loop analytical calculation; in Section 4 we will argue that some of those techniques might be hopefully extended to give a good numerical value for the 4-loop contribution as well.

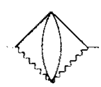
3. The value of c_3, Eq. (11), was obtained by a suitable combination of analytical and algebraic methods. We can just sketch here the main steps of the analytical method (some more details can be found in [8], [14]; see also section 6.5 of [11]): given any scalar 3-loop integral contributing to (g_e-2), we use a suitable combination of hyperspherical variables for the 'outer' loop (or loops) and of a dispersive representation for the 'inner' loops (or loop), obtaining a 'standard integral' in just 4 variables, whose integrand is a logarithm; by judiciously reshuffling the square roots of the integrand and properly ordering the integration variables, the whole expression is seen to consist of integrals over Nielsen polylogarithms, to be processed with by now well established methods. Those techniques apply, at least in principle, to any of the scalar integrals which are present in the contributions of the various Feynman graphs; the direct analytical calculation, one by one, of all the scalar integrals is doable in practice (and in fact all the 3-loop graphs, with the exceptions of the very last, the triple-cross graphs, were evaluated in that way), but turns out to be extremely long and laborious. There are in fact several hundred scalar integrals for each graph and, what is worse, the many scalar integrals with polynomials of the momenta in the numerator give rise to an enormous number of intermediate terms, whose processing requires a substantial amount of (human) direct inspection - besides a lot of more mechanical algebraic elaboration (carried out of course by a computer program; we take here the occasion of acknowledging M. Veltman, M.J. Levine and J. Vermaseren for kindly helping us, through the years, in using their programs `SCHOONSCHIP`[15], `ASHMEDAI`[16] and `FORM`[17];
The last class of 3-loop graphs, the triple-cross graphs, has been completed by a different technique, *i.e.* by heavily using the by now well known integration by parts method [18]. The method, in its essence, generates a very large number of identities between the scalar amplitudes occurring in the Feynman graphs — or rather between their extension from 4 to continuous n dimension. It is extremely long and machine-time consuming, but in principle straightforward, to use those identities for expressing the "most complicated" integral of each identity in terms of the simpler ones occurring in the same identity. There are many integrals (up to a thousand) and even more identities (several thousand) to work out; the systematical exploitation of all the identities permits to express all the occurring scalar integrals as linear combinations of a surprisingly small number of basic (or master) integrals, whose coefficients are ratios of polynomials in n. As one is ultimately interested in the $n = 4$ values of the integrals, we found it convenient to expand everything in powers of $n - 4$, but even the proper bookkeeping of the expansion in $n - 4$ turns out to be rather delicate, and cannot be carried without some care.
The actual choice of the master integrals is to a large extent arbitrary; for

classification purposes, it may be convenient to consider as simpler the 3-loop integrals having less denominators, smaller powers in the denominator and finally smaller powers in the numerator. Following that criterion, we found that all the integrals occurring in the contribution of the triple-cross graphs can be expressed in terms of 17 basic (or master) integrals only; they are listed below, each integral being preceded by an icon describing its topology:

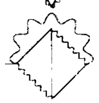
$$J_1 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{p \cdot k_2}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8} \, ,$$



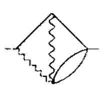
$$J_2 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_1 D_2 D_3 D_4 D_7 D_8} \, ,$$



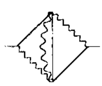
$$J_3 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_1 D_2 D_4 D_5 D_6 D_8} \, ,$$



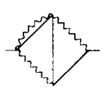
$$J_4 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_2 D_3 D_4 D_6 D_7 D_8} \, ,$$



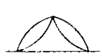
$$J_5 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_1 D_3 D_4 D_5 D_7 D_8} \, ,$$



$$J_6 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_1 D_3 D_5 D_6 D_7 D_8} \, ,$$



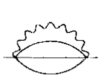
$$J_7 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_2 D_4 D_5 D_6 D_7 D_8} \, ,$$



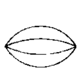
$$J_8 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_1 D_2 D_3 D_4 D_5} \, ,$$

$$J_9 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_2 D_3 D_5 D_6 D_7} \, ,$$

$$J_{10} = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_2 D_4 D_6 D_7 D_8} \, ,$$

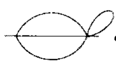


$$J_{11} = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_1 D_3 D_5 D_7} \, ,$$

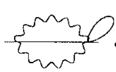


$$J_{12} = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_1 D_2 D_4 D_5} \, ,$$

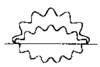
$$J_{13} = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_3 D_5 D_6 D_7} \, ,$$



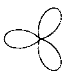
$$J_{14} = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_2 D_3 D_4 D_5} \, ,$$



$$J_{15} = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_3 D_4 D_7 D_8} \, ,$$



$$J_{16} = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_3 D_6 D_7 D_8} \, ,$$



$$J_{17} = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int d^n k_1 d^n k_2 d^n k_3 \, \frac{1}{D_1 D_4 D_5} \, . \qquad (16)$$

where, in $m_e = 1$ units,

$$\begin{aligned} D_1 &= (p-k_1)^2 + 1 - i\epsilon \, , & D_2 &= (p-k_1-k_2)^2 + 1 - i\epsilon \, , \\ D_3 &= (p-k_1-k_2-k_3)^2 + 1 - i\epsilon, & D_4 &= (p-k_2-k_3)^2 + 1 - i\epsilon \, , \\ D_5 &= (p-k_3)^2 + 1 - i\epsilon \, , & D_6 &= k_1^2 - i\epsilon \, , \\ D_7 &= k_2^2 - i\epsilon \, , & D_8 &= k_3^2 - i\epsilon \, . \end{aligned}$$

In [8] we gave a similar table, containing however 18 basic integrals; in the meanwhile we found indeed that the 11-th integral, I_{11} of [8] was in fact a linear combination of the integrals I_{14} and I_{18} of [8]

$$I_{11} = I_{14}\left[\frac{3}{4} - \frac{1}{4}\left(\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + O(\omega^7)\right)\right] - \frac{1}{4} I_{18} \, . \qquad (17)$$

The integral I_{11} of [8] has therefore been dropped, obtaining the 17 integrals of Eq. (16); more precisely, the first 10 integrals, $J_1, ..., J_{10}$ correspond to $I_1, ..., I_{10}$, while $J_{11}, ..., J_{17}$ of the present paper correspond to $I_{12}, ..., I_{18}$ of [8].

As an example of the use of the basic integrals, let us consider the "triple cross" graph of Fig. 1

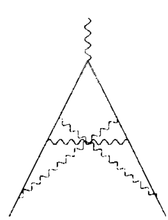


Fig. 1. A triple-cross vertex graph

Its contribution to the electron (g_e-2) in terms of the above 17 basic integrals reads

$$
\begin{aligned}
a_e(\text{triple}-\text{cross}) = \lim_{\omega\to 0}\Bigg[&+ \frac{7}{6}J_1 \\
&+ J_2\left(-\frac{1}{2\omega}+\frac{19}{12}-\frac{1637}{72}\omega\right) \\
&+ J_3\left(-\frac{13}{48\omega}+\frac{253}{144}-\frac{823}{108}\omega+\frac{102797}{2592}\omega^2-\frac{979525}{3888}\omega^3\right) \\
&+ J_4\left(\frac{3}{8\omega}-\frac{5}{24}+\frac{229}{18}\omega\right) \\
&+ J_5\left(-\frac{1}{12\omega}-\frac{7}{18}-\frac{587}{216}\omega+\frac{9133}{324}\omega^2-\frac{340685}{1944}\omega^3\right) \\
&+ J_6\left(-\frac{1}{24\omega}+\frac{55}{72}-\frac{827}{108}\omega+\frac{19075}{648}\omega^2-\frac{874721}{3888}\omega^3\right) \\
&+ J_7\left(\frac{19}{24\omega}-\frac{25}{18}+\frac{632}{27}\omega-\frac{56983}{648}\omega^2+\frac{1488295}{1944}\omega^3\right) \\
&+ J_8\left(\frac{1}{8\omega^2}-\frac{5}{9\omega}+\frac{1585}{216}-\frac{36581}{1296}\omega+\frac{1051253}{3888}\omega^2-\frac{743606}{729}\omega^3\right) \\
&+ J_9\left(\frac{11}{96\omega^2}-\frac{565}{288\omega}+\frac{467}{36}-\frac{11225}{162}\omega+\frac{1629641}{3888}\omega^2\right. \\
&\qquad \left.-\frac{1068433}{432}\omega^3\right) \\
&+ J_{10}\left(-\frac{11}{72\omega}-\frac{613}{216}-\frac{187}{324}\omega-\frac{234293}{3888}\omega^2+\frac{1054147}{23328}\omega^3\right) \\
&+ J_{11}\left(-\frac{11}{512\omega^3}+\frac{525}{1024\omega^2}-\frac{6601}{2304\omega}+\frac{577597}{27648}-\frac{4981223}{41472}\omega\right. \\
&\qquad \left.+\frac{46026487}{62208}\omega^2-\frac{1632171647}{373248}\omega^3\right) \\
&+ J_{12}\left(\frac{7}{96\omega^2}+\frac{13}{192\omega}+\frac{2375}{576}-\frac{95}{432}\omega+\frac{339883}{2592}\omega^2+\frac{60055}{3888}\omega^3\right) \\
&+ J_{13}\left(\frac{11}{192\omega^2}-\frac{1051}{1152\omega}+\frac{16657}{1728}-\frac{107779}{3456}\omega+\frac{9308803}{31104}\omega^2\right. \\
&\qquad \left.-\frac{120014225}{93312}\omega^3\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ J_{14}\left(\frac{1}{4\omega^2} + \frac{7}{36\omega} + \frac{6229}{432} + \frac{187}{27}\omega + \frac{1934333}{3888}\omega^2 + \frac{1580447}{11664}\omega^3\right) \\
&+ J_{15}\left(\frac{11}{96\omega^2} - \frac{1075}{576\omega} + \frac{8179}{864} - \frac{10925}{192}\omega + \frac{2919977}{7776}\omega^2 \right. \\
&\qquad \left. - \frac{49397413}{23328}\omega^3\right) \\
&+ J_{16}\left(-\frac{17}{24\omega^2} + \frac{1429}{144\omega} - \frac{12223}{216} + \frac{146749}{432}\omega - \frac{15117413}{7776}\omega^2\right) \\
&+ J_{17}\left(-\frac{11}{512\omega^3} + \frac{1379}{1536\omega^2} - \frac{1025}{768\omega} + \frac{103331}{2304} - \frac{28067}{512}\omega \right. \\
&\qquad \left. + \frac{63877391}{41472}\omega^2 - \frac{73084675}{31104}\omega^3\right)\Bigg]. \qquad (18)
\end{aligned}
$$

It is important to stress that the 17 basic integrals are sufficient to express the (g_e-2) contribution not only of the other "triple cross" graphs considered in [8], but also of graphs of different topology, such as for instance the "corner ladder" graph of Fig. 2, for which we find

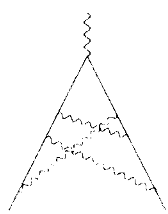


Fig. 2. A corner-ladder vertex graph

$$
\begin{aligned}
&a_e(\text{corner} - \text{ladder}) = \lim_{\omega \to 0}\Bigg[\\
&+ J_2\left(\frac{5}{24\omega} - \frac{125}{72} + \frac{1613}{216}\omega\right) + J_4\left(-\frac{1}{2\omega} + \frac{41}{72} - \frac{245}{27}\omega\right) \\
&+ J_5\left(-\frac{11}{24\omega} + \frac{113}{72} - \frac{1907}{216}\omega + \frac{19811}{648}\omega^2 - \frac{14062}{243}\omega^3\right) \\
&+ J_6\left(\frac{1}{6\omega} - \frac{139}{72} + \frac{295}{54}\omega - \frac{14515}{648}\omega^2 + \frac{12385}{486}\omega^3\right) \\
&+ J_7\left(-\frac{1}{2\omega} + \frac{17}{8} - \frac{11}{2}\omega + \frac{359}{24}\omega^2 - \frac{775}{36}\omega^3\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ J_8\left(\frac{7}{36\omega^2} - \frac{49}{108\omega} + \frac{3289}{1296} - \frac{35533}{3888}\omega + \frac{351305}{5832}\omega^2\right.\\
&\qquad\left. - \frac{2131849}{8748}\omega^3\right)\\
&+ J_9\left(-\frac{5}{48\omega^2} + \frac{29}{96\omega} + \frac{479}{432} + \frac{115}{648}\omega + \frac{583}{324}\omega^2 - \frac{19895}{729}\omega^3\right)\\
&+ J_{10}\left(\frac{1}{18\omega} - \frac{10}{9} + \frac{8917}{1296}\omega - \frac{38543}{3888}\omega^2 + \frac{260881}{7776}\omega^3\right)\\
&+ J_{11}\left(\frac{3}{128\omega^3} - \frac{173}{768\omega^2} - \frac{293}{2304\omega} - \frac{1471}{2304} - \frac{140287}{10368}\omega + \frac{30301}{972}\omega^2\right.\\
&\qquad\left. - \frac{5669447}{31104}\omega^3\right)\\
&+ J_{12}\left(\frac{17}{96\omega^2} - \frac{23}{216\omega} + \frac{2695}{2592} + \frac{18799}{20736}\omega + \frac{987161}{93312}\omega^2\right.\\
&\qquad\left. + \frac{10495663}{139968}\omega^3\right)\\
&+ J_{13}\left(\frac{1}{48\omega^2} - \frac{653}{576\omega} + \frac{4127}{3456} - \frac{67841}{3456}\omega - \frac{289511}{7776}\omega^2\right.\\
&\qquad\left. - \frac{31221347}{93312}\omega^3\right)\\
&+ J_{14}\left(\frac{31}{72\omega^2} - \frac{173}{432\omega} + \frac{4171}{648} + \frac{24911}{7776}\omega + \frac{684947}{11664}\omega^2\right.\\
&\qquad\left. + \frac{17625431}{69984}\omega^3\right)\\
&+ J_{15}\left(-\frac{5}{48\omega^2} - \frac{65}{144\omega} + \frac{1133}{1728} + \frac{10657}{5184}\omega + \frac{40625}{3888}\omega^2\right.\\
&\qquad\left. + \frac{199327}{11664}\omega^3\right)\\
&+ J_{16}\left(\frac{13}{24\omega^2} - \frac{37}{144\omega} + \frac{89}{864} - \frac{10805}{2592}\omega - \frac{41947}{7776}\omega^2\right)\\
&+ J_{17}\left(\frac{3}{128\omega^3} + \frac{355}{576\omega^2} - \frac{145}{128\omega} + \frac{76855}{10368} - \frac{102685}{15552}\omega + \frac{2157821}{31104}\omega^2\right.\\
&\qquad\left.\left. + \frac{63142129}{139968}\omega^3\right)\right]. \qquad (19)
\end{aligned}
$$

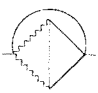
Note that J_1, which is requested for expressing the (g_e-2) contribution due to the "triple cross" graph of Eq. (18) does not appear in Eq. (19), corresponding to a topologically simpler "corner ladder" graph.

At this point, the analytical value of the integrals J_i is needed. Some of them are almost trivial; J_{17}, for instance, is just equal to the cube of the elementary integral $J = i\int d^n q/(q^2 + 1 - i\epsilon)$, J_{14}, J_{15} are the product of J and of a two-loop integral, J_{12}, J_{13} are vacuum–vacuum terms, the others are increasingly more difficult to deal with, up to J_1 which contains the full structure of the "triple-cross" topology. As it is apparent from Eq. (19) and Eq. (18), the expansion in $\omega = (4 - n)/2$ of almost all those integrals is needed. If $C(\omega)$, defined as

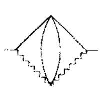
$$C(\omega) = (\pi^\omega \Gamma(1+\omega))^3 \ , n = 4 - 2\omega \ ,$$

is an overall normalization factor, whose limiting value at $\omega = 0$ $(n = 4)$ is 1, the actual analytical values of the integrals are

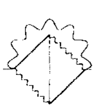
$$J_1 = C(\omega)\left[5\zeta(5) - \frac{1}{2}\pi^2\zeta(3) + O(\omega)\right] ,$$



$$\begin{aligned} J_2 = C(\omega)\Bigg[& 2\frac{\zeta(3)}{\omega} - \frac{13}{90}\pi^4 - \frac{1}{3}\pi^2 + 10\zeta(3) + \omega\Bigg(\frac{385}{2}\zeta(5) \\ & - \frac{85}{6}\pi^2\zeta(3) - \frac{7}{15}\pi^4 - 82\zeta(3) - 4\pi^2\ln 2 + 16\pi^2 - 2C_1 \\ & + 6C_2\Bigg) + O(\omega^2)\Bigg] , \end{aligned}$$



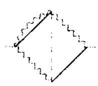
$$\begin{aligned} J_3 = C(\omega)\Bigg[& \frac{1}{3\omega^3} + \frac{7}{3\omega^2} + \frac{31}{3\omega} - \frac{2}{15}\pi^4 - \frac{4}{3}\zeta(3) + \frac{103}{3} \\ & + \omega\Bigg(95\zeta(5) - \frac{25}{3}\pi^2\zeta(3) - \frac{1}{15}\pi^4 - \frac{184}{3}\zeta(3) - 8\pi^2\ln 2 \\ & + \frac{44}{3}\pi^2 + \frac{235}{3} + 4C_2\Bigg) + O(\omega^2)\Bigg] , \end{aligned}$$



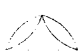
$$\begin{aligned} J_4 = C(\omega)\Bigg[& 2\frac{\zeta(3)}{\omega} - \frac{7}{90}\pi^4 + 2\zeta(3) + \frac{1}{3}\pi^2 + \omega\Bigg(\frac{385}{2}\zeta(5) \\ & - \frac{85}{6}\pi^2\zeta(3) - \frac{7}{15}\pi^4 - 82\zeta(3) - 4\pi^2\ln 2 + 16\pi^2 - 2C_1 \\ & + 4C_2\Bigg) + O(\omega^2)\Bigg] , \end{aligned}$$

$$J_5 = C(\omega)\Big[\frac{1}{6\omega^3} + \frac{3}{2\omega^2} + \frac{1}{\omega}\Big(-\frac{1}{3}\pi^2 + \frac{55}{6}\Big) - \frac{4}{45}\pi^4 - \frac{14}{3}\zeta(3) - \frac{7}{3}\pi^2 + \frac{95}{2} + \omega\Big(-\frac{2}{9}\pi^4 - 44\zeta(3) - \frac{29}{3}\pi^2 + \frac{1351}{6} + 2C_1\Big) + O(\omega^2)\Big] ,$$

$$J_6 = C(\omega)\Big[\frac{1}{3\omega^3} + \frac{7}{3\omega^2} + \frac{31}{3\omega} - \frac{4}{45}\pi^4 + \frac{2}{3}\zeta(3) + \frac{1}{3}\pi^2 + \frac{103}{3} + \omega\Big(\frac{45}{2}\zeta(5) - \frac{7}{2}\pi^2\zeta(3) + \frac{11}{45}\pi^4 + \frac{14}{3}\zeta(3) - 4\pi^2\ln 2 + \frac{14}{3}\pi^2 + \frac{235}{3} + 2C_1\Big) + O(\omega^2)\Big] ,$$



$$J_7 = C(\omega)\Big[\frac{1}{6\omega^3} + \frac{3}{2\omega^2} + \frac{1}{\omega}\Big(-\frac{1}{3}\pi^2 + \frac{55}{6}\Big) - \frac{1}{15}\pi^4 - \frac{8}{3}\zeta(3) - 2\pi^2 + \frac{95}{2} + \omega\Big(\frac{45}{2}\zeta(5) - \frac{17}{6}\pi^2\zeta(3) - \frac{7}{9}\pi^4 - 50\zeta(3) - 4\pi^2\ln 2 + \frac{1}{3}\pi^2 + \frac{1351}{6} + 2C_2\Big) + O(\omega^2)\Big] ,$$



$$J_8 = C(\omega)\Big[-\frac{1}{\omega^3} - \frac{16}{3\omega^2} - \frac{16}{\omega} + 2\zeta(3) - \frac{8}{3}\pi^2 - 20 + \omega\Big(-\frac{3}{10}\pi^4 - \frac{200}{3}\zeta(3) + 16\pi^2\ln 2 - 28\pi^2 + \frac{364}{3}\Big) + \omega^2\Big(-126\zeta(5) + 21\pi^2\zeta(3) + \frac{46}{15}\pi^4 - 512a_4 - \frac{64}{3}\ln^4 2 - \frac{80}{3}\pi^2\ln^2 2 - 776\zeta(3) + 168\pi^2\ln 2 - 188\pi^2 + 1244\Big) + O(\omega^3)\Big] ,$$

$$J_9 = C(\omega)\Big[-\frac{2}{3\omega^3} - \frac{10}{3\omega^2} + \frac{1}{\omega}\Big(-\frac{1}{3}\pi^2 - \frac{26}{3}\Big) - \frac{16}{3}\zeta(3) - \frac{11}{3}\pi^2 - 2 + \omega\Big(-\frac{13}{45}\pi^4 - \frac{248}{3}\zeta(3) + 16\pi^2\ln 2 - \frac{73}{3}\pi^2 + \frac{398}{3}\Big) + \omega^2\Big(-96\zeta(5) - \frac{8}{3}\pi^2\zeta(3) + \frac{3}{5}\pi^4$$

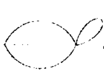
$$-512a_4 - \frac{64}{3}\ln^4 2 - \frac{128}{3}\pi^2\ln^2 2 - \frac{1888}{3}\zeta(3)$$
$$+160\pi^2\ln 2 - 129\pi^2 + 1038\Big) + O(\omega^3)\Big] \, ,$$

$$J_{10} = C(\omega)\Big[-\frac{1}{3\omega^3} - \frac{5}{3\omega^2} + \frac{1}{\omega}\Big(-\frac{2}{3}\pi^2 - 4\Big) - \frac{26}{3}\zeta(3) - \frac{7}{3}\pi^2$$
$$+\frac{10}{3} + \omega\Big(-\frac{35}{18}\pi^4 - \frac{94}{3}\zeta(3) - \pi^2 + \frac{302}{3}\Big) + O(\omega^2)\Big] \, ,$$

$$J_{11} = C(\omega)\Big[\frac{1}{\omega^3} + \frac{7}{2\omega^2} + \frac{253}{36\omega} + \frac{2501}{216} + \omega\Big(-\frac{64}{9}\pi^2 + \frac{59437}{1296}\Big)$$
$$+\omega^2\Big(-\frac{1792}{9}\zeta(3) + \frac{256}{3}\pi^2\ln 2 - \frac{2272}{27}\pi^2 + \frac{2831381}{7776}\Big)$$
$$+\omega^3\Big(\frac{2752}{135}\pi^4 - \frac{8192}{3}a_4 - \frac{1024}{9}\ln^4 2 - \frac{3584}{9}\pi^2\ln^2 2$$
$$-\frac{63616}{27}\zeta(3) + \frac{9088}{9}\pi^2\ln 2 - \frac{49840}{81}\pi^2 + \frac{117529021}{46656}\Big)$$
$$+O(\omega^4)\Big] \, ,$$

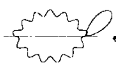
$$J_{12} = C(\omega)\Big[\frac{2}{\omega^3} + \frac{23}{3\omega^2} + \frac{35}{2\omega} + \frac{275}{12} + \omega\Big(\frac{112}{3}\zeta(3) - \frac{189}{8}\Big)$$
$$+\omega^2\Big(-\frac{136}{45}\pi^4 + 256a_4 + \frac{32}{3}\ln^4 2 - \frac{32}{3}\pi^2\ln^2 2$$
$$+280\zeta(3) - \frac{14917}{48}\Big) + O(\omega^3)\Big] \, ,$$

$$J_{13} = C(\omega)\Big[\frac{1}{3\omega^3} + \frac{7}{6\omega^2} + \frac{25}{12\omega} + \frac{8}{3}\zeta(3) - \frac{5}{24} + \omega\Big(-\frac{2}{15}\pi^4$$
$$+\frac{28}{3}\zeta(3) - \frac{959}{48}\Big) + \omega^2\Big(48\zeta(5) - \frac{7}{15}\pi^4 + \frac{50}{3}\zeta(3)$$
$$-\frac{10493}{96}\Big) + O(\omega^3)\Big] \, ,$$

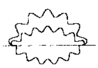


$$J_{14} = C(\omega)\Big[\frac{3}{2\omega^3} + \frac{23}{4\omega^2} + \frac{105}{8\omega} + \frac{4}{3}\pi^2 + \frac{275}{16} + \omega\Big(28\zeta(3)$$

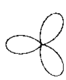
$$-8\pi^2\ln 2 + 10\pi^2 - \frac{567}{32}\Big) + \omega^2\Big(-\frac{62}{45}\pi^4 + 192a_4$$
$$+8\ln^4 2 + 16\pi^2\ln^2 2 + 210\zeta(3) - 60\pi^2\ln 2 + \frac{145}{3}\pi^2$$
$$-\frac{14917}{64}\Big) + O(\omega^3)\Big] ,$$



$$J_{15} = C(\omega)\Big[\frac{1}{2\omega^3} + \frac{7}{4\omega^2} + \frac{1}{\omega}\Big(\frac{1}{3}\pi^2 + \frac{25}{8}\Big) + 4\zeta(3) + \frac{7}{6}\pi^2$$
$$-\frac{5}{16} + \omega\Big(\frac{16}{45}\pi^4 + 14\zeta(3) + \frac{25}{12}\pi^2 - \frac{959}{32}\Big)$$
$$+\omega^2\Big(72\zeta(5) + \frac{8}{3}\pi^2\zeta(3) + \frac{56}{45}\pi^4 + 25\zeta(3) - \frac{5}{24}\pi^2$$
$$-\frac{10493}{64}\Big) + O(\omega^3)\Big] ,$$



$$J_{16} = C(\omega)\Big[-\frac{1}{6\omega^2} - \frac{35}{36\omega} - \frac{1}{3}\pi^2 - \frac{559}{216} + \omega\Big(-\frac{16}{3}\zeta(3)$$
$$-\frac{35}{18}\pi^2 + \frac{2737}{1296}\Big) + \omega^2\Big(-\frac{37}{45}\pi^4 - \frac{280}{9}\zeta(3)$$
$$-\frac{559}{108}\pi^2 + \frac{552041}{7776}\Big) + O(\omega^3)\Big] .$$



$$J_{17} = C(\omega)\Big(-\frac{1}{\omega^3} - \frac{3}{\omega^2} - \frac{6}{\omega} - 10 - 15\omega - 21\omega^2 - 28\omega^3$$
$$+O(\omega^4)\Big) , \qquad (20)$$

where C_1, C_2 are the coefficients of the ω term in the expansion of the integrals

$$M_1 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int \frac{d^nk_1 d^nk_2 d^nk_3\ (p.k_1)}{D_1D_3D_4D_5D_6D_7D_8}$$
$$= C(\omega)\left[-\frac{2}{45}\pi^4 + \zeta(3) + \omega C_1 + O(\omega^2)\right] ,$$
$$M_2 = \left(\frac{-i}{\pi^{n-2}}\right)^3 \int \frac{d^nk_1 d^nk_2 d^nk_3\ (p.k_3)}{D_2D_3D_4D_5D_6D_7D_8}$$
$$= C(\omega)\left[-\frac{1}{30}\pi^4 - \frac{1}{3}\pi^2 + 4\zeta(3) + \omega C_2 + O(\omega^2)\right] .$$

The value of C_1 is not needed (see below); for C_2 we have

$$C_2 = -\frac{173}{4}\zeta(5) + \frac{53}{12}\pi^2\zeta(3) - \frac{2}{15}\pi^4 + 18\zeta(3) + 2\pi^2\ln 2 - 3\pi^2 .$$

Due to an unfortunate misprint, in Ref.[8] the terms containing the constants C_1, C_2 are missing in the *r.h.s.* of the integrals I_2, I_3, I_4, I_5, I_6, I_7 and I_{11}; all the results which follow are however correct, as C_1, C_2 cancel out systematically in all the $(g_e$-2) results of [8] (which are therefore correct despite the misprint!). By substituting the values of the J_i Eq. (20) into Eq. (18) and Eq. (19) the constant C_1 cancels out again, and one obtains

$$a_e(\text{triple} - \text{cross}) =$$

$$\frac{5}{12}\zeta(5) - \frac{4}{9}\pi^2\zeta(3) - \frac{161}{1080}\pi^4 + \frac{8}{3}\left(a_4 + \frac{1}{24}\ln^4 2\right)$$
$$+\frac{32}{9}\pi^2\ln^2 2 + \frac{97}{12}\zeta(3) + \frac{20}{9}\pi^2\ln 2 - \frac{1043}{432}\pi^2 - \frac{1}{48} , \qquad (21)$$

$$a_e(\text{corner} - \text{ladder}) =$$

$$-\frac{215}{24}\zeta(5) + \frac{95}{72}\pi^2\zeta(3) + \frac{41}{180}\pi^4 - \frac{137}{27}\pi^2\ln^2 2 + \frac{160}{9}a_4$$
$$+\frac{20}{27}\ln^4 2 + \frac{69}{4}\zeta(3) - \frac{101}{18}\pi^2\ln 2 + \frac{2401}{2592}\pi^2 - \frac{3017}{864} , \qquad (22)$$

in agreement with Eq. (3) of [8] and Eq. (5) of [9].

Note that all the integrals J_i but J_1 are divergent for $n \to 4$; although for an analytical calculation that fact is not a particular problem (divergent integrals usually have less denominators, a fact which makes the analytical integration simpler), it prevents their direct numerical calculation and makes numerical checks harder. To that aim, it is convenient to express Eq. (18) and Eq. (19) as combination of a different set of other basic integrals N_i, all finite in the $n = 4$ limit, and of as few as possible divergent integrals. It turns out in fact that any scalar integral can be written as a linear combination of the basic integrals, so it is just matter of patience to take a list of finite scalar integrals, to write the equations which express them as a combination of the basic integrals and then solve the equations for the basic integrals in terms of the finite integrals (as matter of fact, some entries of Eq. (20) have been established in that way, by exploiting the analytical results obtained in previous works [8, 9, 14] for many of the scalar amplitudes). In so doing one obtains the formulae

$$
\begin{aligned}
& a_e(\text{triple}-\text{cross}) = \\
& + \frac{449}{192}N_1 + \frac{85}{36}N_2 - \frac{3}{4}N_3 + \frac{1}{12}N_4 - \frac{5}{12}N_5 + \frac{1}{2}N_6 + \frac{1}{4}N_7 \\
& + \frac{5225}{864}N_8 - \frac{5081}{1728}N_9 + \frac{2}{3}N_{10} - \frac{11573}{864}N_{11} - \frac{43}{18}N_{12} \\
& + \frac{2809}{432}N_{13} + \frac{5657}{864}N_{14} - \frac{9797}{432}N_{15} + \frac{1}{48}\lim_{\omega\to 0}\left[\omega^3 J_{17}\right]
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
& a_e(\text{corner}-\text{ladder}) = \\
& - \frac{817}{3456}N_1 - \frac{43}{18}N_2 + \frac{3}{2}N_3 - \frac{1}{3}N_4 + \frac{7}{6}N_6 - \frac{29233}{1728}N_8 \\
& + \frac{32881}{3456}N_9 + \frac{34}{9}N_{10} + \frac{20527}{576}N_{11} + \frac{77}{9}N_{12} - \frac{4657}{864}N_{13} \\
& - \frac{23281}{1728}N_{14} + \frac{16175}{288}N_{15} - N_{16} + \frac{2}{3}L_1 + \frac{3017}{864}\lim_{\omega\to 0}\left[\omega^3 J_{17}\right]
\end{aligned} \tag{24}
$$

where J_{17}, defined in Eq. (16,20), is in fact the only divergent integral, while the N_i are the following triple-cross integrals, finite at $n = 4$,

$$
\begin{aligned}
N_1 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_2)}{D_1D_2D_3D_4D_5D_6D_7D_8} = 5\zeta(5) - \frac{1}{2}\pi^2\zeta(3)\ , \\
N_2 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3}{D_1D_2D_3D_4D_5D_6D_7D_8} = -\frac{1}{6}\pi^4 + 4\pi^2\ln^2 2\ , \\
N_3 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3}{D_2D_3D_4D_5D_6D_7D_8} = -\frac{45}{4}\zeta(5) + \frac{17}{12}\pi^2\zeta(3)\ , \\
N_4 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3}{D_1D_3D_4D_5D_6D_7D_8} = -\frac{45}{4}\zeta(5) + \frac{7}{4}\pi^2\zeta(3)\ , \\
N_5 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3}{D_1D_2D_4D_5D_6D_7D_8} = 25\zeta(5) - \frac{4}{3}\pi^2\zeta(3)\ , \\
N_6 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3}{D_1D_2D_3D_4D_5D_7D_8} = -\frac{5}{4}\zeta(5) + \frac{5}{12}\pi^2\zeta(3)\ ,
\end{aligned}
$$

$$
\begin{aligned}
N_7 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3}{D_1D_2D_3D_4D_5D_6D_8} = -\frac{15}{2}\zeta(5) + \frac{7}{6}\pi^2\zeta(3) \,, \\
N_8 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_1)}{D_1D_3D_4D_5D_6D_7D_8} = -\frac{2}{45}\pi^4 + \zeta(3) \,, \\
N_9 &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_3)}{D_2D_3D_4D_5D_6D_7D_8} = -\frac{1}{30}\pi^4 - \frac{1}{3}\pi^2 + 4\zeta(3) \,, \\
N_{10} &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_2)}{D_1D_2^2D_3D_4D_5D_6D_7D_8} \\
&= \frac{43}{720}\pi^4 - \frac{13}{6}\pi^2\ln^2 2 + \frac{1}{2}\pi^2\ln 2 - \frac{1}{6}\pi^2 + \frac{1}{6}\ln^4 2 + 4a_4 \,, \\
N_{11} &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_2)}{D_1D_2D_3^2D_4D_5D_6D_7D_8} \\
&= -\frac{1}{60}\pi^4 + \pi^2\ln^2 2 - \pi^2\ln 2 + \frac{1}{3}\pi^2 \,, \\
N_{12} &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_2)^2}{D_1D_2D_3D_4D_5D_6D_7^2D_8} = -\frac{1}{15}\pi^4 - \frac{1}{4}\pi^2\zeta(3) \\
&\quad +\pi^2\ln^2 2 - 2\pi^2\ln 2 + \frac{4}{3}\pi^2 + 2\zeta(3) + \frac{5}{2}\zeta(5) \,, \\
N_{13} &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_2)^2}{D_1D_2^2D_3D_4D_5D_6D_7D_8} \\
&= \frac{1}{4}\pi^2\zeta(3) + \frac{3}{4}\pi^2\ln 2 - \frac{11}{12}\pi^2 + \frac{23}{8}\zeta(3) - \frac{5}{2}\zeta(5) \,, \\
N_{14} &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_2)^2}{D_1D_2D_3^2D_4D_5D_6D_7D_8} \\
&= \frac{1}{36}\pi^4 - \frac{1}{4}\pi^2\zeta(3) - \frac{3}{2}\pi^2\ln 2 + \pi^2 - \frac{7}{4}\zeta(3) + \frac{5}{2}\zeta(5) \,, \\
N_{15} &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_1)}{D_1D_2D_3^2D_4D_5D_6D_7D_8} \\
&= \frac{1}{120}\pi^4 - \frac{1}{2}\pi^2\ln^2 2 + \frac{1}{2}\pi^2\ln 2 - \frac{1}{6}\pi^2 - \frac{1}{2}\zeta(3) \,, \\
N_{16} &= \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3\ (p.k_2)}{D_2D_3D_4D_5D_6D_7^2D_8} = -\frac{45}{8}\zeta(5) + \frac{17}{24}\pi^2\zeta(3) \\
&\quad -\frac{19}{180}\pi^4 + \pi^2\ln^2 2 + \zeta(3) - \pi^2\ln 2 + \frac{2}{3}\pi^2 \,;
\end{aligned}
\tag{25}
$$

and L_1 is the scalar integral of the corner-ladder graphs [9]

$$L_1 = \frac{i}{\pi^6}\int \frac{d^4k_1 d^4k_2 d^4k_3}{D_1 D_2 D_3 D_{4a} D_5 D_6 D_7 D_8}$$
$$= \frac{29}{144}\pi^4 - \frac{11}{3}\pi^2 \ln^2 2 + \frac{1}{6}\ln^4 2 + 4a_4 \ ,$$

with

$$D_{4a} = (p - k_1 - k_3)^2 + 1 - i\epsilon \ .$$

Needless to say, by inserting the formulae of Eq. (25) into Eq. (23) and Eq. (24), Eq. (21) and Eq. (22) are recovered.

It is to be noted that Eq. (23), Eq. (24) and the table (25) are much more compact than the formulae of the corresponding Eq. (18), Eq. (19) and (20). The important difference to emphasize is that the integrals N_i are finite at $n = 4$ and simple enough to be calculated numerically without particular problems. Their evaluation is in any case much simpler than the direct numerical evaluation of the whole contribution from the concerned graphs to the electron (g_e-2) expressed in terms of the usual Feynman parameters or the like.

4. In the previous sections we have recalled the need for an independent evaluation of the 4-loop contribution c_4, first of all to provide an independent check of the result of [12] but also, when possible, to reduce the relative numerical error in c_4 below the 1% level, in order to achieve a final theoretical precision in the electron (g_e-2) better than 1ppb. We want to conclude this paper by suggesting that the techniques described in the latter part of the above section, devoted to the analytical evaluation of c_3, might be properly extended to the precise numerical evaluation of c_4 as well.

More in details, the proposal is to work out all the integration by part identities relevant in the 4-loop case, with the aim of expressing all the occurring scalar integrals and the contributions of the various 4-loop graphs to the electron (g_e-2) in terms of a limited number of basic integrals, in close analogy with the results of Eqs (18)–(20); as a guess, we expect a million of identities or more, and hope in a few hundreds basic integrals. As the analytical integration will be possible only for a very small number of basic integrals, we will try to express all the other basic integrals in terms of simple, finite and well behaved 4-loop integrals, easy to evaluate numerically, and then we will express the (g_e-2) contributions in terms of those simple and finite integrals, so obtaining the 4-loop analog of Eqs (23)–(25). Those formulae for the (g_e-2) could be obtained by pure algebraic methods; once the algebraic formulae are established, we can tackle as a second and independent step the problem of the precise numerical evaluation of the occurring "simple and finite" integrals. An appealing feature of the proposed method is indeed the factorization, so to speak, of the calculation in a preliminary algebraic elaboration, to be carried out once for ever — working out the analog of Eqs (23)–(25) — and in an independent numerical integration, to be repeated as many times as needed with increasing precision for obtaining more and more precise values for c_4.

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