NONCOMMUTATIVE DIFFERENTIAL GEOMETRY AND CONNECTIONS ON SIMPLICIAL MANIFOLDS

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For a simplicial manifold we construct the differential geometry structure and use it to investigate linear connections, metric and gravity. We discuss and compare three main approaches and calculate the resulting gravity action functionals.

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1. Introduction

One of the greatest open problems in contemporary theoretical physics is the quantum theory of gravity. Various attempts to create a consistent theory, which would give the right classical limit, have been studied in recent years, including the perturbative approach, supergravity, superstring theories as well as simplicial gravity. The lattice approach to the problem of quantization of gravity has been suggested many years ago by Regge [1] and has been studied extensively (see [2] for reviews and references) since then.

Since there are no fully successful approaches to the quantum theory of general relativity on continuous spaces, which use the path-integral formulation, the approximation of manifolds by discrete structures is an attractive theory, from which we can learn something about the properties of gravity. In this approach the continuous action is seen as a long-distance effective action at the critical point of the discretized model.

The construction of non-commutative geometry [5] has opened a new possibility in this area. Generalization of the tools of differential geometry to the level of algebras has enabled us to consider discrete structures on the

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same footing as continuous manifolds. Therefore, having a discrete lattice or a simplicial manifold, we may investigate the *noncommutative differential geometry* of such objects. This has already been the subject of several research papers [6, 7], which were mostly devoted to the analysis of structures and gauge theories for such geometries.

Only recently the topic of non-commutative Riemannian geometry and linear connections has been investigated for a series of non-standard geometries [8], including matrix geometries and quantum planes.

In our paper we shall attempt to use these tools to the specific geometry obtained from a discrete lattice of a simplicial manifold, trying to find out whether our attempt to derive the structures of Riemannian geometry and gravity from non-commutative geometry would give a reasonable answer. Of course, the question, which we would try to answer is whether one may obtain in this way a prescription for an action, which could correspond to the classical Einstein–Hilbert action.

Since there is no generally approved definition of the linear connection for non-commutative geometry we would try to use all possibilities that have been proposed for various models. Therefore our investigation would additionally do the job of comparing the proposed definitions, in particular it would indicate which one can be successfully applied to this geometry.

The paper is organized as follows. First, we give an introduction of the differential structures on a simplicial manifold, illustrating it with few simple examples. The detailed construction of the differential algebra is presented in the appendix.

Following the definition of the metric we present an example of gauge theory on a simplicial manifold, deriving the Yang–Mills action. Finally for three definitions of linear connections we present their application to the discussed geometry.

2. Algebras and the first order differential calculus

Let us assume that we have a *simplicial manifold* of dimension n — *i.e.*, a simplicial complex constructed in such a way that the set of simplices which contain a vertex (point of a lattice) are homeomorphic to a n-dimensional ball.[1] Furthermore, we assume that the simplicial manifold, to which we should often refer to as *triangulation* is oriented. An orientation of a simplex, shortly speaking, is an assignment of a sign to an ordered set of its vertices. The orientation of a p-simplex induces, naturally, an orientation of all its lower-dimensional walls (sub-simplices). Now, we say that a simplicial manifold is orientable, if one can orient all simplices in such a way, that if

[1] We need this to avoid subtle problems with considering *pseudomanifolds*.

two p-simplices share a $p-1$-dimensional wall, then the induced orientations of this wall are opposite.

In our algebraic approach towards the differential structures we shall begin with identifying the points of the lattice with the generators of the commutative algebra of functions. To each point p of the triangulation we assign a function p, which is 1 at p and vanishes elsewhere.

Clearly, we have

$$ps = \delta_{ps} p. \tag{1}$$

where δ_{ps} is the usual Kronecker delta function. For finite lattices the identity of the algebra could be expressed as a sum of all the generators

$$1 = \sum_{p} p. \tag{2}$$

The construction of the differential algebra shall be based upon the geometrical structure of the lattice, and we shall identify links with one-forms and simplices of the higher dimension with higher order differential forms. Here, we should only sketch the results, for details of mathematical construction we refer the reader to the appendix.

Let Γ be a free vector space of *oriented links* on our lattice. There exists a natural bimodule structure on Γ, given by

$$s \; {}_{p}\!\nearrow^{q} = \delta_{sp} \; {}_{p}\!\nearrow^{q} , \qquad {}_{p}\!\nearrow^{q} \; s = \delta_{sq} \; {}_{p}\!\nearrow^{q} , \tag{3}$$

The involution operation (complex conjugation) of the algebra $\mathcal{A}$ extends to Γ

$$\left({}_{p}\!\nearrow^{q} \right)^{*} = - \; {}_{q}\!\nearrow^{p} \tag{4}$$

We identify the above constructed bimodule Γ with the one-forms of the differential algebra over $\mathcal{A}$. It remains to find an appropriate linear operator $d : \mathcal{A} \to \Gamma$ (external derivative). There exist a unique (up to an automorphism of Γ) linear map $d : \mathcal{A} \to \Gamma$, such that,

- d obeys the Leibniz rule: $d(fg) = (df)g + f(dg)$, $f, g \in \mathcal{A}$,
- $\ker d = \mathbb{C}$
- $\operatorname{Im} d$ generates Γ

First, alone from the Leibniz rule we obtain the following restrictions:

$$pdp + dpp = dp, \qquad pds + dps = 0,\ p \neq s. \tag{5}$$

A linear map $\tilde{d}$, which satisfies them must be of the form

$$\tilde{d}(p) = \sum_q C_{pq} \left(\overset{p}{\underset{q}{\nearrow}} - \overset{q}{\underset{p}{\nearrow}} \right), \tag{6}$$

where C_{pq} are some arbitrary symmetric constants ($C_{pq} = C_{qp}$). In particular, if we define a bimodule automorphism ρ of Γ:

$$\rho \left(\overset{q}{\underset{p}{\nearrow}} \right) = C_{pq} \overset{q}{\underset{p}{\nearrow}} \tag{7}$$

then $\tilde{d}$ could be written as a composition of ρ with a linear map d, $\tilde{d} = \rho \circ d$, where d is as follows:

$$d(p) = \sum_q \left(\overset{p}{\underset{q}{\nearrow}} - \overset{q}{\underset{p}{\nearrow}} \right). \tag{8}$$

Clearly, d is the desired map, which satisfies all our requirements. Additionally, we find that d is compatible with the previously introduced $*$-structure on $\Omega^1(\mathcal{A})$ (4), *i.e.*:

$$(df)^* = d(f^*). \tag{9}$$

Finally, let us demonstrate that the definition of d is quite natural and leads, as expected, to the finite difference along oriented links, being the lattice analogues of partial derivatives. For an arbitrary $\Phi \in \mathcal{A}$, $\Phi = \sum_p p\,\Phi_p$, we have

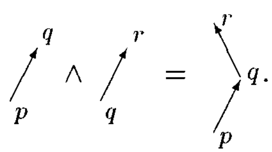
$$d\Phi = \sum_{p,q} \overset{q}{\underset{p}{\nearrow}} \, (\Phi_q - \Phi_p), \tag{10}$$

2.1. Higher order differential forms

The construction of higher order differential forms can be done in many different ways, which can lead to quite distinct differential algebras. The point of view we take in our examples is to relate the differential structure with the geometry of the triangulation. As the details of the construction are rather formal, we present here only results and the rules for the multiplication of one-forms and the action of the external derivative d.

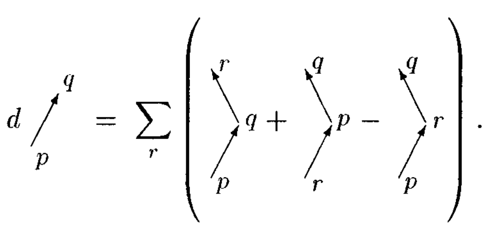
First, let us show how to construct higher-order differential forms by building the product of links. Of course, the multiplication must be done over $\mathcal{A}$, therefore the product of two links will vanish unless the end-point of the first link coincides with the start-point of the second link. Moreover, the requirement that two-forms correspond to two-dimensional simplices enforces that the product of the links vanishes unless these links belong to some 2-simplex.

For simplicity, we shall denote the product of two links in the following pictorial way



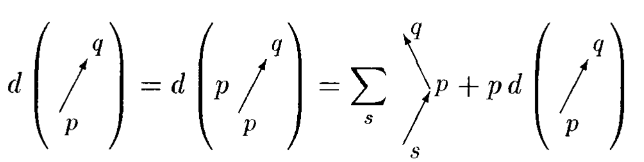
(11)

Now, we define the external derivative d



(12)

Quite easily one may verify that it satisfies the graded Leibniz rule for the multiplication by functions from the left and right-hand side



(13)

$$d\left(p \nearrow q \right) = d\left((p \nearrow q)\, q \right) = \sum_s (p \nearrow q \nwarrow s)\, q + d\left(p \nearrow q \right) q\,, \tag{14}$$

$$s \neq p: \quad 0 = d\left(s\,(p \nearrow q) \right) = -\,(s \nearrow p \nwarrow q) + s\, d\left(p \nearrow q \right), \tag{15}$$

$$s \neq q: \quad = d\left((p \nearrow q)\, s \right) = (p \nearrow q \nwarrow s) - d\left(p \nearrow q \right) s\,. \tag{16}$$

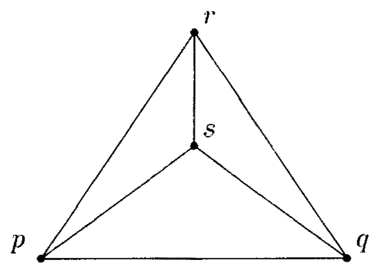
Next, calculate d^2,

$$d^2(p) = \sum_q d\left(q \nearrow p \; - \; p \nearrow q \right) =$$

$$= \sum_{q,r} \left((q \nearrow p \nwarrow r) + (r \nearrow q \nwarrow p) - (q \nearrow r \nwarrow p) \right) - \left((p \nearrow q \nwarrow r) + (r \nearrow p \nwarrow q) - (p \nearrow r \nwarrow q) \right),$$

and we can see that the first component in the first bracket cancels with the second component in the second bracket, and the remaining components cancel each other within the brackets.

Example 2.1 *To illustrate our construction let us choose a simple piece of the triangulation of a plane, as drawn on the picture below,*



As it is a two-dimensional triangulation of the plane (so that all points p, q, r, s lie in the same plane) we see that the following wedge products of links vanish (as well as their conjugates):

$$0 = \underset{p}{\overset{r}{\rangle}} q = \underset{q}{\overset{p}{\rangle}} r = \underset{r}{\overset{q}{\rangle}} p\,.$$

The rest, i.e., products of links that are (different) sides of the triangle of the lattice are linearly independent. Had the figure shown the surface of a tetrahedron then p, q, r would be in one triangle and the above products of links would not vanish.

Finally, let us follow another illustrating calculation, for an arbitrary one form,

$$\omega = \sum_{p,q} \underset{p}{\overset{q}{/}}\ \omega_{pq}$$

calculate $d\omega$,

$$d\omega = \sum_{p,q,r} \omega_{pq} \left(\underset{r}{\overset{q}{\rangle}} p + \underset{p}{\overset{r}{\rangle}} q - \underset{p}{\overset{q}{\rangle}} r \right). \tag{17}$$

After rearranging the order of components of this sum we obtain the following convenient expression

$$d\omega = \sum_{p,q,r} \overset{r}{\underset{p}{\Big\rangle}} q\,(\omega_{pq} + \omega_{qr} - \omega_{pr})\,. \tag{18}$$

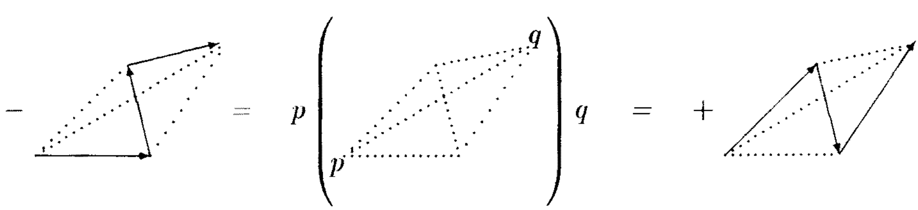
Remark 2.1 *Of course, the cohomology of the complex defined by the sequence:*

$$\mathcal{A} \xrightarrow{d} \Omega^1(\mathcal{A}) \xrightarrow{d} \Omega^2(\mathcal{A})$$

depends on the topology of the triangulated manifold.

For the products of more than two links we have an additional feature, which we must take into account. There might exist different (a priori) products with the same start- and end-points. As we wish that this situation does not happen (see appendix for motivation and formal construction) we should identify the corresponding products up to a sign, which originates from the orientation of the simplex. Then every non-vanishing product of n links could be written (in a symbolic way) as a n-dimensional simplex multiplied by the start-point (from the left) and the end-point (from the right). Below we present the example of a three-dimensional triangulation.

Example 2.2 *We have two possibilities for the products of three links in a simplex, which have the same start- and end-points. The relation between them follows from the above description:*



3. Metric and distances

In this section we shall discuss the concept of *metric* and distances on our lattice. The definition we use, has been introduced and discussed throughout several papers [6–8].

Define metric g as a bimodule morphism:

$$g : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \mathcal{A} . \tag{19}$$

We shall write $g(\omega, \xi)$ instead of $g(\omega \otimes_{\mathcal{A}} \xi)$ to shorten our notation and make it resembling the classical differential geometry. We say that the metric is *Hermitian* if for all $\omega, \xi \in \Omega^1(\mathcal{A})$:

$$g(\omega, \xi)^* = g(\xi^*, \omega^*). \tag{20}$$

This guarantees that $g(\omega, \omega^*)$ and $g(\omega^*, \omega)$ are self-adjoint elements [2] of $\mathcal{A}$. For the discrete geometry of a triangulation the metric has quite a simple form and it associates a real number g_{pq} to every oriented link from p to q. This follows easily from the definition (19) (g being a bimodule morphism):

$$g\left(\overset{q}{\underset{p}{\nearrow}} \otimes_{\mathcal{A}} \overset{s}{\underset{q}{\nearrow}} \right) = p g_{pq} \delta_{ps}, \tag{21}$$

where g_{pq} are complex coefficients. However, from the hermicity (20) we immediately get that $g_{pq} = \overline{g_{pq}}$.

Observe that the general definition does not require the symmetry of the metric (whatever it may mean). Moreover, in general we might have $g_{pq} \neq g_{qp}$, which would tell us that the metric structure is associated with oriented links.[3]

Example 3.1 *Let* $\Psi = \sum_{pq} \overset{q}{\underset{p}{\nearrow}} \psi_{pq}$ *be an arbitrary one-form, then*

$$g(\Psi, \Psi^*) = \sum_{p,q} p g_{pq} |\psi_{pq}|^2, \tag{22}$$

$$g(\Psi^*, \Psi) = \sum_{p,q} p g_{pq} |\psi_{qp}|^2, \tag{23}$$

and we see that $g(\Psi, \Psi^*)$ *cannot be equal to* $g(\Psi^*, \Psi)$ *for every* Ψ*, no matter what relation between the coefficients of the metric we choose (apart from* $g \equiv 0$*, of course).*

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[2] Note that they might be, however, different!

[3] An intuitive picture of that could be that from the point of view of the metric going from point p to q is not the same as going from q to p!

We shall say that g is *positive* if $g(\omega, \omega^*)$ and $g(\omega^*, \omega)$ are positive elements of $\mathcal{A}$ for every $\omega \in \Omega^1(\mathcal{A})$. The metric g on discrete geometries (as in (21)) is positive if and only if every $g_{pq} \geq 0$.

We shall say that the metric is non-degenerate if $g(\omega, \omega^*) = 0$ implies $\omega = 0$. Of course, the metric on discrete geometries is non-degenerate only if $g_{pq} > 0$ for all links $p \to q$.

To solve the apparent problem in the choice between $g(\omega, \omega^*)$ and $g(\omega^*, \omega)$ let us observe that it is not the algebra-valued expression, like the metric, but rather a number-valued expression (*i.e.* the integrated metric) that is important for physics. Let $\int$ be a trace on the algebra A, such that $\int 1 = 1$, $\int(a^*) = (\int a)^*$ and $\int aa^* = 0$ only for $a = 0$.

Example 3.2 *For the algebra of functions on the discrete space, such a trace is of the form*

$$\int a = \sum_p \mu_p a_p, \tag{24}$$

where $a_p = pa$ *and* $\mu_p > 0$ *(the latter are arbitrary positive numbers, such that* $\sum_p \mu_p = 1$*).*

Suppose now that we demand that the integrated metric is independent of the order of the arguments, *i.e.*:

$$\int g(\psi, \psi^*) = \int g(\psi^*, \psi).$$

Then it follows directly

$$\frac{\mu_p}{\mu_q} = \frac{g_{qp}}{g_{pq}}. \tag{25}$$

The above condition could be seen as a compatibility condition between the measure on the manifold and the metric. Clearly, not every metric is admissible. Indeed, for any closed loop along the links of the simplicial manifolds the product of metric coefficients along one orientation of the loop must be equal to the product along the opposite orientation. Suppose we have a metric satisfying this condition. Then there exists a unique measure determined by this metric. The construction follows directly from the relation (25), for some point x let us fix $\mu_x = 1$. Then, for every point y connected with x with a link we determine that $\mu_y = g_{xy}/g_{yx}$. Then we repeat the procedure until we reach all the points of the lattice. Of course, since there are many possible ways to reach a point at some distance from x we must ensure that this procedure is well-defined. This is the case due to the property of the metric: for two different ways we see that the end results coincide because for the loop made out of them, the product of coefficients of g is the same along either of its orientations. Finally we must rescale the measure to have $\int 1 = 1$.

We shall see now that these procedure might be used the other way round. Suppose we have a measure. Then the relation (25) fixes the ratio between g_{pq} and g_{qp} for any two points connected with a link. What remains a free parameter is a real number, which fixes their values. Now, we have a clear understanding of an admissible metric for a discrete geometry. It contains two pieces of information about the manifold: the measure (which can be interpreted as a positive function on vertices) and the association of a positive real number to every unoriented) link, we shall later denote this number $r_{pq} = r_{qp}$, then $g_{pq} = \mu_q r_{pq}$.

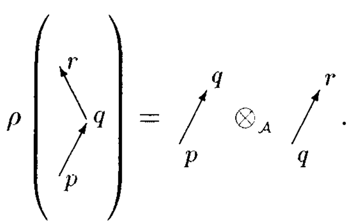
The above definition of the metric leads, however, to some difficulties if we attempt to extend it to higher-order forms. Of course, it is quite natural to propose the following extension of g to

$$\Omega^1(\mathcal{A})^{\otimes_{\mathcal{A}} 4}:$$
$$g_2(\omega_1 \otimes_{\mathcal{A}} \omega_2 \otimes_{\mathcal{A}} \omega_3 \otimes_{\mathcal{A}} \omega_4) = g\left(\omega_1 g(\omega_2, \omega_3)\omega_4\right), \tag{26}$$

though, of course, we may use another definition:

$$g_2'(\omega_1 \otimes_{\mathcal{A}} \omega_2 \otimes_{\mathcal{A}} \omega_3 \otimes_{\mathcal{A}} \omega_4) = g(\omega_1, \omega_2) g(\omega_3, \omega_4). \tag{27}$$

Of course, if g is hermitian (20) so are g_2 and g_2', moreover, for positive g we can easily see that g_2 and g_2' become positive.
To use the above definition for the two-forms rather than $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ we need to have a bimodule embedding $\rho : \Omega^2(\mathcal{A}) \hookrightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$, such that $\pi \circ \rho = \mathrm{id}$, where π is the projection $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A})$. Generally, there is an ambiguity in defining such map, however, in the case of the triangulation we might use the natural embedding ρ:



(28)

Then the definition for the metric on the space of two-forms would be the following:

$$g(u, v) := g_2(\rho(u), \rho(v)), \tag{29}$$

for every two-forms u, v. Let us observe that we are left only with the possibility of g_2 in our definition, as $g \circ \rho \equiv 0$ and hence g_2' vanishes.

As an example of a physical theory, in which we have to use differential structures and the metric, we shall consider a simple nonabelian gauge theory on the triangulation.

Example 3.3 *Let us consider an algebra of matrix-valued functions on our lattice with the gauge group being the group of unitary elements of this algebra. The gauge potential is a matrix-valued, anti-self-adjoint one-form* $A = -A^*$*:*

$$A = \sum_{p,q} \overset{q}{\underset{p}{\nearrow}} A_{pq}, \qquad A_{pq} = A^{\dagger}_{qp}, \tag{30}$$

The gauge transformation acts on A *in the usual way, the transformation rule for its coefficients being:*

$$A'_{pq} = U^{\dagger}_p A_{pq} U_q + U^{\dagger}_p (U_q - U_p), \tag{31}$$

where U_p *denotes the value of gauge transformation at point* p*. The transformation (31) becomes more transparent if we introduce a field* $\Phi_{pq} = 1 + A_{pq}$*:*

$$\Phi'_{pq} = U^{\dagger}_p \Phi_{pq} U_q. \tag{32}$$

The curvature $F = dA + A \wedge A$ *is a matrix-valued hermitian two-form, with coefficients:*

$$F = \sum_{p,q} F_{pqr} \; \left(p \nearrow q \nwarrow r \right), \tag{33}$$

where

$$F_{pqr} = (\Phi_{pq}\Phi_{qr} - \Phi_{pr}) = F^{\dagger}_{rqp}, \tag{34}$$

and its gauge transformation is $F' = U^{\dagger} F U$*:*

$$F'_{prq} = U^{\dagger}_p F_{prq} U_q. \tag{35}$$

The Yang–Mills Lagrangian for the theory would be:

$$\begin{aligned} \mathcal{L}_{\mathrm{YM}} &= \mathrm{Tr}\; g\,(\rho(F), \rho(F^*)) = \\ &= \mathrm{Tr} \sum_{p,q,r} p g_{pq} g_{qr} F_{pqr} F^{\dagger}_{pqr} \end{aligned} \tag{36}$$

This Lagrange function is real-valued and gauge invariant. Observe that if we express it in terms of the field Φ *we get three components, first one, quartic in* Φ*:*

$$\sum_{\triangle(p,q,r)} g_{pq} g_{qr} \mathrm{Tr}\; \Phi^{\dagger}_{pq} \Phi_{pq} \Phi_{qr} \Phi^{\dagger}_{qr},$$

quadratic in Φ:

$$\sum_{\Delta(p,q,r)} g_{pq} g_{qr} \mathrm{Tr}\ \Phi_{pr} \Phi^{\dagger}_{pr},$$

and cubic in Φ:

$$\sum_{\Delta(p,q,r)} g_{pq} g_{qr} \mathrm{Tr}\ \Phi_{pq} \Phi_{qr} \Phi_{rp} + \mathrm{h.c.}$$

If one restricts the space of all possible connections to unitary *connections satisfying $\Phi^{\dagger}_{pq}\Phi_{pq} = 1$ (this condition is gauge invariant and therefore acceptable, also all pure gauge connections are of this type) the first two components contribute only to constant terms (volume of space), and only the third one gives us the lattice action for a restricted pure gauge theory on triangulations.*

4. Linear connections

The general concept of Riemannian geometry and linear connections is still not well understood in the framework of noncommutative geometry. Several propositions have been used, starting from the idea of left-linear connections [6], bimodule connections [8] or projective-bimodule connections [9]. The problem is related with the bimodule structure of the space of one-forms, which is the crucial obstacle for extending the gauge theory formalism of connections to the Riemannian geometry case.

In this section we shall attempt to give a thorough discussion of these three main approaches, using the corresponding propositions for definitions of linear connection, torsion and curvature, for the example of the simplicial geometry.

Our attempt is to derive the Riemannian geometry of triangulated manifold, treated as a base space of theory, not as an approximation of a continuous object.

4.1. Left-linear connection

A left-linear connection ∇_{L} is a map: $\Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$, which satisfies:

$$\nabla_{\mathrm{L}}(a\omega) = a(\nabla_{\mathrm{L}}\omega) + da \otimes_{\mathcal{A}} \omega\,. \tag{37}$$

This map easily extends to a degree 1 map:
$\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$, with the following property:

$$\nabla_{\mathrm{L}}\,(\omega\eta) \;=\; (d\omega)\eta + (-1)^{|\omega|}\omega(\nabla_{\mathrm{L}}\eta), \tag{38}$$

In particular, we have that ∇_{L}^2 is a left-module endomorphism of degree 2 of $\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$.

Let π be the projection $\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^*(\mathcal{A})$. Then the torsion, $T = \pi \circ \nabla_{\mathrm{L}} - d \circ \pi$ is a left module homomorphism $\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^*(\mathcal{A})$.

Similarly, as we have done for the left-connection, we may introduce right-linear connections and the corresponding torsion and curvature.

Let us assume that the differential algebra is a star-algebra (*i.e.* there exists a conjugation, which (graded) commutes with d). If ∇_{L} is a left-connection then $\nabla_{\mathrm{R}} = \pm * \nabla_{\mathrm{L}} *$ is a right connection, the sign chosen accordingly for the degree of the space of forms it acts on.

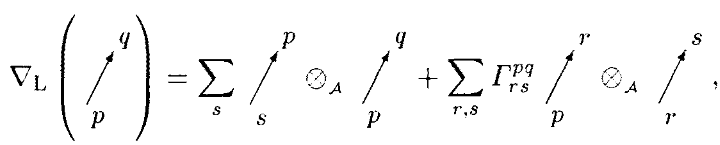
To verify it, let us take $\omega \in \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^n(\mathcal{A}), \rho \in \Omega^m(\mathcal{A})$, then:

$$\begin{aligned}\nabla_{\mathrm{R}}(\omega\rho) &= (-1)^{m+n+1}(*\nabla_{\mathrm{L}}*)(\omega\rho) = (-1)^{m+n+1} * \nabla_{\mathrm{L}}(\rho^*\omega^*) \\ &= (-1)^{m+n+1}\left((-1)^m d\rho^*\omega^* + (-1)^m \rho^* \nabla_{\mathrm{L}}(\omega^*)\right)^* \\ &= (-1)^{n+1}\omega d\rho + (-1)^{n+1}(*\nabla_{\mathrm{L}}*)(\omega)\rho \\ &= (-1)^{n+1}\omega d\rho + (\nabla_{\mathrm{R}}\omega)\rho\end{aligned}$$

which ends the proof.

Of course, since π is a star-homomorphism, we immediately see that for a torsion-free left connection ∇_{L}, the conjugated right-linear connection ∇_{R} is a also torsion free.

For the considered examples of differential calculus on triangulations we shall have:



(39)

where Γ^{pq}_{rs} are arbitrary complex numbers.

The torsion constraint $T = 0$ reads:

$$\Gamma^{pq}_{qr} = 1\,, \qquad \Gamma^{pq}_{rq} = -1\,,$$

for p, q, r forming a triangle,

$$\Gamma^{pq}_{rs} = 0$$

for p, r, s forming a triangle, $s \neq q$.

4.2. Metric compatibility condition

Having linear connections and the metric we may pose the question about the notion of compatibility between these two structures. There is no default answer by now, therefore we should use one of natural possible suggestions. We say that a left linear connection ∇_{L} (and the corresponding right-linear connection ∇_{R}) are compatible with the metric g if the following holds for all one-forms ω, η:

$$dg(\omega, \eta) = \tilde{g}_{\mathrm{L}}(\nabla_{\mathrm{L}}\omega, \eta) - \tilde{g}_{\mathrm{R}}(\omega, \nabla_{\mathrm{R}}\eta)\,, \tag{40}$$

where $\tilde{g}$ are the extensions of g, for instance:

$$\tilde{g}_{\mathrm{L}}(\omega_1 \otimes_{\mathcal{A}} \omega_2, \eta) = \omega_1 g(\omega_2, \eta).$$

Notice that the above constraint includes a lot more that in the classical case. This is because the left-hand side is by definition middle-linear (*i.e.* depends only on $\omega \otimes_{\mathcal{A}} \eta$, whereas for the right-hand side this may not hold for general left-linear connections).

Let us verify what is the outcome of the above introduced metric-compatibility condition for the simplicial geometry. After simple calculations we obtain

$$\Gamma^{pq}_{sr} g_{sr} = \overline{\Gamma}^{sr}_{pq} g_{pq}, \tag{41}$$

(no summation over repeated indices).

One may verify that this is also a sufficient condition for the right-hand side of (40) to have the same middle-linearity property as the left-hand side.

This condition has a very surprising aftermath. Consider a triangle p, q, r, and use the torsion constraint as well as the metric compatibility condition. Then one gets

$$g_{rq} = g_{pq}\,, \tag{42}$$

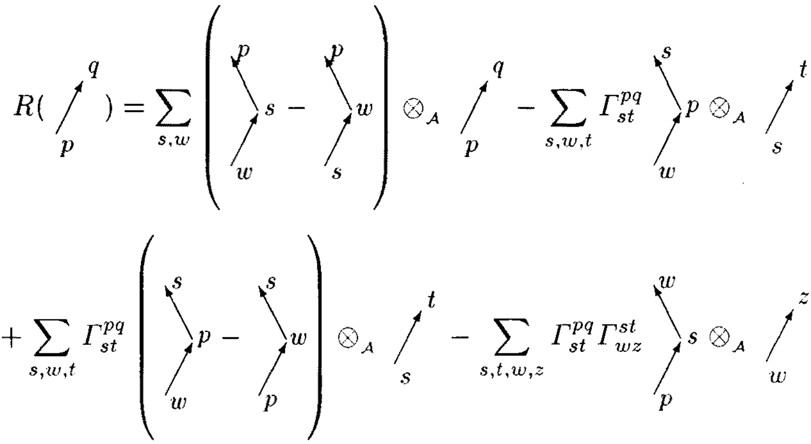
$$g_{qr} = \overline{\Gamma}^{qr}_{pq} g_{pq}\,, \tag{43}$$

and from the first relation (and $g_{pq} = r_{pq}\mu_q$) we immediately see that all r_{pq} must be equal to each other. Hence the metric is now restricted only to an overall constant r and the measure μ!. Furthermore, we obtain an expression for some of the nontrivial symbols Γ (notice that it follows immediately that Γ^{qr}_{pq} must be real).

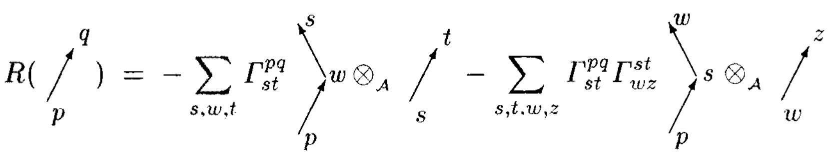
To summarize, for p, q, r being the vertices of a triangle the torsion and metric compatibility constraint give us together the following relations:

$$\begin{aligned} \Gamma^{pq}_{qr} &= 1\,, \\ \Gamma^{ps}_{qr} &= 0\,, s \neq q\,, \\ \Gamma^{pq}_{rq} &= -1\,, \\ \Gamma^{qr}_{pq} &= \frac{\mu_r}{\mu_q}\,. \end{aligned}$$

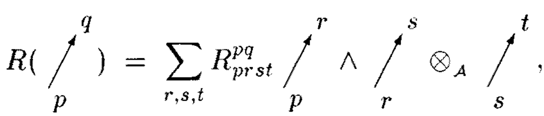
Having calculated the torsion and metric compatibility conditions we shall finally attempt to derive the curvature. From the general formula, the curvature for any left-linear connection on our geometry is:



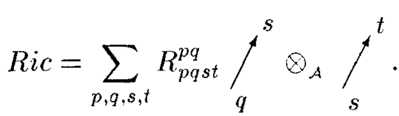
which reduces to the following compact expression:

 (44)

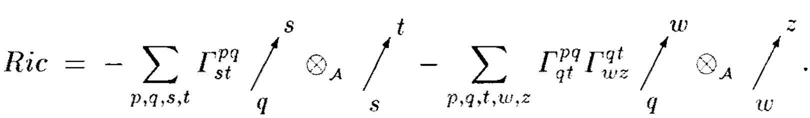
Next, we can introduce a Ricci tensor as a trace of the map R:



then:



In our case we get:

 (45)

The scalar of curvature could now be introduced as a function obtained by composing $g \circ Ric$, we have $R = \sum_q qR_q$:

$$R_q \;=\; -\sum_{p,w} g_{qw}\Gamma^{pq}_{sq} - \sum_{p,w,t} g_{qw}\Gamma^{pq}_{qt}\Gamma^{qt}_{wq}. \tag{46}$$

Remember that we have fixed only some of the Christoffel symbols and the rest of them are still arbitrary. Therefore we might divide R_q into two parts, first, which we may already calculate:

$$R^0_q = \sum_{(p,w;q)} \mu_w - \sum_{\frac{(p,t;q)}{(w,t;q)}} \mu_w \frac{\mu_t}{\mu_q} - \sum_{(p,t;q)} \mu_w \frac{\mu_p}{\mu_q}. \tag{47}$$

The part, which contains auxiliary fields is as follows:

$$R^a_q = -\sum_{\frac{[p,t;q]}{[w,t;q]}} \mu_w \Gamma^{pq}_{qt}\Gamma^{qt}_{wq} - \sum_{\frac{(p,t;q)}{[w,t;q]}} \mu_w \Gamma^{qt}_{wq} - \sum_{\frac{[p,t;q]}{(w,t;q)}} \mu_w \frac{\mu_q}{\mu_t}\Gamma^{pq}_{qt} - \sum_{(p,q;w)} \mu_w \Gamma^{qw}_{wq}\,. \tag{48}$$

where the brackets indicate that the sum is over point forming certain triangle and square brackets denote that the sum is over points do not forming a triangle. Of course, point q is fixed and p, q, w always make a triangle.

This result is not very promising. Though we have been able to define the connection, the metric and the relations between them, the connection still contains some auxiliary fields, which appear also in the resulting action functional.

4.3. Cuntz and Quillen linear connections

Recently Cuntz and Quillen have proposed a definition for linear connections on projective bimodules. Such connection is a pair of left and right linear connections (as defined in the previous section), with the restriction that they are simultaneously a right (and respectively left) module homomorphism:

$$\nabla^{\rm CQ}_{\rm L}(amb) = a(\nabla^{\rm CQ}_{\rm L} m)b + da \otimes_A mb. \tag{49}$$

Of course, our considerations that used the conjugation structure on the bimodule are still valid, therefore we shall still consider a pair of left and right connections that are related by $\nabla_{\rm R} = \pm * \nabla_{\rm L} *$.

For the case of triangulation geometry it means the following:

$$\nabla^{\rm CQ}_{\rm L} \overset{q}{\underset{p}{\nearrow}} = \sum_s \overset{p}{\underset{s}{\nearrow}} \otimes_A \overset{q}{\underset{p}{\nearrow}} + \sum_s \Gamma^{pq}_{sq} \overset{s}{\underset{p}{\nearrow}} \otimes_A \overset{q}{\underset{s}{\nearrow}}\,. \tag{50}$$

However, the torsion constraint cannot be satisfied in its form $T = \pi \circ \nabla - d \circ \pi$ and we have to modify it to satisfy the general properties arising from the Leibniz rule:

$$T = \pi \circ (\nabla_{\mathrm{L}}^{\mathrm{CQ}} + \nabla_{\mathrm{R}}^{\mathrm{CQ}}) - d \circ \pi. \tag{51}$$

We need to modify it and take the connection part of torsion as $\pi \circ (\nabla_{\mathrm{L}}^{\mathrm{CQ}} + \nabla_{\mathrm{R}}^{\mathrm{CQ}})$, which is the only operator that shares the property of Leibniz rule for left and right multiplication with d.

The torsion $T = 0$ constraint then gives us the following restriction on coefficients Γ

$$\Gamma^{pq}_{rq} + \overline{\Gamma}^{qp}_{rp} = -1, \tag{52}$$

for all p, q, r forming a triangle.

The metric compatibility condition (40) shall remain the same

$$\Gamma^{pq}_{sq} g_{sq} = \overline{\Gamma}^{sq}_{pq} g_{pq}, \tag{53}$$

however, since we have no further relations there would be no restriction on the metric itself (in particular r_{pq} are not fixed in this case). The relations between Γ and the metric have more than one solution. One can easily find that if Γ and Γ' satisfy (53) and (52) for a fixed metric, then $\alpha\Gamma + (1-\alpha)\Gamma'$ does as well, $0 \le \alpha \le 1$.

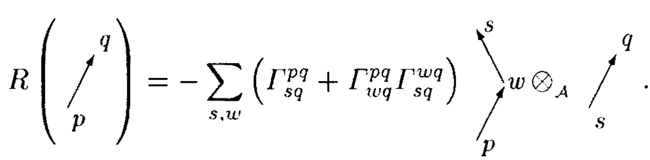
Note that the solutions might not be even real, however. should at least one Γ^{pq}_{rq} be real this would enforce all Γ to be real. Moreover, using simple arguments one can show that the space of all possible connections compatible with a given metric is one-dimensional, in particular if one fixes its one elements Γ^{pq}_{rs} then all others could be calculated.

An example of a real connection can be found quite easily, one verifies that

$$\Gamma^{pq}_{sq} = -\frac{r_{pq}}{r_{pq} + r_{sq}}. \tag{54}$$

satisfies (52) and (53).

For the Cuntz–Quillen the curvature $(\nabla_{\mathrm{L}}^{\mathrm{CQ}})^2$ (or equivalently $(\nabla_{\mathrm{R}}^{\mathrm{CQ}})^2$) is a bimodule morphism $\Omega^1 \to \Omega^2 \otimes_{\mathcal{A}} \Omega^1$:



The Einstein–Hilbert action becomes

$$\mathcal{S} = -\sum_{p,q,s} \mu_q g_{qs} \Gamma^{pq}_{sq},$$

and it appears that, alone, it might not even be real-valued.

What seems to be a bigger problem, is the auxiliary parameter in the connection. Since the metric determines it only up to a complex parameter (note that the dependence is not linear), we would like to have at least the action independent of such parameter. However the above action does not satisfy our demand.

In the approach to gravity on the simplicial manifold based on Cuntz–Quillen connections we have encountered a significant problem of an auxiliary parameter. Of course, this result is much better than in the case of left-linear connections, however unless solved in a satisfactory way, we cannot apply it to investigate gravity.

4.4. Bimodule connections

Finally we shall discuss another proposition for the construction of linear connections in the framework of noncommutative geometry, which uses concepts of the generalized symmetry operator σ [8]. Let us remind the basic assumptions. We assume that there exists a bimodule automorphism of $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that $\pi \circ (1+\sigma) \equiv 0$ and we define ∇ to be a *bimodule connection* if ∇ is a left-linear connection in the sense of our earlier considerations and additionally satisfies:

$$\nabla(ma) = \nabla(m)a + \sigma(m \otimes_{\mathcal{A}} da). \tag{55}$$

Before we go on with calculating torsion and curvature in this scheme we shall give our choice for σ. For the geometry of triangulations we have a natural candidate for σ. From the property $\pi \circ (1+\sigma) = 0$ we immediately get the following:

$$\sigma\left(\overset{q}{\underset{p}{\nearrow}} \otimes_{\mathcal{A}} \overset{r}{\underset{q}{\nearrow}} \right) = \left\{ \begin{array}{ll} -1 & \text{if } p, r, q \text{ form a triangle} \\ 1 & \text{otherwise} \end{array} \right\}. \tag{56}$$

Of course, we could have chosen some other constant, not necessarily 1, in the second case, however, it is quite convenient to have normalization $\sigma^2 \equiv \mathrm{id}$. By no means is this choice of σ unique, however the study of all possible choices would be an impossible task, besides as we try to keep our considerations as close to the classical case as possible, we use the argument that the above chosen σ is a generic for all triangulations and therefore we were justified in our selection.

An arbitrary bimodule connection has the form:

$$\nabla\left(\overset{q}{\underset{p}{\nearrow}}\right) = \sum_s \overset{p}{\underset{s}{\nearrow}} \otimes_A \overset{q}{\underset{p}{\nearrow}} - \sum_r \sigma\left(\overset{q}{\underset{p}{\nearrow}} \otimes_A \overset{r}{\underset{q}{\nearrow}}\right) + \sum_r \Gamma^{pq}_{rq}\, \overset{r}{\underset{p}{\nearrow}} \otimes_A \overset{q}{\underset{r}{\nearrow}} . \tag{57}$$

The torsion constraint is just as it was in the case of left-linear connections:

$$\Gamma^{pq}_{rq} = -1, \text{for } p, q, r \text{ forming a triangle} . \tag{58}$$

The metric compatibility condition also remains the same:

$$\Gamma^{pq}_{sq} g_{sq} = \overline{\Gamma}^{sq}_{pq} g_{pq} , \tag{59}$$

and by using it and (58) we again reach the conclusion that all r_{pq} are equal to each other.

One of the nice properties of bimodule connections is that they could be extended to the tensor product of forms, for $\omega_1 \otimes_A \omega_2 \in \Omega^1(\mathcal{A}) \otimes_A \Omega^1(\mathcal{A})$ we define the following:

$$\nabla'(\omega_1 \otimes_A \omega_2) = \nabla(\omega_1) \otimes_A \omega_2 + (\sigma \otimes_A \mathrm{id})\,(\omega_1 \otimes_A (\nabla\omega_2)) . \tag{60}$$

In the discussion of bimodule connections for some other geometries [8], it was proposed that the metric should satisfy an additional symmetry relation of the form:

$$(g \otimes_A \mathrm{id}) \circ \nabla' = d \circ g = (\mathrm{id} \otimes_A g) \circ \nabla', \tag{61}$$

where ∇' is as in (60). Surprisingly, it appears that neither equation could be satisfied in our triangulation geometry and a simple counter-example to (61) can be constructed using p, q, s such that they form a triangle.

Finally we may proceed and calculate the curvature tensor, Ricci tensor and the scalar of curvature. To make the formula simpler we introduce the following notation, let δ_{pqr} be 1 if p, r, q form a triangle and 0 otherwise. Then η_{prq} would be $1 - 2\delta_{pqr}$. Using this notation we rewrite the curvature $R = \nabla^2$ as the following map:

$$R\left(\overset{q}{\underset{p}{\nearrow}}\right) = \sum_{r,s\neq q} (\eta_{pqs} - \eta_{pqs}\delta_{qrs}) \left(\underset{p}{\nearrow} q \overset{r}{\nwarrow}\right) \otimes_A \overset{s}{\underset{r}{\nearrow}}$$

$$\sum_{r,s} (-\eta_{pqs} + \eta_{qrs}) \; {}_{p}\!\nearrow^{r}\!\nwarrow^{q} \otimes_A {}_{q}\!\nearrow^{s}$$

$$\sum_{r,s} (\delta_{psq} - \delta_{pqr}\delta_{qrs} + \delta_{pqs}\delta_{rq}) \; {}_{p}\!\nearrow^{r}\!\nwarrow^{s} \otimes_A {}_{s}\!\nearrow^{q} \,.$$

Though ∇ respected the bimodule structure, R is not a bimodule map, it is, however, completely determined by the geometric structure of the simplicial manifold. Using the same procedure as in the case of left-linear connections we get:

$$Ric = \sum_{p,r,s} {}_{q}\!\nearrow^{r} \otimes_A {}_{r}\!\nearrow^{s} \; (\delta_{pqr}\eta_{pqs} - \delta_{pqr}\eta_{pqs}\delta_{qrs} + \delta_{pqr}\delta_{sq}) \,. \tag{62}$$

The scalar of curvature function becomes:

$$R_q = \sum_{p,r} \mu_r \delta_{pqr} \tag{63}$$

and the action would have the following form:

$$S = \sum_{p,r,q} \mu_r \mu_q \delta_{prq} \,. \tag{64}$$

This could be easily rewritten in a more symmetric way:

$$S = \sum_{\triangle} \mu(\triangle) \,, \tag{65}$$

where

$$\mu(\triangle) = \mu_p \mu_q \mu_r \left(\frac{1}{\mu_p} + \frac{1}{\mu_q} + \frac{1}{\mu_r} \right) .$$

This action also seems to be an interesting candidate for further investigations, with two possibilities opening. First, we may treat μ as a dynamical field and investigate it on a fixed simplicial manifold. The other choice is to fix the measure and change the geometry, similarly as in the random triangulation simulations of quantum gravity.

One good candidate for a measure is $\mu_p = n_p/N(n+1)$, where n_p denotes the number of n-dimensional simplices at point p, n is the dimension of the simplicial manifold. and N is the total number of n-dimensional simplices. Clearly,

$$\int 1 = \sum_p \mu_p = \sum_p \frac{n_p}{N(n+1)} = 1,$$

as the integration just counts (locally) n-dimensional simplices.

A further investigation of the proposed action and the properties of the resulting model would certainly prove worthwhile. Especially the most interesting would be the critical behavior of the determined model and its relations with the models of discrete gravity constructed and tested up to now.

5. Conclusions

The noncommutative differential geometry is a powerful tool, which has we have used to study the simplicial manifold from the differential geometry point of view.

Our main task was to investigate the linear connections for this models in hope that we might obtain a satisfactory action candidate for simplicial gravity. Comparing the three possible approaches we reach the conclusion that neither of the proposed definitions for linear connections is flawless. Only for strictly *bimodule* objects, *i.e.* bimodule connections we were able to get the action, which depends only on the *metric* in the general sense. This actions are definitely worth further investigations, in particular it would be interesting to find what is their relation with the gravity actions obtained as a discretized approximation of the continuous Hilbert–Einstein action.

We have shown also that the metric, understood as a certain bimodule map with some additional properties, agrees with picture of the metric arising from the Dirac operator, indeed in both approaches we were able to sort out of the metric its *distance* and its *measure* part. Note, that the additional symmetry constraint on the metric, as proposed by some authors cannot be introduce in case of discrete geometry.

Simplicial manifolds are not only a nice model of discretized geometry, relevant for testing physical models but they might be a candidate for the model of space-time structure at Plank scale lengths. Therefore the investigation of gravity models and their geometry is important not only for testing various definitions of linear connections but also for our understanding of fundamental structure of space-time.

The author would like to thank Mario Paschke for comments and remarks.

Appendix

Differential algebra on simplicial manifolds

We present here the precise mathematical description of the differential algebra on a simplicial manifold. As a first step we remind the construction of the universal differential envelope of the algebra $\mathcal{A}$ of complex valued functions on the lattice made of vertices M.

Let $\Omega^n\mathcal{A}$ be the vector space of maps from $M^{n+1} \to \mathbf{C}$, *i.e.* functions of $n+1$ variables on M, which vanish on any subset of M^{n+1} containing a two-dimensional diagonal:

$$\mathrm{diag}(M \times M) = \{m, m\} \in M \times M, \quad m \in M.$$

Of course, $\Omega^0\mathcal{A} = \mathcal{A}$. Then let $\Omega\mathcal{A}$ be a direct sum of all $\Omega^n\mathcal{A}$ from $n = 0$ to $n = \infty$ (an inductive limit). One introduces the following multiplication law between elements of $\Omega\mathcal{A}$, for $f \in \Omega^p\mathcal{A}$ and $g \in \Omega^w\mathcal{A}$ we define $fg \in \Omega^{p+w}\mathcal{A}$ to be the following function:

$$(fg)(a_0, \ldots a_{p+w}) = f(a_0, \ldots, a_p)g(a_p, \ldots, a_{p+w}).$$

It remains to verify that fg satisfies the additional condition. Indeed, if we assume that for some $0 \leq i \leq p+w$ we have $a_i = a_{i+1}$ (which means that then the arguments belong to a subset of M^{p+w+1} containing the two-dimensional diagonal), we would have either f vanishing (for $i \leq p-1$) or g vanishing (for $i \geq p+1$).

Of course, the above product is associative and non-commutative (apart from the case when f and g are both from $\mathcal{A}$). The differential structure on $\Omega\mathcal{A}$ is given by the external derivative operator d, for $f \in \Omega^p\mathcal{A}$ we define $df \in \Omega^{p+1}$:

$$df(a_0, \ldots, a_{p+1}) = \sum_{i=0}^{p+1}(-1)^i f(a_0, \ldots, \hat{a}_i, \ldots, a_{p+1}). \tag{66}$$

where $\hat{a}_i$ denotes that we throw away the i-th entry (since f is a function of only $p+1$ arguments). The operator d satisfies the graded Leibniz rule, for $f \in \Omega^p\mathcal{A}$ and $g \in \Omega^w\mathcal{A}$ we have:

$$d(fg) = (df)g + (-1)^p f(dg),$$

the proof of which we present below:

$$\begin{aligned} d(fg)(a_0, \ldots, a_{p+w+1}) &= \textstyle\sum_{i=0}^{p+w+1}(-1)^i(fg)(a_0, \ldots, \hat{a}_i, \ldots, a_{p+w+1}) \\ &= \textstyle\sum_{i=0}^{p}(-1)^i f(a_0, \ldots, \hat{a}_i, \ldots, a_{p+1})g(a_{p+1}, \ldots, a_{p+w+1}) \\ &+ \textstyle\sum_{i=0}^{w+1}(-1)^{i+p+1} f(a_0, \ldots, a_p)g(a_p, \ldots, \hat{a}_{i+p}, \ldots, a_{p+w+1}) = \ldots \end{aligned}$$

to recover the desired expression we must add to the last component of the sum the following term:

$$(-1)^p f(a_0, \ldots, a_p) g(a_{p+1}, \ldots, a_{p+w+1})$$

and its opposite, $(-1)^{p+1} f(a_0, \ldots, a_p) g(a_{p+1}, \ldots, a_{p+w+1})$ (so that they cancel together) to the first component. Then, grouping the elements in the sums together we obtain:

$$\ldots = (df)g + (-1)^p f(dg).$$

The usual complex conjugation on $\mathcal{A}$ extends to the universal algebra $\Omega\mathcal{A}$ if we define, for $f \in \Omega^p\mathcal{A}$:

$$(f^\star)(a_0, a_1 \ldots, a_{p-1}, a_p) = (-1)^p \bar{f}(a_p, a_{p-1}, \ldots, a_1, a_0),$$

so that $(df)^\star = (-1)^p d(f^\star)$ for a form f of degree p.

The universal differential algebra is, however, too big to describe the simplicial geometry. In fact, it makes use only of 0-dimensional structures, vertices, and does not rely on the higher-dimensional objects. It is useful for the practical purposes coming from universality property. In the case of the simplicial manifold it is easier to construct an universal object and then proceed with some quotient construction. We shall apply this procedure to the simplicial geometry.

Simplicial ideals

Let us consider the subset of $\Omega^p\mathcal{A}$ consisting of all functions, which vanish on all points $\{a_0, \ldots, a_p\} \in M^{p+1}$ such that all points are different from each other and belong to one simplex of dimension p. Of course, such functions form a vector subspace of $\Omega^p\mathcal{A}$ and, moreover, this subspace remains invariant under multiplications by elements of $\mathcal{A}$ from both sides, hence it is a subbimodule of $\Omega^p\mathcal{A}$ over $\mathcal{A}$, which we will denote as $\mathcal{M}^p$. Clearly, $\mathcal{M}^0 = 0$, for k greater then the dimension of the manifold we set $\mathcal{M}^k = \Omega^k\mathcal{A}$. We shall now demonstrate that $d\mathcal{M}^p \subset \mathcal{M}^{p+1}$. Let us take $f \in \mathcal{M}^p$ and evaluate df on points $\{a_0, \ldots, a_{p+1}\}$. Due to the definition (66) it is sufficient to check that each component of the sum vanishes, *i.e.*:

$$f(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{p+1}) = 0.$$

This is true, as all points $\{a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{p+1}\}$ are different from each other by assumption. From the construction of simplices we know that if $\{a_0, \ldots, a_p\}$ belongs to a $p+1$-dimensional simplex then $\{a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_p\}$ belong to one of its wall's, so to a p dimensional simplex. Since f belongs to $\mathcal{M}^p$ the expression above vanishes and indeed $df \in \mathcal{M}^{p+1}$. It remains to verify that the product of two forms f and g of degrees p and w, respectively, lies in $\mathcal{M}^{p+w}$ whenever one of them is in the $\mathcal{M}^i$. Taking $p+w+1$ points, different from each other and belonging to a $p+w$ dimensional simplex, we know that the first $p+1$ of them (or the last $w+1$) are again different from each other and belong to a p dimensional (or w-dimensional, respectively) subsimplex. Therefore the product of f and g will vanish provided that either f or g are in $\mathcal{M}^i$. Using similar simple arguments we observe that $(\mathcal{M}^i)^\star = \mathcal{M}^i$.

The last observation finishes our construction as we have shown that the direct sum:

$$\mathcal{I} = \oplus_{i=0}^{\infty} \mathcal{M}^i,$$

is a differential (star) ideal of the algebra $\Omega\mathcal{A}$, *i.e.*, $d\mathcal{I} \subset \mathcal{I}$. We can construct the quotient algebra:

$$\Omega_s(\mathcal{A}) = \Omega/\mathcal{I},$$

which will have a well-defined star and differential structure. The projection $\pi : \Omega\mathcal{A} \to \Omega_s(\mathcal{A})$ is a morphism of differential algebras. Below we state some important properties of $\Omega_s(\mathcal{A})$:

- $\Omega_s^0(\mathcal{A}) = \mathcal{A}$
- $\Omega_s(\mathcal{A})$ is finite dimensional, $\Omega_s^k(\mathcal{A}) = 0$ for k greater than the dimension of the simplicial manifold.
- There exist a natural embedding: $i : \Omega_s(\mathcal{A}) \hookrightarrow \Omega\mathcal{A}$, constructed in the following way. If $\omega \in \Omega_S^p(\mathcal{A})$ we take $i(\omega)$ to be defined as the following function:

$$i(\omega)(a_0,\ldots,a_p) = \begin{cases} F(a_0,\ldots,a_p) & \text{if } \{a_0,\ldots,a_p\} \text{ are vertices of a } p\text{-dimensional simplex, different from each other, and } F \in \pi^{-1}(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

- There exists a graded trace, *i.e.*, a map $\Omega_s^n(\mathcal{A}) \to \mathbf{C}$, such that $\int df = 0$.

The first two observations follow directly from the construction, so we concentrate on the remaining two. It is easy to see that the definition of $i(\omega)$ does not depend on the choice of $F \in \pi^{-1}(\omega)$, so that i is a well-defined map. Of course, $\pi \circ i = \mathrm{id}$, however, clearly it is neither an algebra nor a differential map ($d \circ i \neq i \circ d$). However, if we introduce a multiplication rule on the image of i just as the image of the multiplication in the quotient:

$$i(\omega)i(\rho) := i(\omega\rho),$$

we easily obtain an algebra structure on $i(\Omega_S(\mathcal{A}))$. This product is quite simple and we shall discuss it in a more detailed way for the forms which generate $\Omega_s(\mathcal{A})$. In the same way we could transport the external derivative d, defining $d(i(f)) = i(df)$.

From now on, we shall very often work with the image of a differential form under the map i rather than with the element of the quotient, using the image of multiplication and d.

The construction of the graded trace is as follows. First, define the map $\int : \Omega^n \mathcal{A} \to \mathbf{C}$:

$$\int f = \sum_{\mathrm{A}} \sum_{\sigma} (-1)^{\sigma} i(f) \left(a_{\sigma(0)}, \ldots, a_{\sigma(n)} \right),$$

where σ denotes the permutation of $n+1$ elements and the first sum is over all n-dimensional simplices.

We shall prove now that the above defined operation has all properties of a graded trace. Clearly, it is linear and $\int f^* = \overline{\int f}$. Calculate $\int df$:

$$\int df = \sum_{\mathrm{A}} \sum_{\sigma} (-1)^{\sigma} i(df) \left(a_{\sigma(0)}, \ldots, a_{\sigma(n)} \right)$$
$$= \sum_{\mathrm{A}} \sum_{\sigma} (-1)^{\sigma} \sum_{i=0}^{n} (-1)^i f \left(a_{\sigma(0)}, \ldots, a_{\sigma(i-1)}, a_{\sigma(i+1)}, \ldots, a_{\sigma(n)} \right) = \ldots .$$

Now, let us choose among all simplices A and B, such that they are glued along an $n-1$-dimension simplex $W = \{w_1, \ldots, w_{n-1}\}$. We shall now investigate all elements of the sum, which have the form $f(w_\tau(0), \ldots, w_{\tau(n-1)})$ for a certain permutation τ of $n-1$ elements. Obviously, one contribution comes from simplex A and another one from B:

$$\sum_i \sum_{\sigma_A}{}' (-1)^{\sigma_A} \quad (-1)^i f \left(w_{\tau(0)}, \ldots, w_{\tau(n-1)} \right)$$
$$+ \sum_i \sum_{\sigma_B}{}' (-1)^{\sigma_A} (-1)^i f \left(w_{\tau(0)}, \ldots, w_{\tau(n-1)} \right),$$

where the prime over sum indicate that we are summing only over such σ_A that satisfy: (for a given i):

$$\{a_{\sigma_A(0)}, \ldots, a_{\sigma_A(i-1)}, a_{\sigma_A(i+1)}, \ldots, a_{\sigma_A(n)}\} \equiv \{w_\tau(0), \ldots, w_{\tau(n-1)}\}$$

with the same property, respectively for the sum over σ_B.

Now we can calculate the sums. For the simplex A the signs $(-1)^{\sigma_A}$ and $(-1)^i$ give together a sign, which corresponds to the orientation of $\tau(W)$ induced by A, so after summing all elements we get n multiplied by this sign. Similarly, for the second sum we get the same but with the sign of orientation of W induced by B. However, since the manifold is orientable, the orientation induced on $\tau(W)$ by A is opposite to the one induced by the orientation of B. hence this two sums add up to zero. Because we can use this argument for all subsimplices W of all pairs A, B this proves the desired property of the graded trace.

Differential algebra and links

Before we complete our construction with some further quotient construction (to eliminate overcounting of simplices) we shall give here a relation between the above formal derivation of the differential algebra and the picture of links.

We already know that the generators of the algebra $\mathcal{A}$ could be identified with points with the help of Kronecker delta functions: $p(s) = \delta_{ps}$. We prove that this extends to higher order forms.

Let f be an element of $\Omega^p_S(\mathcal{A})$ and Q be a p-dimensional simplex $\{q_0, \ldots, q_p\}$. Then $i(f)$ could be written in a unique way as:

$$i(f) = \sum_Q \sum_\sigma C^Q_\sigma \ \delta_{\sigma(Q)}, \tag{67}$$

where σ is a permutation of $p+1$ elements, $\mathbf{C}^Q_\sigma$ are complex numbers and $\delta_{\sigma(Q)}$ is the multiple Kronecker delta function:

$$\delta_{\sigma(Q)}(b_0, \ldots, b_p) = \delta_{q_{\sigma(0)} b_0} \delta_{a_{\sigma(1)} b_1} \cdots \delta_{a_{\sigma(p-1)} b_{p-1}} \delta_{a_{\sigma(p)} b_p}.$$

The proof is straightforward. First, by construction we know that $i(f)$ vanishes on the set of arguments that are not different vertices of a p-dimensional simplex. Let us fix a simplex Q, and look at $i(f)$ on the subset Q^{p+1}. Then, the only non-vanishing values of $i(f)$ are for permutations of different vertices. After summing over all p-dimensional simplices Q we obtain the formula (67).

We see now that the functions $\delta_{\sigma(Q)}$ generate the image of $\Omega_S(\mathcal{A})$. The multiplication between them, transported by i from $\Omega_S(\mathcal{A})$ is quite simple:

$$\delta_{\sigma(Q)} \delta_{\tau(P)} = \begin{cases} \delta_{\sigma(Q)+\tau(P)} & \text{if } P \cup Q \text{ is a } p+q\text{-simplex} \\ 0 & \text{otherwise} \end{cases} \tag{68}$$

We introduce now in a formal way a link. Let l be the following one-form, for any points p and q in a 1-dimensional simplex, we define the link one-form:

$$l^q_p(b_0, b_1) = \delta_{pb_0} \delta_{qb_1}.$$

We have used here the identification of $\Omega_S(\mathcal{A})$ with its image under i and, in fact, we have defined only $i(l^p_q)$, however, this is sufficient as i is injective. In our "pictorial" notation:

$$l^k_p = \overset{q}{\underset{p}{\nearrow}} .$$

Next we shall use the result (67) and the presentation of one-forms by links and the rules for multiplication (68). Since any multiple Kronecker delta function could be written as products of functions for two entries (which are used to define the link) we could rewrite (67) using the links.

Then every p-form f could be written as:

$$f = \sum_Q \sum_\sigma C^Q_\sigma \overset{q_{\sigma(1)}}{\underset{q_{\sigma(0)}}{\nearrow}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \overset{a_{\sigma(p)}}{\underset{a_{\sigma(p-1)}}{\nearrow}} . \tag{69}$$

This follows directly from (67) and the rules for product of forms (68). Now, it is easy to verify the rules for the differentiation of zero and one-forms, which we have presented earlier in the language of links. As an example we shall derive here formally the action of d on a single link.

Let l^q_p be this link. First, let us take $i(l^q_p)$ being the Kronecker delta function, then calculate d of it, in the universal algebra $\Omega\mathcal{A}$:

$$\left(d\delta_{\{p,q\}}\right)(b_0, b_1, b_2) = \delta^{b_1}_p \delta^{b_2}_q - \delta^{b_0}_p \delta^{b_2}_q + \delta^{b_0}_p \delta^{b_1}_q .$$

Let us proceed the quotient operation. The nontrivial contributions to $\pi(d\delta_{\{p,q\}})$ come only from such functions, which do not vanish on $\{b_0, b_1, b_2\}$ being vertices of a triangle T. Hence, we can write:

$$\pi\left(d\delta_{\{p,q\}}\right) = \pi\left(\sum_{r:\{r,p,q\}\in T} \left(\delta^{b_0}_r \delta^{b_1}_p \delta^{b_2}_q - \delta^{b_0}_p \delta^{b_1}_r \delta^{b_2}_q + \delta^{b_0}_p \delta^{b_1}_q \delta^{b_2}_r \right) \right) .$$

Now, we observe that the components of the sum are actually in the image of i and each δ function corresponds to a link. Using the previously introduced notation for the products of two links we obtain:

$$d \overset{q}{\underset{p}{\nearrow}} = \sum_{r:\{r,p,q\}\in T} \left({}_{r}\!\nearrow p \nwarrow^{q} + {}_{p}\!\nearrow q \nwarrow^{r} - {}_{p}\!\nearrow r \nwarrow^{q} \right) . \tag{70}$$

To finish our construction we have to add one more relation. Observe that the vector space of p-forms has a dimension equal to $N_p(p+1)!$, where N_p is the number of p-simplices on our manifold. As a bimodule is generated by $N_p(p-1)!$ elements. Therefore, only for $p=1$ and $p=2$ the number of simplices corresponds to the number of forms of the same dimension. For $p>2$ we could have different p-forms with the same "end-points".

Our task would be to reduce the differential algebra in such a way that the bimodule of p-forms is generated by N_p elements. Hence, we would have a one-to-one correspondence between p-simplices and p-forms. Since for $p=1,2$ this is already the case, the relations would start on the level of three-forms.

For every p-simplex $A=\{a_0,\ldots,a_p\}$ we introduce the set of p-forms:

$$\mathcal{J}^p_A = \overset{a_1}{\underset{a_0}{\nearrow}} \otimes_{\mathcal{A}} \cdots \overset{a_p}{\underset{a_{p-1}}{\nearrow}} - (-1)^{\sigma} \overset{a_{\sigma(1)}}{\underset{a_0}{\nearrow}} \otimes_{\mathcal{A}} \cdots \overset{a_p}{\underset{a_{\sigma(p-1)}}{\nearrow}}, \tag{71}$$

where σ is a permutation of $p-1$ elements. Of course, both components of the sum are different (for $\sigma \neq \mathrm{id}$), so $\mathcal{J}_S$ is nonempty. Let $\mathcal{J}^p$ be the subbimodule of $\Omega^p_S(\mathcal{A})$ generated by all $\mathcal{J}^p_S$. We shall prove that the direct sum of all $\mathcal{J}^p$ is a differential ideal of $\Omega_S(\mathcal{A})$.

First, let us take $f \in \Omega^w_S(\mathcal{A})$ and $g \in \mathcal{J}^p$. Because of linearity we might restrict ourselves to the case when $g \in \mathcal{J}^p_A$ and f is just one of the generators (so that $i(f) = \delta_{\sigma Q}$ for some simplex Q and permutation σ. The product, fg will, of course, vanish unless $\sigma(q_w) = a_0$ and $A \cup Q$ are a $w+p$-dimensional simplex. In the latter case it is quite obvious that the product would be of the form (71) and therefore in $\mathcal{J}^{w+p}_{A\cup Q}$. So the direct sum of all $\mathcal{J}^p_S$ (summed both for all $p=1,\ldots,n$ and all p-dimensional simplices S) would be an ideal of $\Omega_S(\mathcal{A})$.

It remains to verify that $\mathcal{J}$ is a differential ideal, *i.e.* $d\mathcal{J}^p \subset \mathcal{J}^{p+1}$. Again, by linearity, we need to prove it for $\mathcal{J}^p_A$, for a simplex A. Let us calculate:

$$d\left(\overset{a_1}{\underset{a_0}{\nearrow}} \otimes_{\mathcal{A}} \cdots \overset{a_p}{\underset{a_{p-1}}{\nearrow}} - (-1)^{\sigma} \overset{a_{\sigma(1)}}{\underset{a_0}{\nearrow}} \otimes_{\mathcal{A}} \cdots \overset{a_p}{\underset{a_{\sigma(p-1)}}{\nearrow}} \right) = \ldots$$

of course, to get the result we must use both the graded Leibniz rule as well as the rule for the action of d on a single link (70). It is a simple exercise to see that the action of d would give us a sum, consisting of the above expression multiplied by a link at front of it, at the end, or with one link in the middle split into the product of two links (start and end-point remaining the same). Since the same applies to each of the two components of the above sum, the general form of the result is also the same, and therefore the result would lie in the sum of $\sum_Q \mathcal{J}_Q^{p+1}$ for such $p+1$-simplices Q that contain A. Knowing that $\mathcal{J}$ is a differential ideal we may construct the quotient $\Omega^*(\mathcal{A}) = \Omega_S(\mathcal{A})/\mathcal{J}$, which is the differential algebra we define as the one corresponding to the geometry of the simplicial manifold. Of course, the difference between $\Omega_S(\mathcal{A})$ and $\Omega^*(\mathcal{A})$ starts only at the level of three forms, so in most physical applications (which end at the level of two-forms) there is no necessity to distinguish between them.

Linear representation and Dirac operator approach

Quite often in noncommutative geometry the starting point of the approach is not the differential structure, on which one build metric and other structures but rather a representation of the algebra on some Hilbert space and a Dirac operator, which gives both the metric and differential structure. Here we should briefly present the relevant construction for the simplicial manifold.

Let $\mathcal{H}$ be the Hilbert space of dimension $\#\mathcal{A}$ (we take the number of simplices to be finite), and let Ψ_p denote a vector in $\mathcal{H}$ associated with point p. Then there exist a natural representation of the algebra $\mathcal{A}$ on the Hilbert space given by:

$$\pi(f)\Psi_p = f(p)\Psi_p.$$

Let D denote an operator on $\mathcal{H}$, which does not commute with the algebra and is self-adjoint. Then, we may define the differential of the function as a commutator $i[D, f]$. Let's calculate it for the generators of the algebra:

$$\begin{aligned}[D,p]\Psi_q &= Dp\Psi_q - pD\Psi_q \\ &= \delta_{pq}D\Psi_q - pD_q^w\Psi_w \\ &= (\delta_{pq}D_q^w - \delta_{pw}D_q^w)\Psi_w\,. \end{aligned} \tag{72}$$

To verify that it corresponds to our previous formulae, let us define a link operator L_{pq} as the following isometry:

$$L_{pq}\Psi_w = \delta_{wq}\Psi_p.$$

Then, we can rewrite (72) as:

$$\left(\sum_s D_p^s L_{sp} - \sum_s D_s^p L_{ps}\right)\Psi_q, \tag{73}$$

which agrees with the abstract construction (the only difference is that the links are rescaled). Of course, *a priori* the operators L_{pq} might be constructed for any pair of points p, q and we must add the some information about the actual geometry of the simplicial manifold — either by using only some of them (for p, q connected by a link) or, preferably, assume that the Dirac operator D has non-zero entries D^p_q only and only if p is connected with q. The link operators L_{pq} form are an extension of $\mathcal{A}$:

$$L_{pq}L_{rs} = L_{ps}\delta_{qr} + s\delta_{qr}\delta_{ps} .$$

Equivalently, we may use $L_{pp} \equiv p$, then the above identity becomes simpler $L_{pq}L_{rs} = L_{ps}\delta_{qr}$.

Since D is self-adjoint and $L^\dagger_{pq} = L_{qp}$ we see that dp is, as an operator, self-adjoint.

Can we now recover the metric from the Dirac operator? Let us take two points p and q, $p \neq q$, which simultaneously we may interpret as operators on $\mathcal{H}$. Then, following the classical correspondence we might look for the following identity:

$$\int g(dp, dq) = \mathrm{Tr}\left([D, p][D, q]\right). \tag{74}$$

Of course, since our Hilbert space is finite dimensional we are using the standard trace. Now, the left-hand side is as follows:

$$\int g(dp, dq) = \mu_p g_{pq}$$

whereas the right-hand side is equal:

$$\mathrm{Tr} \sum_s \left(D^s_p D^q_r L_{sq} - D^p_s D^q_s L_{pq} + D^p_q D^q_r L_{pr} \right) = \ldots$$

Since Tr L_{wz} is zero unless $w = z$ when it is 1, we obtain:

$$\ldots = 2D^q_p D^p_q .$$

Now we can see that having the Dirac operator D we recover the metric as it was defined earlier. Moreover, as the above expression is symmetric in p and q (note that since D is self-adjoint it must also be real) we obtain the relation $\mu_p g_{pq} = \mu_q g_{qp}$, which we have postulated.

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