THE TIME-ASYMMETRIC FOKKER-TYPE INTEGRALS AND THE RELATIVISTIC MECHANICS ON THE LIGHT CONE

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The class of two-particle relativistic models which are described by the time-asymmetric Fokker-type integral of general form is considered. The manifestly covariant description of such models is constructed in the framework of the canonical formalism with constraints. By means of certain gauge fixing, the time-asymmetric models are reduced to the Bakamjian–Thomas model supplemented by a space-time interpretation. The corresponding two-body problem is reduced to quadratures.

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Introduction

Among various approaches to the *relativistic direct interaction theory* (RDIT), the formalism of Fokker-type action integrals is most closely related with field-theoretical descriptions of particle interaction [1–4]. This feature permits to construct models of particle systems with interactions of transparent physical meaning. Unfortunately the Fokker formalism gives rise to integral- or difference-differential equations of motion which complicate to great extent an analysis of the Fokker-type models.

On the other hand, mathematically simpler approaches to RDIT, namely those which provide a prediction of particle evolution determined by Newton-like equations, lead to phenomenologically, rather than theoretically, substantiated models [5–7].

There exists a relativistic two-particle model which possesses advantages of both of the above classes of approaches to RDIT. This model was originated by Fokker in the framework of his formalism [1] and it was later elaborated in more detail by Staruszkiewicz [8], Rudd and Hill [9] and Künzle [10]. Physically their model describes the following particle interaction mediated by a massless linear vector (electromagnetic) field: the advanced field of the first particle acts on the second particle and the retarded field of the second particle acts on the first particle. In this model a one-to-one correspondence of points of two particle world lines appears naturally, namely, of those points which are separated by an isotropic interval. This correspondence allows to reduce the Fokker integral to a single-time action, *i.e.* to reformulate the model into the framework of the Lagrangian and then Hamiltonian formalism [11, 10] and, finally, to solve the equations of motion [9, 10, 12].

The structure of the above model points the way to its generalizations. The same 1-to-1 correspondence arises in the Fokker integrals of the more general form, namely, in those integrals whose integrand contains the retarded or advanced Green's function of the d'Alembert equation. Fokker actions of such a kind produce a wide class of relativistic two-particle models [13] which can be called (following the Refs [12, 14]) *time-asymmetric models*. Some of them permit a field-theoretical interpretation of interaction and thus can be of special interest in physical applications.

There exist other ways leading to time-asymmetric models which are based on mathematical rather than physical considerations.

One of these was proposed by Künzle [10, 15] in the framework of manifestly covariant predictive mechanics. He has shown that the time-asymmetric models form the only class of relativistic models which do not fall under the purview of the well-known *no-interaction theorem* [16].

Another such approach starts from the 3-dimensional Lagrangian formalism in various forms of relativistic dynamics. Usually the form of relativistic dynamics is defined by means of a family of spacelike or isotropic hypersurfaces determining the simultaneity relation in the Minkowski space $\mathbb{M}_4$ [17]. A possible generalization of forms of relativistic dynamics is the isotropic form proposed in Ref. [13]. It is based on the family of isotropic hypersurfaces which are the future- or past-oriented light cones with the vertices located on one of the particle world lines. In this case the simultaneity relation is determined not in whole $\mathbb{M}_4$ but only on points of the particle world lines. This relation establishes for them a one-to-one correspondence which is the same as in time-asymmetric models. The attractive feature of the isotropic form is the ability to construct interaction Lagrangians which depend only on the first order derivatives (unlike other forms of dynamics which require dependence of Lagrangians on higher derivatives of all orders up to infinity [17]).

It is shown in Ref. [13] that all the ways mentioned above lead to different descriptions of the same class of the time-asymmetric models. The existence of simple relations between these descriptions unifies them into the common approach which can be referred to as the *relativistic two-particle mechanics on the light cone*.

In the case of two-dimensional space-time the light cone degenerates to the light front and the mechanics on the light cone becomes the front form of two-dimensional relativistic dynamics. The general formulation of the latter is proposed in Ref. [18] in the frameworks of both the Lagrangian and Hamiltonian formalisms. The dynamics of time-asymmetric models in this case is rather simple and was well studied with several examples both on classical [8, 9, 14, 19] and quantum [20–22] levels.

In the more realistic case of the 4-dimensional Minkowski space the dynamics of concrete time-asymmetric models has been studied far less. Indeed, the above mentioned vector model is the only model of such a kind which was considered in literature. It is complicated enough, and its mechanical analysis given in Refs [10, 12] seems to be work of art rather than application of some standard method. An analysis of other time-asymmetric models, especially those which permit a field-theoretical interpretation, is not expected to be simpler. Nevertheless just such models can be useful in physical applications and they need further elaboration.

For this purpose the convenient calculation scheme which allows the study of time-asymmetric models is proposed here. Following this scheme we specify the time-asymmetric model by means of the choice of relevant (for instance, physically tractable) time-asymmetric Fokker-type action integral. Then, we reformulate equivalently the chosen model to the well-known Bakamjian–Thomas (BT) model [23] and, finally, reduce it to quadratures by means of the standard Hamilton–Jacobi method. Because the BT model uses the noncovariant canonical centre-of-mass variables, the proposed scheme includes the relation of covariant particle positions with canonical variables. Having the solution of Hamiltonian equations and using this relation we can build particle world lines in the Minkowski space.

The scheme is based on the results of a previous work [13] (in collaboration with Tretyak), and it consists in successive reformulations of time-asymmetric models into the frameworks of various formalisms: manifestly covariant Lagrangian mechanics, Hamiltonian formalism with constraints and three-dimensional Hamiltonian mechanics (*i.e.* BT model).

The Hamiltonian formalism with constraints is the most important link of the scheme. It is based on the two Poincaré-invariant first class constraints. One of them is purely kinematic and it determines the simultaneity on the light cone. The another constraint is the mass-shell constraint which determines the dynamics of the system. The general structure of this constraint is found and the connection of its concrete form with a choice of the original Fokker integral (or Lagrangian) is established.

Calculatingly, the hamiltonization of the time-asymmetric model implies some algebraic problem (namely, the set of three equations) to be solved. It is done for certain models exactly. In other cases some approximation method (for example, expansions in a coupling constant) can be used. Except this problem all steps of the proposed scheme are equally applicable to the arbitrary time-asymmetric model.

We note that not only the Fokker approach, but all the intermediate formalisms entering into the proposed scheme, can be considered as original ones for the formulation of time-asymmetric models. For instance, in the framework of canonical formalism with constraints, the dynamics of models is determined by the choice of the mass-shell constraint; within the BT description it is determined by means of the function of the total mass. In any case, the variety of the time-asymmetric models is as wide as the class of functions depending on three arguments. Besides, the possibility to choose different descriptions for the same model is convenient for comparison with other models known in literature.

This paper is organized as follows. In Section 1 the most general form of the Fokker-type action integrals leading to the class of time-asymmetric models is proposed, and a structure of those which correspond to the field-theoretically tractable models is shown. In Section 2 the reformulation of the time-asymmetric models into the framework of manifestly covariant Lagrangian formalism is performed, and Noether's integrals of motion are obtained as consequences of Poincaré-invariance of the models. In Section 3 the transition to the manifestly covariant description in the framework of canonical formalism with constraints is done. In Appendix A the model which permits the exact construction of the mass-shell constraint is represented. The gauge freedom generated by the pair of first class constraints is analyzed in Section 4, and the transition to three-dimensional Hamiltonian description is discussed. In Section 5 the BT description of an arbitrary time-asymmetric model is constructed. The corresponding two-body problem is reduced to quadratures in Section 6.

In Appendix B the free-particle mechanics on the light cone is considered, and its relation to the standard Hamiltonian description of the free-particle system is found.

1. Time-asymmetric Fokker-type action integrals

We start with the most general Fokker-type action integral for a two-particle systems which has the following form [3, 4]:

$$I = -\sum_{a=1}^{2} m_a \int d\tau_a \sqrt{\dot{x}_a^2} - \int\!\!\int d\tau_1 d\tau_2 \, \Phi \,, \tag{1}$$

here m_a $(a = 1, 2)$ is the rest mass of the a-th particle; $x_a^\mu(\tau_a)$ $(\mu = 0, \ldots, 3)$ are the covariant coordinates of the a-th particle on the Minkowski space

$\mathbb{M}_4$; τ_a is an arbitrary evolution parameter on the a-th world line; $x^\mu \equiv x_1^\mu - x_2^\mu$; $\dot{x}_a^\mu \equiv dx_a^\mu/d\tau_a$;

$$\Phi \equiv \sqrt{\dot{x}_1^2}\sqrt{\dot{x}_2^2}\,U(x, u_1, u_2)\,, \tag{2}$$

where U is an arbitrary scalar function of 4-vectors x and $u_a \equiv \dot{x}_a/\sqrt{\dot{x}_a^2}$. We choose the time-like Minkowski metrics, *i.e.*, $\| \eta_{\mu\nu} \| = \mathrm{diag}(+,-,-,-)$, and put the light speed to be unit.

There exists a physically important class of Fokker-type integrals which permit a field-theoretical interpretation of the interaction between particles [3, 4]. For this class the function U describing the interaction mediated by the tensor field of rank n is given by

$$U^{(n)} = g_1 g_2 (u_1 \cdot u_2)^n G(x)\,, \tag{3}$$

where g_a are the charges of particles, and $G(x)$ is the symmetrical Green's function of relevant wave equation.

The choice $n = 1$ and $G(x) = \delta(x^2)$ on the right-hand side (r.-h.s) of Eq. (3) corresponds to the well known Fokker–Wheeler–Feynman action for the electromagnetic interaction [1, 2]. Its time-asymmetric counterpart was introduced by Fokker [1] and later was developed by Staruszkiewicz [8], Rudd and Hill [9]. They proposed to replace the symmetrical Green's function by

$$G_\eta(x) = 2\Theta(\eta x^0)\delta(x^2) \tag{4}$$

which is the retarded (for $\eta = +1$) or advanced ($\eta = -1$) Green's function of d'Alembert equation. The resulting model yields ordinary differential equations of motion which were analysed in Refs [8, 9, 10, 12].

A natural generalization of this model leads to the action integral (1) with the function Φ of the form:

$$\Phi = \tilde{\Phi} G_\eta\,, \tag{5}$$

where $\tilde{\Phi}$ is an arbitrary regular function of the form (2). Let us express this function in a more explicit form.

Since the Green's function (4) does not vanish if and only if

$$x^2 = 0, \quad \eta x^0 > 0\,, \qquad i.e., \qquad \eta x^0 = |\boldsymbol{x}|\,, \tag{6}$$

where $\boldsymbol{x} \equiv (x_i = -x^i)$ $(i = 1, 2, 3)$, the function $\tilde{\Phi}$ does not depend on the scalar argument x^2. Next let us suppose that the action (1), (5) is determined only on timelike world lines (*i.e.*, $\dot{x}_a^2 > 0$) which are parametrized by well defined evolution parameters (*i.e.*, $\dot{x}_a^0 > 0$). Then, using Eq. (6) one can easily prove the following inequalities:

$$\dot{x}_1 \cdot \dot{x}_2 > 0\,, \tag{7}$$

$$\eta \dot{x}_a \cdot x > 0\,, \qquad a = 1,2\,. \tag{8}$$

Finally, the function $\tilde{\Phi}$ can be put into the following form:

$$\tilde{\Phi} = (\dot{x}_1 \cdot x)(\dot{x}_2 \cdot x)\, \Psi\left(\frac{\sqrt{\dot{x}_1^2}}{\eta \dot{x}_1 \cdot x}, \frac{\sqrt{\dot{x}_2^2}}{\eta \dot{x}_2 \cdot x}, \frac{\dot{x}_1 \cdot \dot{x}_2}{(\dot{x}_1 \cdot x)(\dot{x}_2 \cdot x)}\right), \tag{9}$$

where Ψ is an arbitrary regular function of the indicated positive arguments. The general structure of Ψ determines the class of models which we take under consideration. Especially, the choice of Ψ in the form

$$\Psi^{(n)} = g_1 g_2 \left(\frac{\sqrt{\dot{x}_1^2}}{\eta \dot{x}_1 \cdot x} \frac{\sqrt{\dot{x}_2^2}}{\eta \dot{x}_2 \cdot x}\right)^{1-n} \left(\frac{\dot{x}_1 \cdot \dot{x}_2}{(\dot{x}_1 \cdot x)(\dot{x}_2 \cdot x)}\right)^n \tag{10}$$

corresponds to the n-rank tensor generalization of the vector model.

2. Single-time Lagrangian formalism

The Fokker action (1) is parametrically invariant with respect to each of the parameters τ_1 and τ_2. Thus the two of eight functions $x_1^\mu(\tau_1)$, $x_2^\mu(\tau_2)$ to be found (one for each particles) remain undetermined within the variational problem. The structure of the function Φ allows to fix partially this function arbitrariness in natural manner. Let us require of the condition (6) to be identity if $\tau_1 = \tau_2$. This means, firstly, that both world lines are parametrized with a common evolution parameter, for instance, with τ_1; and, secondly, that a simultaneity relation for points of world lines is set. Since the condition (6) can be treated as the equation of a past- or future-oriented light cone (it depends both on what is the value of $\eta = \pm 1$ and what point x_1 or x_2 is chosen to be the vertex of the cone), this parametrization can be called naturally the *single-time description on the light cone.* Following Ref. [11] the Green's function G_η in the second term of the action (1) can be written down in the form:

$$2\,\Theta\left[\eta\left(x_1^0(\tau_1) - x_2^0(\tau_2)\right)\right] \delta\left[\left(x_1(\tau_1) - x_2(\tau_2)\right)^2\right] = \frac{\delta(\tau_1 - \tau_2)}{\left|\dot{x}_2(\tau_2) \cdot \left(x_1(\tau_1) - x_2(\tau_2)\right)\right|}\,. \tag{11}$$

Integrating explicitly this term over τ_2 one can reduce the functional (1) to the single-time action

$$I = -\int d\tau \tilde{L} \tag{12}$$

with the Lagrangian $\tilde{L} \equiv L|_{\mathrm{T}\mathbb{K}}$, where

$$L \equiv \sum_{a=1}^{2} m_a \sqrt{\dot{x}_a^2} + \frac{\tilde{\Phi}}{|\dot{x}_2 \cdot x|} . \tag{13}$$

The Lagrangian $\tilde{L}$ is defined on the first prolongation $\mathrm{T}\mathbb{K}$ of the 7-dimensional configuration manifold $\mathbb{K} \subset \mathbb{M}_4^2 \equiv \mathbb{M}_4 \times \mathbb{M}_4$ described by the equation (6). The corresponding variational problem gives rise to second order differential equations and thus the transition to the usual Hamiltonian description is straightforward.

The action (12) is a parametrically invariant functional with respect to the common evolution parameter τ. Hence the Lagrangian $\tilde{L}$ (as well as L) is a first order homogeneous function in particle velocities. This fact together with the condition (6) enables to remove redundant degrees of freedom which correspond to the time variables x_1^0, x_2^0. An explicit elimination of these variables, both partial and complete (the latter leads to the ordinary Lagrangian description in the 6-dimensional configuration space) breaks the manifest covariance of the description and makes the hamiltonization procedure cumbersome.

It is more convenient to renew a manifest covariance by means of transition to the Lagrangian description on the 8-dimensional configuration space $\mathbb{M}_4^2$. For this purpose an unconditional extremum problem is modified in favour of an equivalent conditional extremum problem of the action

$$I' = -\int d\tau \, (L + \lambda x^2) \tag{14}$$

with the Lagrangian (13) defined on $\mathrm{T}\mathbb{M}_4^2$. Here the Lagrangian multiplier λ is introduced to take into account the condition (6) as the holonomic constraint (the boundary constraint $\eta x^0 > 0$ is meant also).

For the sake of construction of the Hamiltonian description it is desirable to put the Lagrangian (13) in a more convenient form. Let us parametrize the space $\mathbb{M}_4^2$ by the collective variables

$$y^\mu \equiv \tfrac{1}{2}(x_1^\mu + x_2^\mu) , \qquad x^\mu \equiv x_1^\mu - x_2^\mu \tag{15}$$

(the external and internal variables respectively) in terms of which

$$x_a^\mu = y^\mu + \tfrac{1}{2}(-)^{\bar{a}} x^\mu , \qquad a = 1, 2 , \quad \bar{a} = 3 - a . \tag{16}$$

Taking into account the inequalities (8) and the differential consequence of the constraint (6), *i.e.*,

$$\dot{x}\cdot x = 0\,, \tag{17}$$

the following positive function can be introduced and written down in a few ways:

$$\theta \equiv \eta\dot{y}\cdot x = \eta\dot{x}_1\cdot x = \eta\dot{x}_2\cdot x > 0\,. \tag{18}$$

Then the Lagrangian (13) takes the following form:

$$L = \theta F(\sigma_1, \sigma_2, \delta), \tag{19}$$

$$F \equiv \sum_{a=1}^{2} m_a\sigma_a + \Psi(\sigma_1, \sigma_2, \delta)\,, \tag{20}$$

where an interacting term Ψ, and thus the total expression F, may be an arbitrary function of positive arguments:

$$\sigma_a \equiv \frac{\sqrt{\dot{x}_a^2}}{\theta} = \frac{\sqrt{\dot{y}^2 + (-)^{\bar{a}}\dot{y}\cdot\dot{x} + \frac{1}{4}\dot{x}^2}}{\theta}, \qquad a = 1,2\,,$$

$$\delta \equiv \frac{\dot{x}_1\cdot\dot{x}_2}{\theta^2} = \frac{\dot{y}^2 - \frac{1}{4}\dot{x}^2}{\theta^2}. \tag{21}$$

Especially, for Eq. (10) we have

$$\Psi^{(n)} = g_1g_2(\sigma_1\sigma_2)^{1-n}\delta^n. \tag{22}$$

Poincaré-invariance of both the Lagrangian (19) and the constraint (6) leads to the existence of ten Noether's integrals of motion. These are the total momentum P_μ of the system and the angular momentum tensor

$$J_{\mu\nu} = y_\mu P_\nu - y_\nu P_\mu + \Omega_{\mu\nu}, \tag{23}$$

where

$$\Omega_{\mu\nu} \equiv x_\mu w_\nu - x_\nu w_\mu, \tag{24}$$

$$P_\mu \equiv \frac{\partial L}{\partial\dot{y}^\mu} = \left(\frac{F_1'}{\sigma_1} + \frac{F_2'}{\sigma_2} + 2F_\delta'\right)\frac{\dot{y}_\mu}{\theta} + \frac{1}{2}\left(\frac{F_1'}{\sigma_1} - \frac{F_2'}{\sigma_2}\right)\frac{\dot{x}_\mu}{\theta} + (F - \sigma_1F_1' - \sigma_2F_2' - 2\delta F_\delta')\eta x_\mu\,, \tag{25}$$

$$w_\mu \equiv \frac{\partial L}{\partial\dot{x}^\mu} = \frac{1}{2}\left(\frac{F_1'}{\sigma_1} - \frac{F_2'}{\sigma_2}\right)\frac{\dot{y}_\mu}{\theta} + \frac{1}{4}\left(\frac{F_1'}{\sigma_1} + \frac{F_2'}{\sigma_2} - 2F_\delta'\right)\frac{\dot{x}_\mu}{\theta}\,, \tag{26}$$

and $F_a' \equiv \partial F/\partial\sigma_a$ $(a = 1,2)$, $F_\delta' \equiv \partial F/\partial\delta$.

Besides, the Lagrangian (19) satisfies the identity:

$$\dot{y}\cdot P + \dot{x}\cdot w - L = 0\,, \tag{27}$$

which is the consequence of parametric invariance of the action (14).

3. Canonical formalism with constraints

The Lagrangian description in the configuration space $\mathbb{M}_4^2$ enables a natural transition to the manifestly covariant Hamiltonian description with constraints [24] on the 16-dimensional phase space $\mathrm{T}^*\mathbb{M}_4^2$. First of all consider general features of such a description.

Let us parametrize the space $\mathrm{T}^*\mathbb{M}_4^2$ by the position variables y^μ, x^μ and conjugated momenta P_μ, w_μ, and introduce the standard Poisson brackets $[\ldots,\ldots]$. Then the above integrals of motion P_μ and $J_{\mu\nu}$ become the generators of the canonical realization of the Poincaré group in $\mathrm{T}^*\mathbb{M}_4^2$.

By virtue of the parametric invariance of the Lagrangian description the canonical Hamiltonian vanishes (as follows from the identity (27)) and the dynamics of a system is determined by the Poincaré-invariant constraint which can be called the *mass-shell constraint* in analogy to the single-particle case.

Besides, the kinematical constraint (6) is carried into the Hamiltonian description too, so that both these constraints are the primary ones. The constraint (6) allows to remove a redundant internal coordinate, for one x^0. A conjugated momentum variable (w_0 in present instance) is obviously unobservable and is sooner or later subject to elimination by means of a secondary constraint which can be found to provide a self-consistency of the description. Instead, it is more convenient to construct the Hamiltonian description is such a form which uses at the beginning the observables only. The latters are meant as quantities which do not depend on the redundant momentum variable. Of course, there exists arbitrariness in a choice of redundant variables. Nevertheless it is possible to formulate the observability condition unambiguously. Namely, *the function $f(y,x,P,w)$ is observable if it satisfies the condition*:

$$[f, x^2] \approx 0\,, \tag{28}$$

where sign "$\approx$" denotes an equality on the light cone (6). Integrating the condition (28) one concludes that an observable can be an arbitrary function of the covariant arguments y^μ, x^μ, P_μ and $v_\mu \equiv P^\nu \Omega_{\nu\mu}/P\cdot x$ (see (24)) which form 15 independent quantities (because $P\cdot v \equiv 0$).

The covariant particle coordinates x_a^μ and the canonical generators P_μ, $J_{\mu\nu}$ are evidently the observables (in the sense of the definition (28)). This means that the description in terms of observables only provides the complete physically important information concerning the classical motion of a system. Hence it is natural to require of the Hamiltonian equations of observable motion to be expressed in terms of observables only. This requirement is fulfilled if the function on the left-hand side (l.-h.s.) of the mass-shell constraint is observable. Besides the x^2 there exist 4 independent Poincaré-invariant functions of the observables. They are P^2, v^2, $P\cdot x$

and $v \cdot x \approx w \cdot x$. Thus the general structure of the mass-shell constraint is as follows:

$$\phi(P^2,\ v^2,\ P\cdot x,\ v\cdot x) = 0. \tag{29}$$

Since ϕ satisfies the condition (28), the corresponding Hamiltonian equations of motion guarantee that the constraint (6) holds, and they do not produce any secondary constraints. It follows from this fact that an extra constraint for redundant momentum variable would arise as the primary constraint only and thus the set of equations (25), (26) would be twice degenerated. For the present this extra constraint may be not taken into account because it has no physical meaning while the constraints (29) and (6) are considered as a pair of first class constraints which determine completely the dynamics of the observables.

Above rather general considerations lead to the Hamiltonian mechanics which embraces the class of particle systems as wide as the original Fokker or Lagrangian formalism does. Indeed, the mass-shell constraint (29) can be considered as an equality which determines implicitly one of the argument of ϕ as a function of three other arguments. And so, the variety of all possible models in both the original Fokker and resulting Hamiltonian formalisms contains (except few degenerate cases) an arbitrary function of three arguments.

Now we follow step-by-step the hamiltonization procedure of an arbitrary time-asymmetric model which is given originally by means of its Fokker or Lagrangian formulation. For this purpose consider the relations (25)–(26) as the set of equations for the particle velocities to be found. In the general case the rank defect of this set is 1 (as a consequence of the parametric invariance of the description). In order to obtain the mass-shell constraint in the desirable form (29) it is necessary to decrease the rank of the set (25)–(26) by 1. This is possible owing to the holonomicity of the constraint (6). Indeed, its differential consequence (17) sets an additional relation for velocities which can be taken into account on the r.-h.s. of Eqs (25)–(26). After this is done one can easily see that the 4 arguments of the function ϕ indicated in Eq. (29) can be expressed in terms of 3 independent functions σ_1, σ_2, δ (21) of coordinates and velocities. Especially,

$$P\cdot x = \eta\left(\frac{F_1'}{\sigma_1} + \frac{F_2'}{\sigma_2} + 2F_\delta'\right), \tag{30}$$

$$v\cdot x = \frac{\eta}{2}\left(\frac{F_1'}{\sigma_1} - \frac{F_2'}{\sigma_2}\right), \tag{31}$$

$$P^2 = \sum_{a=1}^{2}\left(\frac{F_a'}{\sigma_a} + F_\delta'\right)^2\sigma_a^2 + 2\left(\frac{F_1'}{\sigma_1} + F_\delta'\right)\left(\frac{F_2'}{\sigma_2} + F_\delta'\right)\delta$$

$$+ 2\left(\frac{F_1'}{\sigma_1} + \frac{F_2'}{\sigma_2} + 2F_\delta'\right)(F - \sigma_1 F_1' - \sigma_2 F_2' - 2\delta F_\delta')\,, \qquad (32)$$

while the remaining quantity v^2 obeys the relation

$$v^2 + \frac{(v\cdot x)^2 P^2}{(P\cdot x)^2} + \frac{1}{\eta P\cdot x}\left(\frac{F_1'}{\sigma_1}\frac{F_2'}{\sigma_2} - F_\delta'^2\right) \times \left(\frac{P^2}{\eta P\cdot x} - 2F + \sigma_1 F_1' + \sigma_2 F_2' + 2\delta F_\delta'\right) = 0\,. \qquad (33)$$

Eliminating σ_1, σ_2, δ from the set of equations (30)–(33) one can find the mass-shell constraint in the form (29). Notice that besides the (30)–(33) one can obtain one more relation which determines the unobservable (in sense of (28)) quantity $P\cdot w$ in terms of σ_1, σ_2, δ. This makes it possible to find the above mentioned subsidiary primary constraint which has no physical meaning and thus will be omitted from further consideration.

The possibility to construct the mass-shell constraint (29) in an explicit form depends on how successful can be excluded the quantities σ_1, σ_2, δ from the relations (30)–(33). For example, if the set of equations (30)–(32) permits the existence of a positive solution for σ_1, σ_2, δ in terms of $P\cdot x$, $v\cdot x$, P^2, its substitution into the l.-h.s of (33) yields the mass-shell constraint sought. In certain models the quantity δ falls out of the equations (30),(31) and (33). In these cases only solution of equations (30) and (31) with respect to σ_1, σ_2 is needed, which simplifies the construction of the mass-shell constraint. The example of such a model which corresponds to the arbitrary superposition of the scalar, vector and confinement interactions is represented in Appendix A.

4. Transition to the three-dimensional Hamiltonian description

The scheme of the transition from the manifestly covariant description of a canonical system with constraints to its three-dimensional formulation is well known in literature [25, 26]. It consists in the reduction of an original phase space to a space of less dimensions (to be the reduced phase space $\mathbb{P}$), determined by means of relevant number of pairs of second class constraints. The latters serve also for constructing of the Dirac brackets $[\ldots,\ldots]^*$ which being restricted to the $\mathbb{P}$ induce on $\mathbb{P}$ the symplectic structure, *i.e.*, the nondegenerated Poisson brackets $\{\ldots,\ldots\}$. Final step of the reduction procedure consists in parametrization of $\mathbb{P}$ by such variables in terms of which the Poisson brackets take the standard form. For this purpose it is convenient to use the Shanmugadhasan method [25], that is, to perform in original phase space a canonical transformation which reduces the set of constraints to the canonical form. The latter is that form in which at least one constraint of each pair of second class constraints means the vanishing of that new canonical variable which must be eliminated. The remaining new variables parametrize the space $\mathbb{P}$ in desirable way, *i.e.*, they are canonical with respect to the induced Poisson brackets.

When first class constraints are present, arbitrary (in principle) gauge fixing constraints of the same number can be added in order to use the above reduction procedure. In this case a proper choice of gauge fixing constraints can simplify to a great extent the Shanmugadhasan transformation and/or a final description in the reduced space $\mathbb{P}$.

In our case the manifestly covariant Hamiltonian description on the 16-dimensional space $T^*\mathbb{M}_4^2$ is based on three constraints. One of them is a first class constraint and two other form a pair of second class constraints. Thus such a description can be reduced to the description on the 12-dimensional phase space $\mathbb{P}$. It has been shown above that the dynamics of such a system can be determined in physically equivalent way by the pair of first class constraints, namely, the mass-shell constraint (29) and the holonomic constraint (6). During the reduction procedure this fact allows to replace the above mentioned second class constraint of nonphysical meaning by an arbitrary (not necessarily Poincaré-invariant) gauge fixing constraint

$$\psi(y, x, P, w) = 0\,, \tag{34}$$

where ψ obeys the only condition

$$[\psi, x^2] \neq 0\,. \tag{35}$$

The constraint (34), which together with (6) removes a pair of redundant internal variables (for instance, x^0 and w_0), has purely formal meaning because its explicit form does not influence both the dynamics of a system and the structure of final three-dimensional description.

In order to remove a redundant pair of external variables (for instance, y^0 and P_0) we need one more gauge fixing constraint which complements the mass-shell constraint and completes the whole set of constraints to the second class ones. This constraint breaks the parametric invariance of the description and fixes the evolution parameter in terms of observables. Therefore it can be defined as follows:

$$\chi(y, x, P, v, t) = 0\,, \tag{36}$$

where the function χ is only restricted by the conditions

$$[\chi, \phi] \neq 0\,, \qquad \partial\chi/\partial t \neq 0\,; \tag{37}$$

here and hereafter t denotes the evolution parameter fixed by the constraint (36).

The gauge fixing constraint (36) (as well as the previous one (34)) does not influence the dynamics of a system, but its choice (together with the structure of the constraints (29) and (6)) determines specific features of final description, namely, the reduced phase space $\mathbb{P}$ (as a submanifold of $T^*\mathbb{M}_4^2$), the induced Poisson brackets, and a possible choice of variables, in terms of which these brackets take the canonical form. An explicit form of observables (*i.e.*, the covariant particle positions, the generators of the Poincaré group *etc.*) being functions of canonical variables of the space $\mathbb{P}$, depends on a choice of the constraint (36) too. Thus using the arbitrariness of this choice one can effectively influence the structure of the final description.

Note that the more special choice of the constraint (36),

$$\chi(y, x, P_0, t) = 0 \tag{38}$$

(here the function on the l.-h.s. is arbitrary and it is chosen to satisfy the conditions (37)), allows to avoid the well-known *no–interaction theorem* [16], that is, to pass to such a three-dimensional Hamiltonian description of time-asymmetric models in which the spatial covariant particle positions become the canonical variables.

The three-dimensional Hamiltonian description in terms of covariant variables is desirable in various aspects. For example, it simplifies the introduction of the interaction with external fields and allows the position representation on the quantum-mechanical level. But this description is not convenient for solving a two-body problem, because it does not provide a relevant separation of external and internal degrees of freedom. Below we propose the transition to another three-dimensional description in which the desirable separation is achieved using relativistic centre-of-mass variables.

5. Three-dimensional Hamiltonian description in terms of relativistic centre-of-mass variables

We look for variables in terms of which the motion of a two-particle system as a whole can be separated naturally from its internal evolution. It is convenient to start this search in the framework of the manifestly covariant Hamiltonian description (in $T^*\mathbb{M}_4^2$). We note, that the above defined external and internal variables y^μ, P_μ and x^μ, w_μ do not solve this task. Indeed, the manifestly covariant Hamiltonian equations of motion for external variables y^μ,

$$\dot{y}^\mu \sim [y^\mu, \phi] = \partial\phi/\partial P_\mu\,, \tag{39}$$

predict an intricate external evolution due to the general structure of the function ϕ (29) including a particle interaction. It is desirable to replace the y^μ by another variables Q^μ which describe the motion of a system as a whole like the motion of single particle with 4-momentum P_μ, *i.e.*,

$$\dot{Q}^\mu \sim P^\mu . \tag{40}$$

Such variables can be introduced by means of a canonical transformation in $\mathrm{T}^*\mathbb{M}_4^2$,

$$(y^\mu,\ P_\mu,\ x^\mu,\ w_\mu) \mapsto (Q^\mu,\ P_\mu,\ \rho^\mu,\ \omega_\mu)\,, \tag{41}$$

which does not change the total 4-momentum P_μ and provides a dependence of the function ϕ on new external variables through the P^2 only.

The transformation (41) can be naturally determined by means of the generating function:

$$W(y, P, x, \omega) = P_\mu y^\mu + \omega_\nu \Lambda\left(\frac{P}{|P|}\right)^\nu_{\ \mu} x^\mu\,, \tag{42}$$

where $|P| \equiv \sqrt{P^2}$, and $\|\Lambda(P/|P|)^\nu{}_\mu\| \in \mathrm{SO}(1,3)$ obeys the condition:

$$\Lambda^\mu{}_\nu P^\nu = \delta^\mu_0 |P|\,. \tag{43}$$

The matrix Λ describes the Lorentz transformation into the centre-of-inertia reference frame. The condition (43) fixes Λ up to an arbitrary matrix of spatial rotation (which can depend on P). Omitting details we note that this arbitrariness allows to get as the result of the following reduction procedure various Hamiltonian forms of dynamics (see, for example, [26, 27]).

Let us choose Λ as a pure boost, *i.e.*,

$$\|\Lambda^\mu{}_\nu\| = \left\| \begin{array}{c|c} \frac{P_0}{|P|} & \frac{P_j}{|P|} \\ \hline \frac{P_i}{|P|} & \delta_{ij} + \frac{P_i P_j}{|P|(|P|+P_0)} \end{array} \right\| . \tag{44}$$

Then we obtain the final description in the framework of the Bakamjian–Thomas model, that is, the three-dimensional Hamiltonian description in the instant form of dynamics formulated in terms of the centre-of-mass variables.

Let us write down explicitly the canonical transformation generated by the function (42),

$$y^\mu = Q^\mu - \tfrac{1}{2} S_{\lambda\nu} \Lambda^\lambda{}_\tau \frac{\partial \Lambda^{\nu\tau}}{\partial P_\mu}\,, \tag{45}$$

$$x^\mu = \rho^\nu \Lambda_\nu{}^\mu\,, \qquad w_\mu = \omega_\nu \Lambda^\nu{}_\mu\,, \tag{46}$$

where

$$S_{\mu\nu} \equiv \Lambda_\mu{}^\tau \Lambda_\nu{}^\sigma \Omega_{\tau\sigma} = \rho_\mu \omega_\nu - \rho_\nu \omega_\mu \,, \tag{47}$$

and express the constraints (6) and (29) in terms of new variables:

$$\rho^2 = 0, \quad \eta\rho^0 > 0 \,, \qquad i.e., \qquad \rho^0 - \eta|\boldsymbol{\rho}| = 0 \,, \tag{48}$$

$$P^2 - M^2(S_{0\sigma}S^{0\sigma},\, \rho^0,\, \rho\cdot\omega) = 0 \,. \tag{49}$$

The function M in the l.-h.s. of Eq. (49) is the positive solution of the equation:

$$\phi(M^2, M^2 S_{0\sigma}S^{0\sigma},\, M\rho^0,\, \rho\cdot\omega) = 0 \tag{50}$$

and has a sense of the total mass of the system. Under the general analysis of the description (when ϕ is meant arbitrary) the mass M can be considered as an arbitrary function of the indicated scalar combinations of the internal canonical variables ρ^μ, ω_μ. It is obvious that the mass-shell constraint (49) satisfies the condition (40) regardless of the internal dynamics of the system. So, we have the desirable separation of variables.

Now we perform the transition to the three-dimensional Hamiltonian description following the scheme proposed in Sec. 4. Let us choose the gauge fixing constraint (36) in the following form:

$$Q^0 - t \equiv y^0 + \mathrm{tr}(\Lambda\, \Omega\, \partial\Lambda^{\mathrm{T}}/\partial P_0) - t = 0 \,, \tag{51}$$

while the constraint (34) remains arbitrary.

Following the Shanmugadhasan method [25] we perform the canonical transformation

$$\begin{aligned}(Q^0,\, Q^i,\, P_0,\, P_i,\, \rho^0,\, \rho^i,\, \omega_0,\, \omega_i)& \\ \mapsto\; (\bar{Q}^0,\, Q^i,\, P_0,\, P_i,\, \bar{\rho}^0,\, \rho^i,\, \omega_0,\, \pi_i)&\end{aligned} \tag{52}$$

which is determined by the generating function:

$$\tilde{W} = P_0(Q^0 - t) + P_i Q^i + \omega_0(\rho^0 - \eta|\boldsymbol{\rho}|) + \pi_i \rho^i \,. \tag{53}$$

It has the following explicit form:

$$\bar{Q}^0 = Q^0 - t \,, \quad \bar{\rho}^0 = \rho^0 - \eta|\boldsymbol{\rho}| \,, \quad \pi_i = \omega_i - \eta\omega_0\rho_i/|\boldsymbol{\rho}| \tag{54}$$

(the remaining variables do not change). This transformation reduces the set of constraints to the canonical form (*i.e.*, two constraints (48) and (51) among four ones read: $\bar{\rho}^0 = 0$ and $\bar{Q}^0 = 0$). Besides, due to the explicit dependence of the transformation (52)–(54) on t, the Hamiltonian $H = P_0$ appears. The following reduction of the description onto the 12-dimensional phase space $\mathbb{P}$ parametrized by the canonical variables Q^i, P_i, ρ^i, π_i ($i = 1, 2, 3$) is straightforward and leads to the BT model [23] with the well-known canonical generators of the Poincaré group,

$$\begin{aligned} H &= P_0 = \sqrt{M^2 + \boldsymbol{P}^2}, \quad P_i, \\ J_i &= \varepsilon_{ij}{}^k Q^j P_k + S_i, \\ K_i &= -tP_i + Q_i H + \frac{(\boldsymbol{P} \times \boldsymbol{S})_i}{M + H}. \end{aligned} \tag{55}$$

Here $\boldsymbol{S} \equiv \boldsymbol{\rho} \times \boldsymbol{\pi}$ is the total spin (the internal angular momentum) of the system, and $M(\boldsymbol{\rho}, \boldsymbol{\pi})$ is the total mass of the system which determines its dynamics in the reduced space $\mathbb{P}$. In our case M is the positive solution of the equation:

$$\phi(M^2, \ -M^2\rho^2\pi^2, \ \eta M\rho, \ -\boldsymbol{\rho}\cdot\boldsymbol{\pi}) = 0, \tag{56}$$

which we call the *mass-shell equation* (here $\rho \equiv |\boldsymbol{\rho}|$, $\pi \equiv |\boldsymbol{\pi}|$).

Besides the canonical realization of the Poincaré group, the reduction scheme proposed in Sect. 4 permits to obtain the covariant coordinates of particles x_a^μ as functions of the canonical variables, what makes it possible to build particle world lines in the Minkowski space $\mathbb{M}_4$. Using constraints in the r.-h.s. of Eqs (16) and (44)–(47) it is easy to get the x_a^μ explicitly,

$$x_a^\mu = X^\mu + \left[\Lambda^{\mathrm{T}}\left(\frac{P}{M}\right)\right]^\mu{}_\nu e_a^\nu(\boldsymbol{\rho}, \boldsymbol{\pi}), \tag{57}$$

where

$$X^0 = t, \tag{58}$$

$$X^i = Q^i - \frac{(\boldsymbol{P} \times \boldsymbol{S})^i}{M(M+H)} \tag{59}$$

are the well-known Pryce centre-of-inertia variables [28], and

$$\begin{aligned} e_a^0 &= \tfrac{1}{2}(-)^{\bar{a}}\eta\rho, & a &= 1, 2, \\ e_a^i &= \tfrac{1}{2}(-)^{\bar{a}}\rho^i + \eta\rho\pi^i/M, & \bar{a} &= 3 - a. \end{aligned} \tag{60}$$

The formulae (57)–(60) correspond to the special choice of general expressions for the covariant coordinates which are proposed in Refs [29, 30] for a space-time interpretation of the BT model considered *a priori*.

In Appendix B we treat the free-particle system as the time-asymmetric model, and we find the relation of its BT description to the standard Hamiltonian description of the free particles in the instant form of dynamics.

6. Reduction of the relativistic two-body problem to quadratures

In the BT model the 12-dimensional phase space $\mathbb{P}$ can be expanded naturally onto the external and internal subspaces $\mathbb{P} = \mathbb{P}_{\rm ex} \times \mathbb{P}_{\rm in}$, where the $\mathbb{P}_{\rm ex}$ is parametrized by the external variables Q_i, P_i and the $\mathbb{P}_{\rm in}$ is parametrized by the internal variables ρ_i, π_i.

Due to Poincaré invariance of the description it is sufficient to choose the centre-of-inertia reference frame in which

$$\boldsymbol{P} = \boldsymbol{0}\,, \quad \boldsymbol{K} = \boldsymbol{0}\,, \tag{61}$$

so that

$$\boldsymbol{Q} = \boldsymbol{0}\,, \quad \boldsymbol{X} = \boldsymbol{0}\,, \tag{62}$$

and then to consider the subspace $\mathbb{P}_{\rm in}$ only. At this point the problem is reduced to the rotation invariant problem of some effective single body. The corresponding phase trajectory lies in the plane which is orthogonal to $\boldsymbol{S}$. For its description it is natural to use the polar coordinates,

$$\boldsymbol{\rho} = \rho\boldsymbol{\epsilon}_\rho\,, \quad \boldsymbol{\pi} = \pi_\rho\boldsymbol{\epsilon}_\rho + S\boldsymbol{\epsilon}_\varphi/\rho\,; \tag{63}$$

here $S \equiv |\boldsymbol{S}|$; the unit vectors $\boldsymbol{\epsilon}_\rho$, $\boldsymbol{\epsilon}_\varphi$ are orthogonal to $\boldsymbol{S}$, they form together with $\boldsymbol{S}$ a right triplet of vectors and can be decomposed in the usual manner in terms of the Cartesian unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, *i.e.*,

$$\boldsymbol{\epsilon}_\rho = \boldsymbol{i}\cos\varphi + \boldsymbol{j}\sin\varphi, \quad \boldsymbol{\epsilon}_\varphi = -\boldsymbol{i}\sin\varphi + \boldsymbol{j}\cos\varphi\,, \tag{64}$$

where φ is the polar angle.

The mass-shell equation (56) reads:

$$\phi(M^2,\ -M^2(\rho^2\pi_\rho^2 + S^2),\ \eta M\rho,\ -\rho\pi_\rho) = 0\,, \tag{65}$$

and the covariant coordinates of particles (57)–(60) take the form:

$$x_a^0 = t + \tfrac{1}{2}(-)^{\bar a}\eta\rho\,, \tag{66}$$

$$\boldsymbol{x}_a = \left(\tfrac{1}{2}(-)^{\bar a} + \frac{\eta\pi_\rho}{M}\right)\rho\boldsymbol{\epsilon}_\rho + \frac{\eta S\boldsymbol{\epsilon}_\varphi}{M}. \tag{67}$$

Using equation (65) the internal radial momentum π_ρ can be expressed in terms of ρ, M, S and then following the Hamilton–Jacobi method a solution of the Hamilton equations can be locally found in quadratures,

$$t - t_0 = \int d\rho \frac{\partial\pi_\rho(\rho, M, S)}{\partial M}\,, \tag{68}$$

$$\varphi - \varphi_0 = -\int d\rho \frac{\partial\pi_\rho(\rho, M, S)}{\partial S}\,. \tag{69}$$

The function $\pi_\rho(\rho, M, S)$ usually consists of few branches and describes the projection of the phase trajectory onto the subspace of the radial variables $(\rho, \pi_\rho) \equiv \mathbb{P}_\rho \subset \mathbb{P}_{\text{in}}$. Thus the quadratures (68), (69) give a local solution of the problem only, *i.e.*, within the domain of values of ρ in which some branch of the function π_ρ exists. In order to obtain the global solution it is necessary to sew up local solutions in such a way that the resulting curve in $\mathbb{P}_{\text{in}}$ should be a continuous and preferably smooth curve. Generally such a phase trajectory can consist of few isolated continuous components, and physical meaning of some of them turns out to be not clear. This situation occurs often in various relativistic models [12, 19] and one has to be careful when constructing their Hamiltonian description.

Finally we give the direct prescription how to obtain the mass-shell equation when the dynamics of a time-asymmetric model is given originally in the framework of the Fokker or Lagrangian formalism. In this case the function ϕ in the l.-h.s. of Eq. (65) is determined by the structure of the function $F(\sigma_1, \sigma_2, \delta)$ (20). Taking the relations (30)–(33) into account one can represent the corresponding mass-shell equation in the following form:

$$\begin{aligned}\frac{S^2}{\rho^2} &+ \left(F'_\delta - \left(\tfrac{1}{4}M^2 - \pi_\rho^2\right)\frac{\rho}{M}\right) \\ &\times \left(\frac{M}{\rho} - 2F + \sigma_1 F'_1 + \sigma_2 F'_2 + 2\delta F'_\delta\right) = 0\,, \qquad (70)\end{aligned}$$

where σ_1, σ_2, δ must be found from the set of equations:

$$\frac{F'_a}{\sigma_a} + F'_\delta = b_a \equiv \left(\tfrac{1}{2}M + \eta(-)^a \pi_\rho\right)\rho\,, \qquad a = 1, 2\,, \qquad (71)$$

$$b_1^2\sigma_1^2 + b_2^2\sigma_2^2 + 2b_1b_2\delta + 2M\rho(F - \sigma_1 F'_1 - \sigma_2 F'_2 - 2\delta F'_\delta) = M^2\,. \qquad (72)$$

Conclusion

The formalism of the Fokker-type action integrals has arisen as one of early approaches to RDIT. In spite of the close relation of this approach to the field theory its application to the description of concrete physical systems is held up by difficulties due to nonlocality of the equations of motion. Alternatively a variety of other approaches which are more similar mathematically to the nonrelativistic mechanics appears. They make it possible the construction of much simpler (including exactly solvable) but usually phenomenological models of relativistic systems of interacting particles.

The class of time-asymmetric two-particle models proposed here can be considered as the compromise approach which lies on the frontier between the field theory and the relativistic mechanics of directly interacting particles and possesses some of their advantages.

First of all, there exists a subclass of the time-asymmetric Fokker-type integrals which permits a field-theoretical interpretation of the particle interaction and thus it can lead to tractable models of various physical systems. There is also an important possibility to modify these integrals in order to take semiphenomenologically into account such field effects which by now can not be deduced from the first principles (for example, the phenomenon of quark confinement).

Second, the time-asymmetric models are descriptively flexible. It is possible to reformulate them into the framework of the Lagrangian and Hamiltonian formalisms (both in the manifestly covariant and three-dimensional forms) using the covariant coordinates or the centre-of-mass variables. These opportunities are useful for the study of the models and enable their comparison with other models known in literature.

Third, all the time-asymmetric models are exactly solvable: their equations of motion are integrable in quadratures. We plan in next works to study some physically most interesting models.

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Appendix A

The model of particle system with scalar, vector and confinement interactions

Let us choose the function $\Psi(\sigma_1, \sigma_2, \delta)$ on the r.-h.s. of Eq. (20) as follows:

$$\Psi = \alpha\sigma_1\sigma_2 + \beta\delta + \gamma, \tag{A.1}$$

where α, β, γ are arbitrary constants. The first and second terms on the r.-h.s. of Eq. (A.1) correspond to the scalar and vector field-type interactions with the coupling constants α and β respectively, and the third term describes the confinement interaction (when $\gamma > 0$). In the nonrelativistic limit this model leads to the potential $U = (\alpha + \beta)/r + \gamma r$, where r is the distance between particles.

Calculating F_a' and F_δ' for this case and substituting them into the l.-h.s. of Eq. (33) one can see that the latter does not depend on δ. The corresponding set of equations (30)–(31) takes the form:

$$\frac{m_a + \alpha\sigma_{\bar{a}} + \beta\sigma_a}{\sigma_a} = b_a \equiv \eta(\tfrac{1}{2}P \cdot x + (-)^{\bar{a}} v \cdot x)\,, \tag{A.2}$$

and can be easily solved with respect to σ_a:

$$\sigma_a = \frac{(b_{\bar{a}} - \beta)m_a + \alpha m_{\bar{a}}}{(b_1 - \beta)(b_2 - \beta) - \alpha^2}\,. \tag{A.3}$$

Finally, substitution of σ_a (A.3) into the l.-h.s. of (33) gives the mass-shell constraint

$$\phi = \phi_f + \phi_{\rm int} = 0\,, \tag{A.4}$$

where

$$\phi_f = \tfrac{1}{4}P^2 - \tfrac{1}{2}(m_1^2 + m_2^2) + (m_1^2 - m_2^2)\frac{v\cdot x}{P\cdot x} + v^2 \tag{A.5}$$

is the free-particle term, and

$$\begin{aligned}\phi_{\rm int} = & -\frac{2\alpha m_1 m_2 + \beta(P^2 - m_1^2 - m_2^2)}{\eta P\cdot x} - 2\gamma\left(\frac{b_1 b_2}{\eta P\cdot x} - \beta\right)\\ & - (\alpha^2 - \beta^2)\frac{2\alpha m_1 m_2 + (b_1-\beta)m_2^2 + (b_2 - \beta)m_1^2}{\eta P\cdot x((b_1-\beta)(b_2-\beta) - \alpha^2)}\end{aligned} \tag{A.6}$$

describes the interaction.

Appendix B

The free-particle system

The free-particle mass-shell constraint $\phi_f = 0$ (see Eq. (A.5)) takes within the framework of BT description the form

$$\tfrac{1}{4}M^2 - \boldsymbol{\pi}^2 - \tfrac{1}{2}(m_1^2 + m_2^2) - \eta\,(m_1^2 - m_2^2)\frac{\boldsymbol{\pi}\cdot\boldsymbol{\rho}}{M\rho} = 0\,. \tag{B.1}$$

Besides, the requirement of σ_1, σ_2 to be positive restricts the phase space to the domain in which

$$|\boldsymbol{\pi}\cdot\boldsymbol{\rho}| < \tfrac{1}{2}M\rho\,. \tag{B.2}$$

The mass-shell equation is cubic with respect to the function of the total mass $M(\boldsymbol{\rho}, \boldsymbol{\pi})$. Its solution has a complicated form and does not coincide with the standard total mass of the free-particle system in the BT model [31]. Nevertheless, following the reduction scheme proposed in Sec. 4 and integrating the equations of motion (using (B.2)) we come to correct particle world lines.

Here we do not display this analysis which represents rather methodological interest. Instead, we construct the canonical transformation of the internal variables $(\boldsymbol{\rho}, \boldsymbol{\pi}) \mapsto (\boldsymbol{r}, \boldsymbol{k})$ which reduces the free-particle total mass to the standard form.

Let

$$\boldsymbol{\pi} = \boldsymbol{k} - \eta\frac{m_1^2 - m_2^2}{2M}\frac{\boldsymbol{\rho}}{\rho}. \tag{B.3}$$

The substitution of (B.3) into (B.1) leads to a biquadratic equation for M.

$$\tfrac{1}{4}M^2 - \tfrac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4M^2} - \boldsymbol{k}^2 = 0, \tag{B.4}$$

which has four solutions. They are:

$$M(\boldsymbol{k}) = \sum_{a=1}^{2} \varepsilon_a k_{a0} \equiv \sum_{a=1}^{2} \varepsilon_a \sqrt{m_a^2 + \boldsymbol{k}^2}, \tag{B.5}$$

where $\varepsilon_a = \pm 1$ $(a = 1, 2)$. Only one of them has physical meaning (if $\varepsilon_a = 1$). Let us show that nonphysical solutions can be dropped when requiring positivity of M and taking (B.2) into account. Hereafter we set $m_1 > m_2$. Using (B.2) and (B.3) in (B.1) one can obtain the inequalities

$$\tfrac{1}{4}M^2 - m_a^2 \begin{matrix} \le \\ \ge \end{matrix} \boldsymbol{\pi}^2 \begin{matrix} \le \\ \ge \end{matrix} \left(|\boldsymbol{k}| + \frac{m_a^2 - m_{\bar{a}}^2}{2M} \right)^2, \quad \begin{matrix} a = 1, \\ a = 2, \end{matrix} \tag{B.6}$$

which after simple calculations and use of (B.5) become

$$(\varepsilon_1 k_{10} - \varepsilon_2 k_{20})(\varepsilon_a k_{a0} + |\boldsymbol{k}|) \ge 0, \qquad a = 1, 2. \tag{B.7}$$

Requiring the positivity of M and taking (B.7) into account we conclude that $\varepsilon_a = 1$, which corresponds to the standard form of the free-particle total mass.

Now taking an explicit expression for $M(\boldsymbol{k})$ into account one can write down the Eq. (B.3) in the following form:

$$\begin{aligned} \boldsymbol{\pi} &= \boldsymbol{k} + \frac{\eta}{2}(k_{20} - k_{10})\frac{\boldsymbol{\rho}}{\rho} \\ &= \frac{\partial}{\partial \boldsymbol{\rho}}\left(\boldsymbol{k} \cdot \boldsymbol{\rho} + \frac{\eta}{2}(k_{20} - k_{10})\rho \right) \equiv \frac{\partial W(\boldsymbol{\rho}, \boldsymbol{k})}{\partial \boldsymbol{\rho}}. \end{aligned} \tag{B.8}$$

Thus $W(\boldsymbol{\rho}, \boldsymbol{k})$ is the generating function of the transformation sought, and we immediately obtain:

$$\boldsymbol{r} = \frac{\partial W(\boldsymbol{\rho}, \boldsymbol{k})}{\partial \boldsymbol{k}} = \boldsymbol{\rho} + \frac{\eta}{2}\left(\frac{1}{k_{20}} - \frac{1}{k_{10}} \right) \rho \boldsymbol{k}. \tag{B.9}$$

Eqs (B.8) and (B.9) which determine this transformation in an unexplicit form, make it possible to express all dynamical quantities in terms of new variables. Especially the expressions for the functions $\boldsymbol{e}_a$, e_a^0 (64) determining the covariant coordinates of particles can be written down as follows:

$$\begin{aligned} \boldsymbol{e}_a(\boldsymbol{r}, \boldsymbol{k}) &= (-)^{\bar{a}} \left(\frac{k_{\bar{a}0}}{M} \boldsymbol{r} + \frac{e_a^0}{k_{a0}} \boldsymbol{k} \right), \qquad & a = 1, 2, \\ e_a^0(\boldsymbol{r}, \boldsymbol{k}) &= \tfrac{1}{2}(-)^{\bar{a}} \eta \rho(\boldsymbol{r}, \boldsymbol{k}), & \bar{a} = 3 - a, \end{aligned} \tag{B.10}$$

where the function $\rho(\boldsymbol{r},\boldsymbol{k})$ has a cumbersome form but is not essential for the following calculations.

The covariant coordinates of particles (57)–(59), (B.10) do not agree with the standard description of the free particles in the framework of the BT model [31] unlike the canonical generators (55) and the total mass (B.5) (with $\varepsilon_a = 1$). Nevertheless the description obtained here reproduces correctly the free-particle dynamics. In order to prove this statement we perform (following [31]) the canonical transformation from the centre-of-mass variables $\boldsymbol{Q},\boldsymbol{P},\boldsymbol{r},\boldsymbol{k}$ to the particle canonical variables $\boldsymbol{q}_a,\boldsymbol{p}_a$ $(a = 1,2)$. This transformation is defined by the generating function

$$\tilde{W}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{P},\boldsymbol{k}) = \sum_{a=1}^{2}\frac{k_{a0}}{M}\boldsymbol{q}_a\cdot\boldsymbol{P} + (\boldsymbol{q}_1-\boldsymbol{q}_2)\cdot\left(\boldsymbol{k}+\frac{(\boldsymbol{P}\cdot\boldsymbol{k})\boldsymbol{P}}{M(M+H)}\right). \qquad \text{(B.11)}$$

It reduces the canonical generators (55), (B.5) to the standard free expressions in the instant form ot dynamics, *i.e.*,

$$H = \sum_{a=1}^{2} p_{a0} \equiv \sum_{a=1}^{2}\sqrt{m_a^2+\boldsymbol{p}_a^2}, \qquad \boldsymbol{P} = \sum_{a=1}^{2}\boldsymbol{p}_a,$$
$$\boldsymbol{J} = \sum_{a=1}^{2}\boldsymbol{q}_a\times\boldsymbol{p}_a, \qquad \boldsymbol{K} = -t\boldsymbol{P} + \sum_{a=1}^{2}\boldsymbol{q}_a p_{a0}, \qquad \text{(B.12)}$$

The covariant coordinates of particles (57)–(59), (B.10) in terms of new variables read:

$$x_a^0 = t + \Delta_a, \qquad \boldsymbol{x}_a = \boldsymbol{q}_a + \Delta_a\boldsymbol{p}_a/p_{a0}, \qquad \text{(B.13)}$$

where Δ_a are some functions of canonical variables. These formulae describe correct (straight) free-particle world lines, each of them is parametrized by the time t although shifted in time by Δ_a in comparison with the standard description.

REFERENCES

[1] A.D. Fokker, *Z. Phys.* **28**, 386 (1929).

[2] J.A. Wheeler, R.P. Feynman, *Rev. Mod. Phys.* **21**, 425 (1949).

[3] P. Havas, in: *Problems in the Foundations of Physics*, ed. M. Bunge, Springer, Berlin 1971, p.31.

[4] P. Ramond, *Phys. Rev.* **D7**, 449 (1973).

[5] Ph. Droz-Vincent, *Ann. Inst. Henri Poincaré Phys. Theor.* **A27**, 407 (1977).

[6] A. Komar, *Phys. Rev.* **D18**, 3617 (1978).
[7] I.T. Todorov, *Lect. Notes Phys.* **162**, 213 (1982).
[8] A. Staruszkiewicz, *Ann. Phys.* **25**, 362 (1970).
[9] R.A. Rudd, R.N. Hill, *J. Math. Phys* **11**, 2704 (1970).
[10] H.P. Künzle, *Int. J. Theor. Phys.* **11**, 395 (1974).
[11] A. Staruszkiewicz, *Ann. Inst. Henri Poincaré Phys. Theor.* **14**, 69 (1971).
[12] D.E. Fahnline, *J. Math. Phys.* **22**, 1640 (1981).
[13] A. Duviryak, V. Tretyak, *Condensed Matter Physics* **1**, 92 (1993) (in Ukrainian).
[14] P. Stephas, *Phys. Rev.* **D31**, 319 (1985).
[15] H.P. Künzle, *Nuovo Cim.* **B101**, 721 (1988).
[16] D.G. Currie, J.F. Jordan, E.C.G. Sudarshan, *Rev. Mod. Phys.* **35**, 350 (1963).
[17] R.P. Gaida, Yu.B. Kluchkovsky, V.I. Tretyak, *Teor. Mat. Fiz.* **55**, 88 (1983).
[18] S.N. Sokolov, V.I. Tretyak, *Teor. Mat. Fiz.* **67**, 102 (1986).
[19] A.A. Mayorov, S.N. Sokolov, V.I. Tretyak, Preprint IHEP 86-243, Serpukhov, 1988.
[20] S.N. Sokolov, *Dokl. Akad. Nauk SSSR* **221**, 809 (1975).
[21] V.I.Tretyak, V.Ye. Shpytko, Preprint Inst. of Appl. Probl. of Mech. and Math. 13-88, Lviv, 1988 (in Russian).
[22] V.Ye. Shpytko, Preprint ICMP-93-12U, Lviv, 1993 (in Ukrainian).
[23] B. Bakamjian, L.H. Thomas, *Phys. Rev.* **92**, 1300 (1953).
[24] P.A.M. Dirac, *Can. J. Math.* **2**, 129 (1950).
[25] S. Shanmugadhasan, *J. Math. Phys.* **14**, 677 (1973).
[26] N.P. Kliepikov, F.N. Shatnij, *Viest. Mosk. Univ., Fiz., Astr.* **24**, 32 (1983) (in Russian).
[27] A.A. Duviryak, in: *Methods of Studying Differential and Integral Operators*, Naukova Dumka, Kiev, 1989, p. 59 (in Russian).
[28] M.H.L. Pryce, *Proc. Roy. Soc. London* **A195**, 62 (1948).
[29] A.A. Duviryak, Yu.B. Kluchkovsky, *Matemat. Metody i Fiz.-Mekh. Polya* **34**, 100 (1991) (in Ukrainian; transl. in English: *J. Soviet Math.* **66**, 2648 (1993)).
[30] A.A.Duviryak, Yu.B. Kluchkovsky, *Ukr. Fiz. Zh.* **37** 313 (1992).
[31] M. Pauri, G.M. Prosperi, *J. Math. Phys.* **16**, 1503 (1975).