DIFFERENTIAL CALCULUS ON DEFORMED E(2) GROUP *

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Fourdimensional bicovariant differential $*$-calculus on quantum E(2) group is constructed. The relevant Lie algebra is obtained and covariant differential calculus on quantum plane is found.

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1. Introduction

Much attention has been paid recently to the special deformation of Poincaré algebra (group) called κ-deformed Poincaré algebra [1] (group [2]). Many particular results were obtained which gave more deep insight into their structure. One of the most important open problems is to find and classify bicovariant differential $*$-calculi. Partial results were obtained in Ref. [3] where some differential calculi on deformed Minkowski space were considered using bicrossproduct technique developed in [4].

In the present paper, following the approach developed by Woronowicz school [5, 6] we construct fourdimensional bicovariant $*$-calculus for the deformed E(2) group. This group differs trivially from twodimensional κ-Poincaré group so our results apply, *mutatis mutandis*, to this case also. In the subsequent paper we shall consider the bicovariant differential calculi for four dimensional case.

The paper is organized as follows. In Section 2 we describe the bicovariant differential $*$-calculus obtained, according to Woronowicz [5], by an

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appropriate choice of right ideal $\mathcal{R} \subset \ker\varepsilon$. In Section 3 the generalized Lie algebra is found and compared with the deformed (euclidean) κ-Poincaré algebra. It is shown that Woronowicz functionals are expressible in terms of κ-Poincaré algebra; in particular, additional, fourth generator, is related to the Casimir operator of κ-Poincaré. Finally, in Section 4 the covariant differential calculus on κ-plane is constructed and compared with that given in Ref. [3].

To conclude the Introduction let us remind the definition of deformed E(2) group [7, 8]. It is generated by four elements: a, a^*, v_+, $v_- = v_+^*$ subject to the following commutation rules:

$$\begin{aligned}
&[a, v_-] = \frac{i}{\kappa}(I - a)\,, && [a^*, v_-] = \frac{i}{\kappa}\left(a^* - a^{*2}\right)\,,\\
&[a, v_+] = \frac{i}{\kappa}\left(a - a^2\right)\,, && [a^*, v_+] = \frac{i}{\kappa}\left(I - a^*\right)\,,\\
&[v_+, v_-] = \frac{i}{\kappa}(v_- - v_+)\,, && a^*a = aa^* = I\,.
\end{aligned} \tag{1}$$

The coproduct, antipode and counit read, respectively:

$$\begin{aligned}
&\Delta a = a \otimes a\,, && \Delta a^* = a^* \otimes a^*\,,\\
&\Delta v_+ = a \otimes v_+ + v_+ \otimes I\,, && \Delta v_- = a^* \otimes v_- + v_- \otimes I\,,\\
&S(a) = a^*\,, && S(a^*) = a\,,\\
&S(v_+) = -a^* v_+\,, && S(v_-) = -a v_-\,,\\
&\varepsilon(a) = \varepsilon(a^*) = 1\,, && \varepsilon(v_-) = \varepsilon(v_+) = 0\,.
\end{aligned} \tag{2}$$

2. The bicovariant $*$-calculus

We follow closely the Woronowicz school approach [5, 6]. The ideal $\mathcal{R} \subset \ker\varepsilon$ is chosen to be generated by the following nine elements:

$$\begin{aligned}
&(a - I)(a^* - I) = 2I - (a + a^*)\,,\\
&(a - I)v_+\,, \qquad (a^* - I)v_+\,,\\
&(a - I)v_-\,, \qquad (a^* - I)v_-\,,\\
&v_+^2 - \frac{i}{\kappa}v_+\,, \qquad v_-^2 - \frac{i}{\kappa}v_-\,.
\end{aligned} \tag{3}$$

We easily check, that $\mathcal{R}$ is ad-invariant; also $x \in \mathcal{R}$ implies $S(x)^* \in \mathcal{R}$. Therefore, $\mathcal{R}$ determines bicovariant $*$-calculus [5]. This calculus is fourdimensional: $\ker\varepsilon/\mathcal{R}$ is spanned by $a - a^*$, v_+, v_-, $v_+v_- + (i/\kappa)v_+$.

Using standard technique we arrive at the following commutation rules between the elements of algebra and differentials:

$$
\begin{aligned}
&a\mathrm{d}a = \mathrm{d}a a\,, \quad a^*\mathrm{d}a = \mathrm{d}a a^*\,, \quad a\mathrm{d}a^* = \mathrm{d}a^* a\,, \quad a^*\mathrm{d}a^* = \mathrm{d}a^* a^*\,,\\
&v_+\mathrm{d}a = \mathrm{d}a v_+ + \frac{i}{\kappa}(a - I)\mathrm{d}a\,, \quad v_+\mathrm{d}a^* = \mathrm{d}a^* v_+ + \frac{i}{\kappa}(I - a)\mathrm{d}a^*\,,\\
&v_-\mathrm{d}a = \mathrm{d}a v_- + \frac{i}{\kappa}(a^* - I)\mathrm{d}a\,, \quad v_-\mathrm{d}a^* = \mathrm{d}a^* v_- + \frac{i}{\kappa}(a^* - I)\mathrm{d}a^*\,,\\
&a\mathrm{d}v_+ = \mathrm{d}v_+ a - \frac{i}{\kappa}a\mathrm{d}a\,, \qquad a\mathrm{d}v_- = \mathrm{d}v_- a - \frac{i}{\kappa}a^*\mathrm{d}a\,,\\
&a^*\mathrm{d}v_+ = \mathrm{d}v_+ a^* - \frac{i}{\kappa}a\mathrm{d}a^*\,, \qquad a^*\mathrm{d}v_- = \mathrm{d}v_- a^* - \frac{i}{\kappa}a^*\mathrm{d}a^*\,,\\
&v_+\mathrm{d}v_+ = \mathrm{d}v_+ v_+ - \frac{i}{\kappa}\mathrm{d}v_+\,, \qquad v_-\mathrm{d}v_- = \mathrm{d}v_- v_- - \frac{i}{\kappa}\mathrm{d}v_-\,,\\
&v_+\mathrm{d}v_- = -\mathrm{d}v_+ v_- - \frac{i}{\kappa}\mathrm{d}v_+ + \mathrm{d}\left(v_+v_- + \frac{i}{\kappa}v_+\right)\,,\\
&v_-\mathrm{d}v_+ = -\mathrm{d}v_- v_+ - \frac{i}{\kappa}\mathrm{d}v_- + \mathrm{d}\left(v_+v_- + \frac{i}{\kappa}v_+\right)\,.
\end{aligned} \tag{4}
$$

The left invariant forms read

$$
\begin{aligned}
&\omega_0 = \tfrac{1}{2}(a^*\mathrm{d}a - a\mathrm{d}a^*) = a^*\mathrm{d}a = -a\mathrm{d}a^*\,,\\
&\omega_+ = a^*\mathrm{d}v_+\,,\\
&\omega_- = a\mathrm{d}v_-\,,\\
&\widetilde{\omega}_0 = \mathrm{d}\left(v_+v_- + \frac{i}{\kappa}v_+\right) - v_+\mathrm{d}v_- - v_-\mathrm{d}v_+\,.
\end{aligned} \tag{5}
$$

Relations (4)–(5) lead, via Woronowicz construction [5], to the following external algebra :

$$
\begin{aligned}
&\omega_0 \wedge \omega_0 = 0\,,\\
&\widetilde{\omega}_0 \wedge \widetilde{\omega}_0 = 0\,,\\
&\omega_0 \wedge \omega_\pm = -\omega_\pm \wedge \omega_0\,,\\
&\omega_0 \wedge \widetilde{\omega}_0 = -\widetilde{\omega}_0 \wedge \omega_0\,,\\
&\omega_\pm \wedge \widetilde{\omega}_0 = -\widetilde{\omega}_0 \wedge \omega_\pm \pm \frac{2}{\kappa^2}\omega_0 \wedge \omega_\pm\,,\\
&\omega_\pm \wedge \omega_\pm = \mp\frac{i}{\kappa}\omega_0 \wedge \omega_\pm\,,\\
&\omega_- \wedge \omega_+ = -\omega_+ \wedge \omega_- + \frac{i}{\kappa}\omega_0 \wedge (\omega_- - \omega_+)\,.
\end{aligned} \tag{6}
$$

Also, using equations (4), (5) and (6) one easily gets

$$\begin{aligned} \mathrm{d}\omega_0 &= 0\,, \\ \mathrm{d}\tilde{\omega}_0 &= 0\,, \\ \mathrm{d}\omega_\pm &= \mp\omega_0 \wedge \omega_\pm\,. \end{aligned} \tag{7}$$

3. The Lie algebra of deformed E(2) group

It has been shown recently [7–9] that the deformed Lie algebra $\mathfrak{e}(2)$, dual to E(2), has the following structure:

$$\begin{aligned} &[P_1, P_2] = 0\,, \\ &[J, P_1] = iP_2\,, \\ &[J, P_2] = -i\kappa \sinh\left(\frac{P_1}{\kappa}\right)\,, && \text{(8a)} \\ &J^* = J\,, \quad P_1^* = P_1\,, \quad P_2^* = P_2\,, && \text{(8b)} \\ &\triangle P_1 = I \otimes P_1 + P_1 \otimes I\,, \\ &\triangle P_2 = \exp\left(-\frac{P_1}{2\kappa}\right) \otimes P_2 + P_2 \otimes \exp\left(\frac{P_1}{2\kappa}\right)\,, \\ &\triangle J = e^{-\frac{P_1}{2\kappa}} \otimes J + J \otimes \exp\left(\frac{P_1}{2\kappa}\right)\,. && \text{(8c)} \end{aligned}$$

On the other hand, Woronowicz theory [5] provides us with the general construction of Lie algebra once bicovariant calculus is given. The resulting Lie algebra is quadratic so it must differ from one given by equations (8). Therefore the question arises how the both algebras are related to each other. To address this problem we first construct the Woronowicz algebra for E(2). To this end we introduce counterparts of left-invariant vector fields by the formula ([5])

$$\mathrm{d}x = (\chi_0 * x)\omega_0 + (\tilde{\chi}_0 * x)\tilde{\omega}_0 + (\chi_+ * x)\omega_+ + (\chi_- * x)\omega_-\,, \tag{9}$$

where $x \in \mathrm{E}(2)$ and $\chi * x = (\mathrm{id} \otimes \chi)\triangle(x)$.

Using $\mathrm{d}(\mathrm{d}x) = 0$ and equations (6), (7) and (9) we arrive at the following algebra:

$$\begin{aligned} &[\chi_0, \tilde{\chi}_0] = 0\,, \\ &[\chi_\pm, \tilde{\chi}_0] = 0\,, \\ &[\chi_+, \chi_-] = 0\,, \\ &[\chi_0, \chi_+] = \chi_+ + \frac{i}{\kappa}\chi_+^2 + \frac{i}{\kappa}\chi_-\chi_+ - \frac{2}{\kappa^2}\chi_+\tilde{\chi}_0\,, \\ &[\chi_0, \chi_-] = -\chi_- - \frac{i}{\kappa}\chi_-^2 - \frac{i}{\kappa}\chi_-\chi_+ + \frac{2}{\kappa^2}\chi_-\tilde{\chi}_0\,. && \text{(10)} \end{aligned}$$

The $*$-structure can be introduced as follows ([7–9])

$$\chi^*(x) = \overline{\chi(S^{-1}(x^*))} \,. \tag{11}$$

It is not difficult to show that, due to the relation $S(x)^* = S^{-1}(x^*)$ valid in our E(2) algebra, the conjugation rule (11) can be rephrased in Woronowicz formalism as

$$\langle \omega, \chi^* \rangle = -\overline{\langle \omega^*, \chi \rangle} \,. \tag{12}$$

In our case

$$\begin{aligned}
\chi_0^* &= \chi_0 + \frac{i}{\kappa}(\chi_+ - \chi_-)\,, \\
\chi_\pm^* &= -\chi_\mp\,, \\
\tilde{\chi}_0^* &= \tilde{\chi}_0\,.
\end{aligned} \tag{13}$$

Now, it is straightforward to check that equations (10) and (13) are equivalent to equations (8a) and (8b) provided we make the following substitutions:

$$\begin{aligned}
\chi_0 &= \exp\left(-\frac{P_1}{2\kappa}\right)\left(J + \frac{i}{4\kappa}P_2\right)\,, \\
\tilde{\chi}_0 &= -\frac{1}{4}\left(4\kappa^2 \sinh^2\left(\frac{P_1}{2\kappa}\right) + P_2^2\right)\,, \\
\chi_- - \chi_+ &= P_2 \exp\left(-\frac{P_1}{2\kappa}\right)\,, \\
I + \frac{i}{\kappa}(\chi_+ + \chi_-) - \frac{2}{\kappa^2}\tilde{\chi}_0 &= \exp\left(-\frac{P_1}{\kappa}\right)\,.
\end{aligned} \tag{14}$$

Obviously, we should also compare the coalgebra structures. This is done as follows. First, the coproduct for χ's is defined as follows [5] . Let $\omega_i = (\omega_0, \tilde{\omega}_0, \omega_+, \omega_-)$; we define the functionals f_{ij} by

$$\omega_i x = \sum_j (f_{ij} * x)\omega_j \,. \tag{15}$$

Then the coproduct for χ_i is then defined by

$$\Delta\chi_i = \sum_j \chi_j \otimes f_{ji} + I \otimes \chi_i \,. \tag{16}$$

Formulae (15) and (16) allow us to calculate the coproducts for χ's. It is then straightforward to check that Eqs (14) give a coalgebra map. The detailed calculations are rather lengthy and will be not reported here.

4. Differential calculus on the quantum plane

Let us define the quantum plane Π as the algebra with unity generated by two elements x_+, $x_- = (x_+)^*$ subject to the following condition

$$[x_+, x_-] = \frac{i}{\kappa}(x_- - x_+)\,. \tag{17}$$

If we supply Π with the coproduct, antipode and counit:

$$\begin{aligned} \triangle x_\pm &= x_\pm \otimes I + I \otimes x_\pm\,, \\ S(x_\pm) &= -x_\pm\,, \\ \varepsilon(x_\pm) &= 0 \end{aligned} \tag{18}$$

it becomes a quantum subgroup of deformed E(2). Therefore the Woronowicz construction for Π considered as quantum group can be obtained by putting $a \to I$, $v_\pm \to x_\pm$ in all formulae above. As a result we obtain differential calculus determined by the ideal generated by $x_+^2 - \frac{i}{\kappa}x_+$, $x_-^2 - \frac{i}{\kappa}x_-$. The basic rules are as follows:

$$\begin{aligned} x_\pm \mathrm{d}x_\pm &= \mathrm{d}x_\pm x_\pm - \frac{i}{\kappa}\mathrm{d}x_\pm\,, \\ x_\pm \mathrm{d}x_\mp &= -\mathrm{d}x_\pm x_\mp - \frac{i}{\kappa}\mathrm{d}x_\pm + \mathrm{d}\left(x_+x_- + \frac{i}{\kappa}x_+\right)\,. \end{aligned} \tag{19}$$

The invariant forms read:

$$\begin{aligned} \Omega_\pm &= \mathrm{d}x_\pm\,, \\ \tilde{\Omega}_0 &= \mathrm{d}\left(x_+x_- + \frac{i}{\kappa}x_+\right) - x_+\mathrm{d}x_- - x_-\mathrm{d}x_+\,. \end{aligned} \tag{20}$$

The external algebra reads:

$$\begin{aligned} \Omega_+ \wedge \Omega_- &= -\Omega_- \wedge \Omega_+\,, \\ \tilde{\Omega}_0 \wedge \Omega_\pm &= -\Omega_\pm \wedge \tilde{\Omega}_0\,, \\ \Omega_\alpha \wedge \Omega_\alpha &= 0 \qquad \alpha = +,-,0\,. \end{aligned} \tag{21}$$

Let us define the action of E(2) on Π as follows:

$$\begin{aligned} \varrho(I) &= I \otimes I\,, \\ \varrho(x_+) &= a \otimes x_+ + v_+ \otimes I\,, \\ \varrho(x_-) &= a^* \otimes x_- + v_- \otimes I \end{aligned} \tag{22}$$

extended by linearity and multiplicavity. Obviously this action is a homomorphism $\Pi \to \mathrm{E}(2) \otimes \Pi$ and obeys

$$\begin{aligned}(\mathrm{id} \otimes \varrho) \circ \varrho &= (\triangle \otimes \mathrm{id}) \circ \varrho\,, \\ (\varepsilon \otimes \mathrm{id}) \circ \varrho &= \mathrm{id}\,. \end{aligned} \tag{23}$$

Let us now make the following remark. One can easily extend the Woronowicz theory as follows (the notation below is taken from Ref. [5]).

Assume $\mathcal{A}$ is an algebra with unity, $\mathcal{B}$ — quantum group and $\varrho : \mathcal{A} \to \mathcal{B} \otimes \mathcal{A}$ homomorphism obeying (23). We define linear map

$$\tilde{\varrho} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A}$$

as follows: let

$$q = \sum_i x_i \otimes y_i\,, \qquad \varrho(x_i) = \sum_k a_i^k \otimes x_i^k\,, \qquad \varrho(y_i) = \sum_j b_i^j \otimes y_i^j\,,$$

then

$$\tilde{\varrho}(q) = \sum_{i,j,k} a_i^k b_i^j \otimes x_i^k \otimes y_i^j\,.$$

It is then easy to show that $\tilde{\varrho} : \mathcal{A}^2 \to \mathcal{B} \otimes \mathcal{A}^2$ and is given by:

$$\tilde{\varrho}\left(\sum_i x_i \otimes y_i\right) = \sum_i \varrho(x_i)(\mathrm{id} \otimes \mathrm{D})\varrho(y_i)\,.$$

Now, we assume that $\mathcal{N} \subset \mathcal{A}^2$ is a sub-bimodule such that $\tilde{\varrho}(\mathcal{N}) \subset \mathcal{B} \otimes \mathcal{N}$. Then the differential calculus (Γ, d) defined by $\mathcal{N}$ has the following property

$$\sum_k x_k \mathrm{d}y_k = 0 \quad \Rightarrow \quad \sum_k \varrho(x_k)(\mathrm{id} \otimes \mathrm{d})\varrho(y_k) = 0\,.$$

Therefore $\tilde{\varrho}\left(\sum_k x_k \mathrm{d}y_k\right) = \sum_k \varrho(x_k)(\mathrm{id} \otimes \mathrm{d})\varrho(y_k)$ is well-defined linear mapping from Γ into $\mathcal{B} \otimes \Gamma$.

Following the same lines as in Ref. [5] we can easily prove the following properties of $\tilde{\varrho}$:

(i) for $x \in \mathcal{A}$, $y \in \Gamma$

$$\begin{aligned}\tilde{\varrho}(xy) &= \varrho(x)\tilde{\varrho}(y)\,, \\ \tilde{\varrho}(yx) &= \tilde{\varrho}(y)\varrho(x)\,, \end{aligned} \tag{24}$$

(ii)

$$\tilde{\varrho} \circ \mathrm{d} = (\mathrm{id} \otimes \mathrm{d}) \circ \varrho ,$$

(iii)

$$(\mathrm{id} \otimes \tilde{\varrho}) \circ \tilde{\varrho} = (\triangle \otimes \mathrm{id}) \circ \tilde{\varrho}, \tag{25}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \tilde{\varrho} = \mathrm{id} . \tag{26}$$

In our case $\mathcal{A} = \Pi$, $\mathcal{B} = \mathrm{E}(2)$, ϱ is given by equation (22). To check that $\tilde{\varrho}(\mathcal{N}) \subset \mathcal{B} \otimes \mathcal{N}$ we use the results of Ref. [5] which, together with the property *(i)* above imply that it is sufficient to consider elements $r^{-1}\left(I \otimes \left(x_{\pm}^2 - \frac{i}{\kappa} x_{\pm}\right)\right)$. Simple explicit calculations verify the property under consideration.

The action $\tilde{\varrho}$ is given by:

$$\begin{aligned}
&\tilde{\varrho}(\mathrm{d}x_+) = a \otimes \mathrm{d}x_+ , \\
&\tilde{\varrho}(\mathrm{d}x_-) = a^* \otimes \mathrm{d}x_- , \\
&\tilde{\varrho}\left(\mathrm{d}\left(x_+ x_- + \frac{i}{\kappa} x_+\right) - x_+ \mathrm{d}x_- - x_- \mathrm{d}x_+\right) \\
&= I \otimes \left(\mathrm{d}\left(x_+ x_- + \frac{i}{\kappa} x_+\right) - x_+ \mathrm{d}x_- - x_- \mathrm{d}x_+\right) .
\end{aligned} \tag{27}$$

Let us now describe our calculus in terms of real coordinates : $x_+ = x_1 + ix_2$, $x_- = x_1 - ix_2$. We get

$$\begin{aligned}
&[x_1, x_2] = \frac{i}{\kappa} x_2 , \\
&[x_1, \mathrm{d}x_2] = 0 , \\
&[x_2, \mathrm{d}x_1] = -\frac{i}{\kappa} \mathrm{d}x_2 , \\
&[x_1, \mathrm{d}x_1] = \frac{1}{\kappa^2} \Phi , \\
&[x_2, \mathrm{d}x_2] = \frac{1}{\kappa^2} \Phi + \frac{i}{\kappa} \mathrm{d}x_1
\end{aligned} \tag{28}$$

to be compared with $3d$ calculus on $2d$ deformed Minkowski space given by Sitarz [3].

Finally, let us construct the infitesimal transformations. For any linear functional χ we define

$$\chi_{\varrho} = (\chi \otimes \mathrm{id}) \circ \varrho , \chi_{\tilde{\varrho}} = (\chi \otimes \mathrm{id}) \circ \tilde{\varrho} . \tag{29a}$$

The first definition, equation (29a), coincides with the one used by Majid and Ruegg [4], while the second one is equivalent to Sitarz proposal, equation

(23) of Ref. [3]. Indeed, we have

$$\begin{aligned} \chi_{\tilde{\varrho}}(x\mathrm{d}y) &= (\chi \otimes \mathrm{id})(\varrho(x)(\mathrm{id} \otimes \mathrm{d})\varrho(y)) \\ &= [(\chi_{(1)} \otimes \mathrm{id})\varrho(x)](\mathrm{id} \otimes \mathrm{d})[(\chi_{(2)} \otimes \mathrm{id})\varrho(y)] \\ &= \chi_{(1)\varrho}(x)(\mathrm{id} \otimes \mathrm{d})\chi_{(2)\varrho}(y)\,, \end{aligned} \tag{30}$$

where $\triangle\chi = \chi_{(1)} \otimes \chi_{(2)}$.

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