# ELEMENTARY DERIVATION OF A RECENTLY PROPOSED INTEGRAL REPRESENTATION FOR PERMANENTS

K. Zalewski*

Institute of Physics, Jagellonian University
Reymonta 4, 30-059, Cracow, Poland
and
Institute of Nuclear Physcis
Kawiory 26a, 30-055 Cracow, Poland

(Received March 11, 1997)

A recently proposed integral representation for permanents is rederived using only elementary combinatorics. For this proof the assumption that the matrix, for which the permanent is calculated, has an inverse is not necessary.

PACS numbers: 02.10. Eb, 12.40. Ee

The permanent of an $N \times N$ matrix $A$ is the number

$$P_N = \sum_Q \prod_{i=1}^{N} A_{i\,Qi}, \tag{1}$$

where $Q$ is a permutation of the numbers $i = 1, 2, \ldots, N$ and the summation extends over all the $N!$ permutations. This formula is similar to the formula defining the determinant of matrix $A$, but the minus signs in front of the terms where the permutations $Q$ are odd are missing. In practice this makes the handling of permanents more cumbersome than the handling of determinants. The numerical evaluation of permanents, even of moderate size, is usually quite hard. This is unfortunate, because they occur in some physical problems *e.g.* in the description of the Bose–Einstein correlations in high energy multiple particle production processes. The general experience is that even big computers get in trouble with the exact formula soon after $N$ exceeds ten, which is to soon for the applications.

* Supported in part by the KBN grant 2P302-076-07.

A new approach to the evaluation of permanents has been recently proposed by Wosiek [1], who pointed out that, when the inverse of matrix $A$ exists, permanent (1) is equal to the integral

$$P_N = 2^{-N} \int \prod_{i=i}^{N} \frac{dxdy}{2\pi} \exp\left(-\frac{x_i^2+y_i^2}{2}\right) \times \left[\left(\sum_{k=1}^{N} e_{ik}x_k\right)^2 + \left(\sum_{k=1}^{N} e_{ik}y_k\right)^2\right], \tag{2}$$

where matrix $e$, which is defined below, is constructed from the eigenvectors and eigenvalues of matrix $A$. Here and in the following it is assumed that matrix $A$ is real and symmetric. Since there are many methods of approximating complicated integrals — *e.g.* Monte Carlo methods and steepest descent methods — it is plausible that it is easier to get a good approximation to $P_N$ when starting from formula (2) then when starting from formula (1). The examples given in Ref. [1] are very encouraging. In Ref. [1] formula (2) was derived using methods inspired by quantum field theory. In the present note we give an alternative, elementary proof. For this proof the assumption that matrix $A$ has an inverse is no more necessary.

Let us note first that, since matrix $A$ is real and symmetric, it can be diagonalized by an orthogonal transformation

$$O^T A O = \Lambda, \tag{3}$$

where $\Lambda$ is a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_N$. There may be many orthogonal matrices satisfying equation (3). In such cases any of them can be chosen. Multiplying equality (3) by matrix $O$ from the left and by matrix $O^T$ from the right, defining [1]

$$e_{ij} = O_{ij}\sqrt{\lambda_j} \tag{4}$$

and using the orgthogonality property $OO^T = O^T O = 1$ we find

$$A_{ij} = \sum_{k=1}^{N} e_{ik}e_{jk}. \tag{5}$$

Note further that the integral in formula (2) is a linear combination of products of integrals of the type

$$\langle x^{2n}\rangle \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2}\, x^{2n} = (2n-1)!! \quad \text{for} \quad n \geq 1. \tag{6}$$

In particular $\langle x^2 \rangle = 1$. Of course $\langle 1 \rangle = 1$ and for any integer nonnegative $n$ — $\langle x^{2n+1} \rangle = 0$,

We shall also use the fact that to each term in sum (1) corresponds a diagram consisting of $N$ labelled vertices connected by $N$ lines, the $i$-th line connecting the vertices $i$ and $Qi$. Note that a line can connect a vertex to itself. Since each permutation can be decomposed into cycles, each diagram consists of closed loops. For loops containing more than two vertices the two orientations of the loop are distinguishable. Thus *e.g.* for $N = 3$ the term $A_{11}A_{22}A_{33}$ corresponds to three isolated vertices (one-vertex loops), the terms $A_{11}A_{23}A_{32}$, $A_{22}A_{13}A_{31}$ and $A_{33}A_{12}A_{21}$ to the three diagrams consisting each of an isolated vertex and a two-vertex loop and the terms $A_{12}A_{23}A_{31}$ and $A_{13}A_{32}A_{21}$ to the two three-vertex loops with different orientations. Ascribing to each diagram the corresponding product of matrix elements we find that the permanent is the sum of the contributions corresponding to all the diagrams. For a symmetric matrix $A$ the contributions corresponding to loops differing only by their orientation are equal. Thus, an alternative approach is to ignore the loop orientations and to ascribe an additional factor of two to each loop containing more than two vertices. We shall need somewhat more complicated diagrams. Let us label also the lines and ascribe to line labelled $k$ and connecting vertices $i$ and $Qi$ the number $e_{ik}e_{Qi\,k}$. For such labelled diagrams it is still true that the permanent is the sum of contributions corresponding to all the diagrams.

In order to evaluate the integral (2) let us consider first the terms in the expansion of the integrand, where only the variables $x_1, \ldots, x_N$ occur and where each of them occurs exactly twice. In this case each of the integrals (6) equals one and we obtain the contribution to $P_N$

$$\Pi_N^0 = 2^{-N} \sum_Q \prod_{i=1}^{N}{}' \sum_{k=1}^{N} e_{ik}e_{Qi\,k} 2^{N-L(Q)}, \tag{7}$$

where $L(Q)$ is the number of cycles in permutation $Q$, or equivalently the number of loops in the corresponding diagram, the prime over the product sign means that only the terms where each value of the index $k$ occurs exactly twice should be kept. In terms of labelled diagrams, only the diagrams where no two lines have the same label are included and the contribution corresponding to each diagram is reduced by the factor $2^{-L(Q)}$. This factor arises as follows. Consider a loop with $M$ vertices. Each vertex corresponds to one factor from the product in the integrand of formula (2). For $M = 1$ and $M = 2$ there are respectively one and two ways of choosing the necessary $x$-variables. For $M > 2$ at each vertex there are two ways of choosing the necessary $x$-variable, thus the factor is $2^M$. This, however, should be divided between the two loops differing by their orientation. Thus for each loop the factor is $2^{M-1}$. It is seen that this formula works also for $M = 1, 2$. Evaluating the product of the factors corresponding to all the loops in the diagram we find the overall factor $2^{N-L(Q)}$ as given in formula (7).

As mentioned, the diagrams corresponding to formula (7) differ from the diagrams corresponding to formula (1) in two ways. Firstly, the diagrams where groups of lines have equal labels are missing. Secondly, there is an additional factor of $2^{-L(Q)}$ for each diagram. In order to remove the first difference let us start with a diagram, where a group of p lines has different labels and look at the effect of ascribing the same label $l$ to these $p$ lines. It is assumed that none of the other lines has either the label $l$ or the label of any of the original $p$ lines. We will show that as a result the number of diagrams increases by a factor $(2p - 1)!!$. Consider the $2p$ vertices at the ends of the $p$ lines. Of course some labelled vertices may occur in this list twice. All the different connections of these $2p$ vertices by the $p$ lines can be counted as follows. Consider the $(2p)!$ permutations of the $2p$ vertices and in each permutation connect with a line labelled $l$ the first vertex with the second, the third with the fourth *etc.* This gives all the connections, but with much multiple counting. The $p!$ permutations of the lines among themselves and the $2^p$ exchanges of the ends of the lines do not change the diagram. Thus finally there are

$$\frac{(2p)!}{2^p p!} = (2p - 1)!! \tag{8}$$

new diagrams for the single original one, which proves our statement. On the other hand, in the integral this change introduces a factor $\langle x^{2p} \rangle = (2p-1)!!$, which is just enough to give weight one to each of the new diagrams. Applying repeatedly this argument it is seen that summing all the terms, where the expansion of the integrand contains the variables $x_1, \ldots, x_N$ only, one obtains

$$\Pi_N = \sum_Q \prod_{i=1}^{N} \sum_{k=1}^{N} e_{ik} e_{Qik} 2^{-L(Q)}, \tag{9}$$

where the restriction that different line should carry different labels has been removed. The factors $2^{-L(Q)}$ is justified just like in the case when all the lines have different labels.

Let us consider now the contributions from the terms containing the $y$ variables. Including such terms it is easily seen that the diagrams are as before, but that a loop can originate either only from the $x$ integrations, or only from the $y$ integrations. Thus, the result of including the $y$ variables is that each loop should be counted twice, or equivalently that it gets an additional factor of two in its weight. For a complete diagram this yields the additional factor $2^{L(Q)}$, which exactly cancels the unwanted factors in formula (9). This completes the proof of formula (2).

## REFERENCES

[1] J. Wosiek, A simple formula for Bose–Einstein corrections, Jagellonian University preprint TPJU 1/97 and *Phys. Letters* **B** in print.