REAL WAVE EQUATIONS AND EXACTNESS RELATIONS FOR COMPLEXIFIED MAXWELL–DIRAC FIELDS

J.G. Cardoso

Department of Mathematics Centre for Technological Sciences-UDESC
Joinville 89223-100 SC, Brazil

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The real version of the wave equations that control the propagation of Maxwell-Dirac fields in complex Minkowski space is presented. It is particularly shown that the electromagnetic potential turns out to be subject to certain pluriharmonicity conditions whenever the Maxwell fields are taken to carry positive energy. The actual derivation of a set of invariant exactness relations for the entire system is then carried out.

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1. Introduction

One of the purposes of the present paper is to build up a set of real wave equations for complexified Maxwell-Dirac fields. A remarkable result arising in this framework is that if the electromagnetic fields are required at the outset to bear positive energy, then the Maxwell potential turns out to satisfy specific pluriharmonicity relations on the forward tube $\mathbf{CM}^+$ of complex Minkowski space $\mathbf{CM}$. The relevant statements thus appear to be directly associated with the corresponding holomorphicity condition. Our paper is also aimed at carrying out explicitly the derivation of second-order derivative relations for the fields and potential which bring out both the invariance and the exactness of the complete system whenever the Lorentz gauge is effectively adopted [1]. The formulation of the theory in $\mathbf{CM}$ shall obviously be taken for granted from the beginning [2], and will indeed afford the basic blocks upon which the elaboration of our work rests.

There are two somewhat strong motivations lying behind the completion of the pertinent procedures. One is the fact that such a work would certainly enhance some of the striking features of the inner structure of the system. The other is essentially related to the absence of prescriptions which might enable one to establish the invariant exactness of the system in a manifestly methodical way. We have organized the paper as follows: Section 2 recalls the complexified field and wave equations which are of immediate interest to us. In Section 3, we exhibit a symbolic version of the holomorphicity requirements along with the pluriharmonicity relations. There, it will suffice to work out the key potential statements by using the Lorentz gauge. In addition, we will assume that the Dirac fields also satisfy the positive-energy condition. Roughly speaking, the reason underlying this assumption is that it entails the occurrence of real structures possessing a useful symmetry. Section 4 deals with the presentation of the real wave equations. The methods for constructing the derivative relations are given in Section 5. We make some remarks on our procedures in Section 6. Throughout the paper, we utilize the natural system of units wherein $c = \hbar = 1$ as well as the spinor conventions and rules provided by Penrose and Rindler [1]. The set $\mathbf{CM}^+$ will be looked upon as the topological product $\mathbf{RM} \times \mathbf{V}^+$, with the factors involved standing for the real slice of $\mathbf{CM}$ and the (convex) interior of the closure of the future cone of some origin of $\mathbf{RM}$, respectively. For convenience, an arbitrary point $x^a \in \mathbf{CM}$ will be defined as $\xi^a - i\eta^a$, where ξ^a and η^a are both real vectors. The ordinary (holomorphic) partial derivative operator $\partial/\partial x^{AA'}$ on $\mathbf{CM}$ will be denoted by $\nabla_{AA'}$. All the spinor quantities occurring in our statements must be regarded as complex mappings on suitable products between unprimed and primed $\mathbf{SL}(2,\mathbf{C})\otimes\mathbf{SL}(2,\mathbf{C})$ fibers over x^a. In particular, the individual Maxwell and Dirac fields will be written as $\phi_{AB}(x)$, $\theta^{A'B'}(x)$ and $\psi_A(x)$, $\chi^{A'}(x)$ whilst $\Phi_{AA'}(x)$ will stand for the electromagnetic potential. It is evident that the symmetry of the Maxwell fields makes it immaterial to order the indices borne by them. Whence we will not stagger their upper and lower indices when performing any calculations involving them.

2. Complexified equations

The Maxwell part of the complexified theory is constituted by the gauge-invariant field equations

$$\nabla^B_{A'}\phi_{AB}(x) = 2\pi J_{AA'}(x) = \nabla^{B'}_A\theta_{A'B'}(x)\,, \tag{2.1}$$

along with the field-potential relationships

$$\phi_{AB}(x) = \nabla_{A'(A}\Phi^{A'}_{B)}(x)\,, \tag{2.2a}$$

$$\theta_{A'B'}(x) = \nabla_{A(A'}\Phi^{A}_{B')}(x)\,, \tag{2.2b}$$

with $J_{AA'}(x)$ being the (divergenceless) current density of the theory, which is effectively given by

$$J_{AA'}(x) = e[\widetilde{\psi}_{A'}(x)\psi_A(x) + \widetilde{\chi}_A(x)\chi_{A'}(x)]. \tag{2.3}$$

In Eq. (2.3), the quantity e denotes the charge of the Dirac fields and the elements of the pair $(\widetilde{\psi}_{A'}(x), \widetilde{\chi}_A(x))$ are the so-called conjugate Dirac fields (see Ref.[2]). Adopting the Lorentz gauge $A(x) = \nabla_a \Phi^a(x) = 0$ allows one to drop the symmetrization round-brackets from Eqs. (2.2). Hence, this choice turns out to be associated to the ε-tracelessness relations

$$A(x) = \phi_A^A(x) = \theta_{A'}^{A'}(x) = 0, \tag{2.4}$$

which really amount to the symmetry property of the electromagnetic fields. The Dirac equations show up as the statements

$$\mathcal{D}^{AA'}\psi_A(x) = \mu\chi^{A'}(x), \tag{2.5a}$$

$$\mathcal{D}_{AA'}\chi^{A'}(x) = -\mu\psi_A(x), \tag{2.5b}$$

where $\mu = m/\sqrt{2}$ is the normalized rest mass, and $\mathcal{D}_{AA'}$ denotes the (minimal coupling) covariant derivative operator of the entire theory whose defining expression is written as

$$\mathcal{D}_{AA'} = \nabla_{AA'} - ie\Phi_{AA'}(x). \tag{2.6}$$

The wave equations which govern the propagation of photons are written out explicitly as [2]

$$\Box\phi_{AB}(x) = 4\pi e\nabla_{A'(A}[\widetilde{\psi}^{A'}(x)\psi_{B)}(x) + \widetilde{\chi}_{B)}(x)\chi^{A'}(x)], \tag{2.7}$$

$$\Box\theta_{A'B'}(x) = 4\pi e\nabla_{A(A'}[\widetilde{\psi}_{B')}(x)\psi^A(x) + \widetilde{\chi}^A(x)\chi_{B')}(x)], \tag{2.8}$$

or, alternatively, as

$$\begin{aligned}\Box\phi_{AB}(x) = {} & 4\pi e\{\widetilde{\psi}^{A'}(x)\nabla_{A'(A}\psi_{B)}(x) + [\nabla_{A'(A}\widetilde{\chi}_{B)}(x)]\chi^{A'}(x) \\ & + ie[\widetilde{\chi}_{(A}(x)\Phi_{B)A'}(x)\chi^{A'}(x) - \Phi_{A'(A}(x)\widetilde{\psi}^{A'}(x)\psi_{B)}(x)] \\ & -2\mu\widetilde{\chi}_{(A}(x)\psi_{B)}(x)\},\end{aligned} \tag{2.9}$$

$$\begin{aligned}\Box\theta_{A'B'}(x) = {} & 4\pi e\{[\nabla_{A(A'}\widetilde{\psi}_{B')}(x)]\psi^A(x) + \widetilde{\chi}^A(x)\nabla_{A(A'}\chi_{B')}(x) \\ & + ie[\widetilde{\psi}_{(A'}(x)\Phi_{B')A}(x)\psi^A(x) - \Phi_{A(A'}(x)\widetilde{\chi}^A(x)\chi_{B')}(x)] \\ & - 2\mu\widetilde{\psi}_{(A'}(x)\chi_{B')}(x)\},\end{aligned} \tag{2.10}$$

with $\Box = \nabla_{AA'}\nabla^{AA'}$. For the potential, we have the statement

$$\Box\Phi_{AA'}(x) = 4\pi e[\tilde{\psi}_{A'}\ (x)\psi_A(x) + \tilde{\chi}_A\ (x)\chi_{A'}(x)] + \nabla_{AA'}\Lambda(x). \quad (2.11)$$

The propagation of electrons can be thought of as being controlled by the "extended" Klein-Gordon equations

$$(\hat{\Delta} + m^2)\psi_A(x) = 2ie\phi_A^B(x)\psi_B(x), \quad (2.12)$$

$$(\hat{\Delta} + m^2)\chi^{A'}(x) = -2ie\theta_{B'}^{A'}(x)\chi^{B'}(x)\,, \quad (2.13)$$

with the operator involved being expressed as

$$\hat{\Delta} = \mathcal{D}^{AA'}\mathcal{D}_{AA'} = \Box - ie[2\Phi_{AA'}(x)\nabla^{AA'} + \Lambda(x)] - e^2\Phi_{AA'}(x)\Phi^{AA'}(x). \quad (2.14)$$

If we implement this explicit operator expression, Eqs. (2.12) and (2.13) take the simpler form

$$(\Box + m^2)\psi_A(x) = \sigma_A(x)\,, \quad (2.15)$$

$$(\Box + m^2)\chi^{A'}(x) = \Sigma^{A'}(x), \quad (2.16)$$

where the source spinors are given by

$$\sigma_A(x) = 2ie\phi_A^B(x)\psi_B(x) + ie[2\Phi_{BB'}(x)\nabla^{BB'} + \Lambda(x)]\psi_A(x) + e^2\Phi_{BB'}(x)\Phi^{BB'}(x)\psi_A(x), \quad (2.17)$$

and

$$\Sigma^{A'}(x) = -2ie\theta_{B'}^{A'}(x)\chi^{B'}(x) + ie[2\Phi_{BB'}(x)\nabla^{BB'} + \Lambda(x)]\chi^{A'}(x) + e^2\Phi_{BB'}(x)\Phi^{BB'}(x)\chi^{A'}(x). \quad (2.18)$$

A point concerning the formal simplicity of Eqs. (2.15) and (2.16) will be made later in Section 4 when we effectively consider the situation that involves the construction of real structures for the Dirac fields.

3. Harmonicity and pluriharmonicity conditions

It shall now be assumed that the fields which enter into the former system are all holomorphic throughout $\mathbf{CM}^+$. We thus have the symbolic positive-energy requirements

$$\tilde{\nabla}_{CC'}\ \Omega_{\bullet\bullet}(x) = 0, \quad (3.1)$$

and

$$\tilde{\nabla}_{CC'}\ \Upsilon_{\bullet}(x) = 0 \text{ on } \mathbf{CM}^+, \quad (3.2)$$

where $\tilde{\nabla}_{CC'}$ is the antiholomorphic partial derivative operator on **CM**, and $\Omega_{\bullet\bullet}(x)$ (resp. $\Upsilon_{\bullet}(x)$) stands for either $\phi_{AB}(x)$ or $\theta_{A'B'}(x)$ (resp. $\psi_A(x)$ or $\chi^{A'}(x)$). Accordingly, the field set being considered can be split up into its left-handed and right-handed parts. In effect, we have

$$\mathbf{LH} = \{\phi_{AB}(x), \psi_A(x)\}, \tag{3.3}$$

$$\mathbf{RH} = \{\theta_{A'B'}(x), \chi^{A'}(x)\}. \tag{3.4}$$

It follows that, making use of the relation

$$\tilde{\nabla}_{CC'} = \tfrac{1}{2}(\overset{\xi}{\nabla}_{CC'} - i\overset{\eta}{\nabla}_{CC'}), \tag{3.5}$$

together with the splittings

$$\Omega_{\bullet\bullet}(x) = \operatorname{Re}\Omega_{\bullet\bullet}(\xi,\eta) - i\operatorname{Im}\Omega_{\bullet\bullet}(\xi,\eta), \tag{3.6}$$

and

$$\Upsilon_{\bullet}(x) = \operatorname{Re}\Upsilon_{\bullet}(\xi,\eta) - i\operatorname{Im}\Upsilon_{\bullet}(\xi,\eta), \tag{3.7}$$

yields the Cauchy-Riemann equations on $\mathbf{RM}\times\mathbf{V}^+$

$$\overset{\xi}{\nabla}_{CC'}\ \operatorname{Re}\Omega_{\bullet\bullet}(\xi,\eta) = \overset{\eta}{\nabla}_{CC'}\ \operatorname{Im}\Omega_{\bullet\bullet}(\xi,\eta)\,, \tag{3.8}$$

$$\overset{\xi}{\nabla}_{CC'}\ \operatorname{Im}\Omega_{\bullet\bullet}(\xi,\eta) = -\overset{\eta}{\nabla}_{CC'}\ \operatorname{Re}\Omega_{\bullet\bullet}(\xi,\eta), \tag{3.9}$$

along with the ones carrying $\Upsilon_{\bullet}(\xi,\eta)$ in place of $\Omega_{\bullet\bullet}(\xi,\eta)$. We must stress that the spotted "subscripts" borne by the above real structures have been introduced only to effectively denote the spinor indices which label the relevant field components. Therefore, such "symbols" do not appear to play any other role here. Thus, the harmonicity statements for the Maxwell fields, say, are written as

$$(\Box_\xi + \Box_\eta)\operatorname{Re}\Omega_{\bullet\bullet}(\xi,\eta) = 0, \tag{3.10}$$

$$(\Box_\xi + \Box_\eta)\operatorname{Im}\Omega_{\bullet\bullet}(\xi,\eta) = 0, \tag{3.11}$$

where $\Box_\lambda = \overset{\lambda}{\nabla}_{CC'}\overset{\lambda}{\nabla}{}^{CC'}$, with λ denoting either ξ or η. In carrying out the explicit derivation of Eqs. (3.10) and (3.11), it is convenient to employ relations of the type [3]

$$\overset{\xi}{\nabla}_{AA'}\overset{\eta}{\nabla}_{B'}{}^{A} + \overset{\xi}{\nabla}_{B'}{}^{A}\ \overset{\eta}{\nabla}_{AA'} = \varepsilon_{A'B'}\ \overset{\xi}{\nabla}_{CC'}\overset{\eta}{\nabla}{}^{CC'}. \tag{3.12}$$

Notice that the individual pieces carried by the left-hand side of Eq. (3.12), which are symmetric in the indices A' and B', do not vanish identically.

Nevertheless, they turn out to cancel each other when the computation is carried through.

When combined together with the holomorphicity condition (3.1), the bivector relationships (2.2) yield at once a set of pluriharmonicity relations for $\Phi_{AA'}(x)$. More explicitly, we have on $\mathbf{CM}^+$

$$\tilde{\nabla}_{CC'}\ \phi_{AB}(x) = 0 = \tilde{\nabla}_{CC'}\ \nabla_{A'(A}\Phi^{A'}_{B)}(x), \tag{3.13}$$

$$\tilde{\nabla}_{CC'}\ \theta_{A'B'}(x) = 0 = \tilde{\nabla}_{CC'}\ \nabla_{A(A'}\Phi^{A}_{B')}(x). \tag{3.14}$$

Indeed, to see what the typical real patterns of these conditions look like, it suffices to work out either of them by taking up the Lorentz gauge. We thus have the derivative operator

$$\tilde{\nabla}_C{}^{C'}\nabla_A{}^{A'} = \frac{1}{4}\Big[\frac{1}{4}\varepsilon^{C'A'}\varepsilon_{CA}(\Box_\xi + \Box_\eta) + \overset{\xi}{\nabla}_{(C}{}^{(C'}\overset{\xi}{\nabla}_{A)}{}^{A')} + \overset{\eta}{\nabla}_{(C}{}^{(C'}\overset{\eta}{\nabla}_{A)}{}^{A')}$$
$$-i(\varepsilon^{C'A'}\overset{\xi}{\nabla}_{(C}{}^{M'}\overset{\eta}{\nabla}_{A)M'} + \varepsilon_{CA}\overset{\xi}{\nabla}{}^{M(A'}\overset{\eta}{\nabla}_M{}^{C')})\Big], \tag{3.15}$$

since the "crossed" pieces

$$2\overset{\xi}{\nabla}_{(C}{}^{[C'}\overset{\eta}{\nabla}_{A)}{}^{A']}, 2\overset{\xi}{\nabla}_{[C}{}^{(C'}\overset{\eta}{\nabla}_{A]}{}^{A')}$$

are the only ones which survive after the completion of the computation of $\mathrm{Im}\ \tilde{\nabla}_C{}^{C'}\nabla_A{}^{A'}$. Therefore, writing

$$\Phi_{DD'}(x) = \mathrm{Re}\,\Phi_{DD'}(\xi,\eta) - i\,\mathrm{Im}\,\Phi_{DD'}(\xi,\eta), \tag{3.16}$$

yields the pluriharmonicity relations corresponding to Eq. (3.13)

$$\varepsilon_{CA}\Big[\frac{1}{4}(\Box_\xi + \Box_\eta)\,\mathrm{Re}\,\Phi^{C'}_B(\xi,\eta) - \overset{\xi}{\nabla}{}^{M(A'}\overset{\eta}{\nabla}_M{}^{C')}\,\mathrm{Im}\,\Phi_{BA'}(\xi,\eta)\Big]$$
$$+[\overset{\xi}{\nabla}_{(C}{}^{(C'}\overset{\xi}{\nabla}_{A)}{}^{A')} + \overset{\eta}{\nabla}_{(C}{}^{(C'}\overset{\eta}{\nabla}_{A)}{}^{A')}]\,\mathrm{Re}\,\Phi_{BA'}(\xi,\eta)$$
$$-\overset{\xi}{\nabla}_{(C}{}^{M'}\overset{\eta}{\nabla}_{A)M'}\ \mathrm{Im}\,\Phi^{C'}_B(\xi,\eta) = 0, \tag{3.17}$$

and

$$\varepsilon_{CA}\Big[\frac{1}{4}(\Box_\xi + \Box_\eta)\,\mathrm{Im}\,\Phi^{C'}_B(\xi,\eta) + \overset{\xi}{\nabla}{}^{M(A'}\overset{\eta}{\nabla}_M{}^{C')}\,\mathrm{Re}\,\Phi_{BA'}(\xi,\eta)\Big]$$
$$+[\overset{\xi}{\nabla}_{(C}{}^{(C'}\overset{\xi}{\nabla}_{A)}{}^{A')} + \overset{\eta}{\nabla}_{(C}{}^{(C'}\overset{\eta}{\nabla}_{A)}{}^{A')}]\,\mathrm{Im}\,\Phi_{BA'}(\xi,\eta)$$
$$+\overset{\xi}{\nabla}_{(C}{}^{M'}\overset{\eta}{\nabla}_{A)M'}\ \mathrm{Re}\,\Phi^{C'}_B(\xi,\eta) = 0. \tag{3.18}$$

Of course, the patterns associated with Eqs. (3.14) can be built up out of the ones just obtained by combining a trivial primed-unprimed index-interchange rule with an adequate replacement of the real and imaginary parts.

4. Real wave equations

The basic procedure for decoupling Eqs. (2.7) and (2.8) consists in splitting the complex D'Alembertian operator by utilizing the conjugate of the relation (3.5). We have, in effect,

$$\Box = \frac{1}{4}\Delta_{\xi\eta} + \frac{i}{2}\overset{\xi}{\nabla}_{AA'}\overset{\eta}{\nabla}{}^{AA'}, \tag{4.1}$$

with $\Delta_{\xi\eta} = \Box_\xi - \Box_\eta$. Consequently, Eq. (2.7) becomes equivalent to the simultaneous statements

$$\begin{aligned}&\Delta_{\xi\eta}\,\mathrm{Re}\,\phi_{AB}(\xi,\eta) + 2\,\overset{\xi}{\nabla}_{CC'}\overset{\eta}{\nabla}{}^{CC'}\,\mathrm{Im}\,\phi_{AB}(\xi,\eta)\\ &= 8\pi[\overset{\xi}{\nabla}_{A'(A}\ \mathrm{Re}\,J_{B)}{}^{A'}(\xi,\eta) + \overset{\eta}{\nabla}_{A'(A}\ \mathrm{Im}\,J_{B)}{}^{A'}(\xi,\eta)],\end{aligned} \tag{4.2a}$$

and

$$\begin{aligned}&\Delta_{\xi\eta}\,\mathrm{Im}\,\phi_{AB}(\xi,\eta) - 2\,\overset{\xi}{\nabla}_{CC'}\overset{\eta}{\nabla}{}^{CC'}\,\mathrm{Re}\,\phi_{AB}(\xi,\eta)\\ &= 8\pi[\overset{\xi}{\nabla}_{A'(A}\ \mathrm{Im}\,J_{B)}{}^{A'}(\xi,\eta) - \overset{\eta}{\nabla}_{A'(A}\ \mathrm{Re}\,J_{B)}{}^{A'}(\xi,\eta)].\end{aligned} \tag{4.2b}$$

It turns out that, allowing for Eqs. (3.10) and (3.11) yields the explicit wave equations on $\mathbf{RM} \times \mathbf{V}^+$

$$\begin{aligned}\Box_\xi\,\mathrm{Re}\,\phi_{AB}(\xi,\eta) &= 2\pi[\overset{\xi}{\nabla}_{A'(A}\ \mathrm{Re}\,J_{B)}{}^{A'}(\xi,\eta) + \overset{\eta}{\nabla}_{A'(A}\ \mathrm{Im}\,J_{B)}{}^{A'}(\xi,\eta)]\\ &= -\Box_\eta\,\mathrm{Re}\,\phi_{AB}(\xi,\eta),\end{aligned} \tag{4.3a}$$

and

$$\begin{aligned}\Box_\xi\,\mathrm{Im}\,\phi_{AB}(\xi,\eta) &= 2\pi[\overset{\xi}{\nabla}_{A'(A}\ \mathrm{Im}\,J_{B)}{}^{A'}(\xi,\eta) - \overset{\eta}{\nabla}_{A'(A}\ \mathrm{Re}\,J_{B)}{}^{A'}(\xi,\eta)]\\ &= -\Box_\eta\,\mathrm{Im}\,\phi_{AB}(\xi,\eta),\end{aligned} \tag{4.3b}$$

whence we can write down the electromagnetic statements

$$\Delta_{\xi\eta}\,\mathrm{Re}\,\phi_{AB}(\xi,\eta) = 4\pi[\overset{\xi}{\nabla}_{A'(A}\ \mathrm{Re}\,J_{B)}{}^{A'}(\xi,\eta) + \overset{\eta}{\nabla}_{A'(A}\ \mathrm{Im}\,J_{B)}{}^{A'}(\xi,\eta)], \tag{4.4a}$$

and

$$\Delta_{\xi\eta}\,\mathrm{Im}\,\phi_{AB}(\xi,\eta) = 4\pi[\overset{\xi}{\nabla}_{A'(A}\ \mathrm{Im}\,J_{B)}{}^{A'}(\xi,\eta) - \overset{\eta}{\nabla}_{A'(A}\ \mathrm{Re}\,J_{B)}{}^{A'}(\xi,\eta)]. \tag{4.4b}$$

Evidently, the decoupled structures for Eq. (2.8) can be immediately spelt out by applying to Eqs. (4.2) through (4.4) the index-interchange rule referred to earlier.

In what follows, it will be particularly seen that the use of Eqs. (2.15) and (2.16) does in fact simplify the form of the real wave equations for the Dirac fields. In the case of Eq. (2.15), for instance, we have

$$(\Delta_{\xi\eta} + 4m^2)\,\mathrm{Re}\,\psi_A(\xi,\eta) + 2\,\overset{\xi}{\nabla}_{CC'}\overset{\eta}{\nabla}{}^{CC'}\,\mathrm{Im}\,\psi_A(\xi,\eta) = 4\,\mathrm{Re}\,\sigma_A(\xi,\eta)\,, \tag{4.5a}$$

and

$$(\Delta_{\xi\eta} + 4m^2)\,\mathrm{Im}\,\psi_A(\xi,\eta) - 2\,\overset{\xi}{\nabla}_{CC'}\overset{\eta}{\nabla}{}^{CC'}\,\mathrm{Re}\,\psi_A(\xi,\eta) = 4\,\mathrm{Im}\,\sigma_A(\xi,\eta)\,. \tag{4.5b}$$

Setting Eqs. (4.5) upon $\mathbf{RM} \times \mathbf{V}^+$ thus yields the statements

$$(\Box_\xi + m^2)\,\mathrm{Re}\,\psi_A(\xi,\eta) = \mathrm{Re}\,\sigma_A(\xi,\eta) = (m^2 - \Box_\eta)\,\mathrm{Re}\,\psi_A(\xi,\eta)\,, \tag{4.6a}$$

and

$$(\Box_\xi + m^2)\,\mathrm{Im}\,\psi_A(\xi,\eta) = \mathrm{Im}\,\sigma_A(\xi,\eta) = (m^2 - \Box_\eta)\,\mathrm{Im}\,\psi_A(\xi,\eta)\,. \tag{4.6b}$$

Hence, the wave equations for the ψ-field, on the real product space, are written as

$$(\Delta_{\xi\eta} + 2m^2)\,\mathrm{Re}\,\psi_A(\xi,\eta) = 2\,\mathrm{Re}\,\sigma_A(\xi,\eta)\,, \tag{4.7a}$$
$$(\Delta_{\xi\eta} + 2m^2)\,\mathrm{Im}\,\psi_A(\xi,\eta) = 2\,\mathrm{Im}\,\sigma_A(\xi,\eta). \tag{4.7b}$$

In a similar way, we obtain for the χ-field

$$(\Delta_{\xi\eta} + 2m^2)\,\mathrm{Re}\,\chi_{A'}(\xi,\eta) = 2\,\mathrm{Re}\,\Sigma_{A'}(\xi,\eta), \tag{4.8a}$$
$$(\Delta_{\xi\eta} + 2m^2)\,\mathrm{Im}\,\chi_{A'}(\xi,\eta) = 2\,\mathrm{Im}\,\Sigma_{A'}(\xi,\eta). \tag{4.8b}$$

It is perhaps pertinent to look into the structure of the wave equations that involve the scalar operator $\hat{\Delta}$. Splitting $\hat{\Delta} = \mathrm{Re}\,\hat{\Delta} - i\,\mathrm{Im}\,\hat{\Delta}$, and using the unprimed expression

$$\begin{aligned} i\phi_A^B(x)\psi_B(x) &= [\,\mathrm{Im}\,\phi_A^B(\xi,\eta)\,\mathrm{Re}\,\psi_B(\xi,\eta) + \mathrm{Re}\,\phi_A^B(\xi,\eta)\,\mathrm{Im}\,\psi_B(\xi,\eta)] \\ &-i[\,\mathrm{Im}\,\phi_A^B(\xi,\eta)\,\mathrm{Im}\,\psi_B(\xi,\eta) - \mathrm{Re}\,\phi_A^B(\xi,\eta)\,\mathrm{Re}\,\psi_B(\xi,\eta)], \end{aligned} \tag{4.9}$$

we are led to the following formal statements for Eq. (2.12)

$$(\mathrm{Re}\,\hat{\Delta} + m^2)\,\mathrm{Re}\,\psi_A(\xi,\eta) - \mathrm{Im}\,\hat{\Delta}\mathrm{Im}\,\psi_A(\xi,\eta)$$
$$= 2e[\,\mathrm{Im}\,\phi_A^B(\xi,\eta)\,\mathrm{Re}\,\psi_B(\xi,\eta) + \mathrm{Re}\,\phi_A^B(\xi,\eta)\,\mathrm{Im}\,\psi_B(\xi,\eta)]\,, \quad (4.10a)$$

and

$$(\mathrm{Re}\,\hat{\Delta} + m^2)\,\mathrm{Im}\,\psi_A(\xi,\eta) + \mathrm{Im}\,\hat{\Delta}\mathrm{Re}\,\psi_A(\xi,\eta)$$
$$= 2e[\,\mathrm{Im}\,\phi_A^B(\xi,\eta)\,\mathrm{Im}\,\psi_B(\xi,\eta) - \mathrm{Re}\,\phi_A^B(\xi,\eta)\,\mathrm{Re}\,\psi_B(\xi,\eta)]. \quad (4.10b)$$

Once again, the statements corresponding to Eq. (2.13) can be readily obtained from the structures (4.10) by making trivial replacements. In any case, it seems to be worthwhile to write down the explicit pieces occurring in the splitting of $\hat{\Delta}$. We thus have

$$\mathrm{Re}\,\hat{\Delta} = \tfrac{1}{4}\Delta_{\xi\eta} - \frac{e}{2}[\overset{\xi}{\nabla}_{AA'}\ \mathrm{Im}\,\Phi^{AA'}(\xi,\eta) - \overset{\eta}{\nabla}_{AA'}\ \mathrm{Re}\,\Phi^{AA'}(\xi,\eta)]$$
$$-e[\,\mathrm{Im}\,\Phi^{AA'}(\xi,\eta)\,\overset{\xi}{\nabla}_{AA'} - \mathrm{Re}\,\Phi^{AA'}(\xi,\eta)\,\overset{\eta}{\nabla}_{AA'}]$$
$$-e^2[\,\mathrm{Re}\,\Phi^{AA'}(\xi,\eta)\,\mathrm{Re}\,\Phi_{AA'}(\xi,\eta) - \mathrm{Im}\,\Phi^{AA'}(\xi,\eta)\,\mathrm{Im}\,\Phi_{AA'}(\xi,\eta)]\,, \quad (4.11)$$

and

$$\mathrm{Im}\,\hat{\Delta} = -\tfrac{1}{2}\,\overset{\xi}{\nabla}_{AA'}\overset{\eta}{\nabla}{}^{AA'} + \frac{e}{2}[\overset{\xi}{\nabla}_{AA'}\ \mathrm{Re}\,\Phi^{AA'}(\xi,\eta) + \overset{\eta}{\nabla}_{AA'}\ \mathrm{Im}\,\Phi^{AA'}(\xi,\eta)]$$
$$+e[\,\mathrm{Re}\,\Phi^{AA'}(\xi,\eta)\,\overset{\xi}{\nabla}_{AA'} + \mathrm{Im}\,\Phi^{AA'}(\xi,\eta)\,\overset{\eta}{\nabla}_{AA'}]$$
$$-2e^2\,\mathrm{Re}\,\Phi^{AA'}(\xi,\eta)\,\mathrm{Im}\,\Phi_{AA'}(\xi,\eta)\,. \quad (4.12)$$

5. Invariant exactness relations

At this point, the detailed calculations leading to the invariant exactness relations for the system will be carried out. Such relations were shown before in Ref. [1], but we will repair some mistaken numerical factors. Another motivation for introducing these structures is the fact that they make up the work by Cardoso [4] which deals with a null description of the classical dynamics of the fields in **RM**. It will be enough to derive only the relations for the potential and unprimed fields. The structures for the primed fields can be constructed from the others by changing indices and substituting kernel letters appropriately. The equations arising in the formulation of the theory in **CM** will play an auxiliary role herein. They will be used so many times that we will not refer to them explicitly upon performing the relevant computations.

Let us start with the first-order derivative expression for the potential

$$\nabla_A^{A'}\Phi_B^{B'}(x) = \nabla_{(A}^{(A'}\Phi_{B)}^{B')}(x) + \tfrac{1}{2}\varepsilon^{A'B'}\phi_{AB}(x) + \tfrac{1}{2}\varepsilon_{AB}\theta^{A'B'}(x) + \tfrac{1}{4}\varepsilon_{AB}\varepsilon^{A'B'}\Lambda(x). \tag{5.1}$$

Expanding the second-order derivative in terms of its unprimed-index contributions yields

$$\nabla_A^{A'}\nabla_B^{B'}\Phi_C^{C'}(x) = \nabla_{(A}^{A'}\nabla_B^{B'}\Phi_{C)}^{C'}(x) - \tfrac{1}{3}\varepsilon_{AB}\nabla^{MA'}\nabla_{(M}^{B'}\Phi_{C)}^{C'}(x) - \tfrac{1}{3}\varepsilon_{AC}\nabla^{MA'}\nabla_{(M}^{B'}\Phi_{B)}^{C'}(x) - \tfrac{1}{2}\varepsilon_{BC}\nabla_A^{A'}\nabla^{MB'}\Phi_M^{C'}(x). \tag{5.2}$$

The kernel of the second piece occurring on the right-hand side of Eq. (5.2) reads

$$\begin{aligned}\nabla^{MA'}\nabla_{(M}^{B'}\Phi_{C)}^{C'}(x) &= -\tfrac{1}{4}[\varepsilon^{A'B'}\Box\Phi_C^{C'}(x) + \varepsilon^{A'C'}\nabla_C^{B'}\Lambda(x)] - \tfrac{1}{2}\nabla_C^{B'}\theta^{A'C'}(x)\\ &= -\tfrac{1}{2}[\varepsilon^{A'(B'}\nabla_C^{C')}\Lambda(x) + \nabla_C^{B'}\theta^{A'C'}(x)] - \pi\varepsilon^{A'B'}J_C^{C'}(x),\end{aligned} \tag{5.3}$$

whereas the kernel of the fourth piece gives

$$\nabla_A^{A'}\nabla^{MB'}\Phi_M^{C'}(x) = -\nabla_A^{A'}\theta^{B'C'}(x) - \tfrac{1}{2}\varepsilon^{B'C'}\nabla_A^{A'}\Lambda(x). \tag{5.4}$$

Notice that the sum of the second and third pieces is symmetric in the indices B and C. We next carry out the primed-index expansion of each of the pertinent terms. We have

$$\nabla_{(A}^{A'}\nabla_B^{B'}\Phi_{C)}^{C'}(x) = \nabla_{(A}^{(A'}\nabla_B^{B'}\Phi_{C)}^{C')}(x) + \tfrac{1}{3}\varepsilon^{A'(B'}\nabla_{(A}^{C')}\phi_{BC)}(x) + \tfrac{1}{2}\varepsilon^{B'C'}\nabla_{(A}^{A'}\phi_{BC)}(x). \tag{5.5}$$

The ε-pieces entering explicitly into the relations (5.3)–(5.5) are already irreducible. Essentially, this is because of the skew-symmetry of the "metric" spinors. Hence, making use of the simple structure

$$\begin{aligned}\nabla_A^{A'}\theta^{B'C'}(x) &= \nabla_A^{(A'}\theta^{B'C')}(x) + \tfrac{2}{3}\varepsilon^{A'(B'}\nabla_{AD'}\theta^{C')D'}(x)\\ &= \nabla_A^{(A'}\theta^{B'C')}(x) - \frac{4\pi}{3}\varepsilon^{A'(B'}J_A^{C')}(x),\end{aligned} \tag{5.6}$$

and fitting blocks together, we obtain

$$\begin{aligned}\nabla_A^{A'}\nabla_B^{B'}\Phi_C^{C'}(x) &= \nabla_{(A}^{(A'}\nabla_B^{B'}\Phi_{C)}^{C')}(x) + \frac{1}{3}\varepsilon^{A'(B'}\nabla_{(A}^{C')}\phi_{BC)}(x) \\ &+\frac{1}{2}\varepsilon^{B'C'}\nabla_{(A}^{A'}\phi_{BC)}(x) + \frac{1}{3}\varepsilon_{A(B}\varepsilon^{A'(B'}\nabla_{C)}^{C')}\Lambda(x) \\ &+\frac{2\pi}{3}\varepsilon_{A(B}\varepsilon^{A'B'}J_{C)}^{C'}(x) + \frac{1}{3}\varepsilon_{A(B}\nabla_{C)}^{(A'}\theta^{B'C')}(x) \\ &-\frac{4\pi}{9}\varepsilon_{A(B}\varepsilon^{B'(A'}J_{C)}^{C')}(x) + \frac{1}{2}\varepsilon_{BC}\nabla_A^{(A'}\theta^{B'C')}(x) \\ &-\frac{2\pi}{3}\varepsilon_{BC}\varepsilon^{A'(B'}J_A^{C')}(x) + \frac{1}{4}\varepsilon_{BC}\varepsilon^{B'C'}\nabla_A^{A'}\Lambda(x)\,. \quad (5.7)\end{aligned}$$

Let us now consider the electromagnetic relation

$$\begin{aligned}\nabla_A^{A'}\phi_{BC}(x) &= \nabla_{(A}^{A'}\phi_{BC)}(x) - \tfrac{2}{3}\varepsilon_{A(B}\nabla^{DA'}\phi_{C)D}(x) \\ &= \nabla_{(A}^{A'}\phi_{BC)}(x) - \frac{4\pi}{3}\varepsilon_{A(B}J_{C)}^{A'}(x)\,, \quad (5.8)\end{aligned}$$

which corresponds to the unprimed-field version of Eq. (5.6). To derive the relevant second-order derivative structure, it is convenient first to utilize the symbolic four-unprimed-index expression

$$\begin{aligned}\Xi_{ABCD}^{\mathcal{A}'} &= \Xi_{(ABCD)}^{\mathcal{A}'} - \tfrac{1}{4}[\varepsilon_{AB}\Xi^{M\mathcal{A}'}{}_{(MCD)} + \varepsilon_{AC}\Xi^{M\mathcal{A}'}{}_{(MBD)} \\ &+\varepsilon_{AD}\Xi^{M\mathcal{A}'}{}_{(MBC)}] - \tfrac{1}{3}[\varepsilon_{BC}\Xi_A{}^{M\mathcal{A}'}{}_{(MD)} \\ &+\varepsilon_{BD}\Xi_A{}^{M\mathcal{A}'}{}_{(MC)}] - \tfrac{1}{2}\varepsilon_{CD}\Xi_{AB}{}^{M\mathcal{A}'}{}_M\,, \quad (5.9)\end{aligned}$$

with $\mathcal{A}'$ being some clumped primed index. We thus have

$$\begin{aligned}\nabla_A^{A'}\nabla_B^{B'}\phi_{CD}(x) &= \nabla_{(A}^{A'}\nabla_B^{B'}\phi_{CD)}(x) - \tfrac{1}{4}\Big[\varepsilon_{AB}\nabla^{MA'}\nabla_{(M}^{B'}\phi_{CD)}(x) \\ &+\varepsilon_{AC}\nabla^{MA'}\nabla_{(M}^{B'}\phi_{BD)}(x) + \varepsilon_{AD}\nabla^{MA'}\nabla_{(M}^{B'}\phi_{BC)}(x)\Big] \\ &-\tfrac{1}{3}[\varepsilon_{BC}\nabla_A^{A'}\nabla^{MB'}\phi_{MD}(x) + \varepsilon_{BD}\nabla_A^{A'}\nabla^{MB'}\phi_{MC}(x)]. \quad (5.10)\end{aligned}$$

It should be observed that the piece involving ε_{CD} in Eq. (5.9) vanishes because it bears the ε-trace of $\phi_{AB}(x)$. The totally symmetric piece of Eq. (5.10) is also symmetric in the indices A' and B', as can be seen by invoking the fact that the ∇'s commute and performing the simple calculation

$$\begin{aligned}2\nabla_{(A}^{[A'}\nabla_B^{B']}\phi_{CD)}(x) &= \varepsilon^{A'B'}\nabla_{M'(A}\nabla_B^{M'}\phi_{CD)}(x) \\ &= \varepsilon^{A'B'}\nabla_{M'((A}\nabla_{B)}^{M'}\phi_{CD)}(x) = 0. \quad (5.11)\end{aligned}$$

Obviously, the sum carrying the overall factor $-\frac{1}{3}$ can at this stage be formally written as

$$-\tfrac{1}{3}\left[\varepsilon_{BC}\Xi_A{}^{MA'B'}{}_{(MD)}(x)+\varepsilon_{BD}\Xi_A{}^{MA'B'}{}_{(MC)}(x)\right]$$
$$=-\frac{4\pi}{3}\nabla_A^{A'}\varepsilon_{B(C}J_{D)}^{B'}(x), \tag{5.12}$$

while the contribution bearing the factor $-\frac{1}{4}$ reads

$$-\tfrac{1}{4}[\varepsilon_{AB}\Xi^{MA'B'}{}_{(MCD)}(x)+\varepsilon_{AC}\Xi^{MA'B'}{}_{(MBD)}(x)+\varepsilon_{AD}\Xi^{MA'B'}{}_{(MBC)}(x)]$$
$$=\frac{\pi}{6}\varepsilon^{A'B'}[\varepsilon_{AB}\nabla_{D'(C}J_{D)}^{D'}(x)+\varepsilon_{AC}\nabla_{D'(B}J_{D)}^{D'}(x)+\varepsilon_{AD}\nabla_{D'(B}J_{C)}^{D'}(x)]$$
$$-\frac{\pi}{3}[\varepsilon_{AB}\nabla_{(C}^{B'}J_{D)}^{A'}(x)+\varepsilon_{AC}\nabla_{(B}^{B'}J_{D)}^{A'}(x)+\varepsilon_{AD}\nabla_{(B}^{B'}J_{C)}^{A'}(x)]\,, \tag{5.13}$$

since, say,

$$\begin{aligned}\varepsilon_{AB}\nabla^{MA'}\nabla_{(M}^{B'}\phi_{CD)}(x) &= \frac{1}{3}\varepsilon_{AB}[4\pi\nabla_{(C}^{B'}J_{D)}^{A'}(x)-\frac{1}{2}\varepsilon^{A'B'}\Box\phi_{CD}(x)]\\ &= \frac{2\pi}{3}\varepsilon_{AB}[2\nabla_{(C}^{B'}J_{D)}^{A'}(x)-\varepsilon^{A'B'}\nabla_{D'(C}J_{D)}^{D'}(x)]\,.\end{aligned} \tag{5.14}$$

Picking up the part of Eq. (5.10) which is symmetric in A' and B', we then obtain

$$\begin{aligned}\Xi_{ABCD}^{(A'B')}(x) &= \nabla_{(A}^{(A'}\nabla_B^{B')}\phi_{CD)}(x)-\frac{\pi}{3}[\Psi_{AB(CD)}^{(A'B')}(x)+\Psi_{AC(BD)}^{(A'B')}(x)\\ &\quad+\Psi_{AD(BC)}^{(A'B')}(x)]-\frac{4\pi}{3}\Psi_{B(C|A|D)}^{(A'B')}(x)\,,\end{aligned} \tag{5.15}$$

where we have introduced the auxiliary spinor field

$$\Psi_{ABCD}^{A'B'}(x)=\varepsilon_{AB}\nabla_C^{A'}J_D^{B'}(x)=\Psi_{[AB]CD}^{A'B'}(x). \tag{5.16}$$

We have the partial computation

$$-\frac{\pi}{6}[\Psi_{ACBD}^{(A'B')}(x)+\Psi_{ADBC}^{(A'B')}(x)]-\frac{2\pi}{3}[\Psi_{BCAD}^{(A'B')}(x)+\Psi_{BDAC}^{(A'B')}(x)]$$
$$=-\frac{\pi}{6}[\Psi_{ACBD}^{(A'B')}(x)+\Psi_{ADBC}^{(A'B')}(x)]-[\frac{\pi}{6}\Psi_{BCAD}^{(A'B')}(x)+\frac{\pi}{2}\Psi_{BCAD}^{(A'B')}(x)]$$
$$-[\frac{\pi}{6}\Psi_{BDAC}^{(A'B')}(x)+\frac{\pi}{2}\Psi_{BDAC}^{(A'B')}(x)]$$
$$=-\frac{2\pi}{3}\nabla_{(A}^{(A'}\varepsilon_{B)(C}J_{D)}^{B')}(x)-\pi\nabla_{(A}^{(A'}\varepsilon_{B)(C}J_{D)}^{B')}(x)+\frac{\pi}{2}\Psi_{AB(CD)}^{(A'B')}(x)$$
$$=\frac{\pi}{2}\Psi_{AB(CD)}^{(A'B')}(x)-\frac{5\pi}{3}\nabla_{(A}^{(A'}\varepsilon_{B)(C}J_{D)}^{B')}(x)\,. \tag{5.17}$$

The remaining Ψ-contributions of Eq. (5.15) read

$$
\begin{aligned}
&-\frac{\pi}{3}[\Psi^{(A'B')}_{AB(CD)}(x) + \Psi^{(A'B')}_{A(CD)B}(x)] \\
&= -\frac{\pi}{3}[\Psi^{(A'B')}_{AB(CD)}(x) - \nabla^{(A'}_{(C}\varepsilon_{D)(A}J^{B')}_{B)}(x) + \tfrac{1}{2}\varepsilon_{AB}\Psi^{(A'B')M}_{M(CD)}(x)] \\
&= \frac{\pi}{3}\nabla^{(A'}_{(C}\varepsilon_{D)(A}J^{B')}_{B)}(x) - \frac{\pi}{2}\Psi^{(A'B')}_{AB(CD)}(x) ,
\end{aligned}
\tag{5.18}
$$

where the second step involves utilizing the property

$$
\varepsilon_{AB}\Psi^{(A'B')M}_{M(CD)}(x) = \Psi^{(A'B')}_{AB(CD)}(x) .
\tag{5.19}
$$

Consequently, we can reexpress the overall structure (5.15) as

$$
\begin{aligned}
\Xi^{(A'B')}_{ABCD}(x) &= \nabla^{(A'}_{(A}\nabla^{B')}_{B}\phi_{CD)}(x) - \frac{5\pi}{3}\nabla^{(A'}_{(A}\varepsilon_{B)(C}J^{B')}_{D)}(x) \\
&\quad + \frac{\pi}{3}\nabla^{(A'}_{(C}\varepsilon_{D)(A}J^{B')}_{B)}(x) .
\end{aligned}
\tag{5.20}
$$

There are three ways of computing the primed skew-symmetric part of $\Xi^{A'B'}_{ABCD}(x)$. One consists in bringing together the pertinent parts of Eqs. (5.12) and (5.13), effectively using the divergencelessness of $J^{AA'}(x)$ in the form

$$
\Psi_{AB(CD)M'}{}^{M'}(x) = \Psi_{ABCDM'}{}^{M'}(x) = \Psi_{ABDCM'}{}^{M'}(x),
\tag{5.21}
$$

along with the index-displacement rule

$$
\varepsilon_{EF}\Theta^{A'}_{G\mathcal{A}}(x) = 2\varepsilon_{G[F}\Theta^{A'}_{E]\mathcal{A}}(x) = 2\varepsilon_{[E|G|}\Theta^{A'}_{F]\mathcal{A}}(x).
\tag{5.22}
$$

Explicitly, we have

$$
\begin{aligned}
\Xi^{[A'B']}_{ABCD}(x) &= \frac{\pi}{6}\varepsilon^{A'B'}[\Psi_{AB(CD)M'}{}^{M'}(x) + \Psi_{AC(BD)M'}{}^{M'}(x) \\
&\quad + \Psi_{AD(BC)M'}{}^{M'}(x)] - \frac{\pi}{3}[\Psi^{[B'A']}_{AB(CD)}(x) \\
&\quad + \Psi^{[B'A']}_{AC(BD)}(x) + \Psi^{[B'A']}_{AD(BC)}(x) + 4\Psi^{[A'B']}_{B(C|A|D)}(x)] \\
&= \frac{\pi}{3}\varepsilon^{A'B'}[\Psi_{ABCDM'}{}^{M'}(x) + \Psi_{ACBDM'}{}^{M'}(x) + \Psi_{ADBCM'}{}^{M'}(x) \\
&\quad - \Psi_{BCADM'}{}^{M'}(x) - \Psi_{BDACM'}{}^{M'}(x)] \\
&= \pi\varepsilon^{A'B'}\Psi_{ABCDM'}{}^{M'}(x) = \pi\varepsilon^{A'B'}\varepsilon_{AB}\nabla_{M'(C}J^{M'}_{D)}(x) ,
\end{aligned}
\tag{5.23}
$$

where the conservation law (5.21) has been particularly employed to write the last statement. Here, the ordered-index blocks that partake of Eq. (5.22) are ACB and ADB. The presence of this structure actually entails the cancellation of the pieces carrying ε_{BC} and ε_{BD}.

An alternative procedure for obtaining the result expressed by Eq. (5.23) amounts simply to picking up the primed skew part of the derivative of the statement (5.8). We have, in effect,

$$\begin{aligned}\nabla_A^{A'}\nabla_B^{B'}\phi_{CD}(x) &= \tfrac{1}{3}[\nabla_A^{A'}\nabla_B^{B'}\phi_{CD}(x)+\nabla_A^{A'}\nabla_C^{B'}\phi_{BD}(x)+\nabla_A^{A'}\nabla_D^{B'}\phi_{BC}(x)] \\ &\quad -\frac{2\pi}{3}[\Psi_{BCAD}^{A'B'}(x)+\Psi_{BDAC}^{A'B'}(x)]\,,\end{aligned} \tag{5.24}$$

such that the commutativity of the ∇'s yields the relations

$$\begin{aligned}\nabla_A^{[A'}\nabla_B^{B']}\phi_{CD}(x) &= \frac{1}{12}\varepsilon^{A'B'}[\varepsilon_{AB}\Box\phi_{CD}(x) \\ &\quad +\varepsilon_{AC}\Box\phi_{BD}(x)+\varepsilon_{AD}\Box\phi_{BC}(x)] \\ &\quad -\frac{\pi}{3}\varepsilon^{A'B'}[\Psi_{BC(AD)M'}{}^{M'}(x)+\Psi_{BD(AC)M'}{}^{M'}(x)] \\ &= \frac{\pi}{3}\varepsilon^{A'B'}[\Psi_{AB(CD)M'}{}^{M'}(x)+\Psi_{AC(BD)M'}{}^{M'}(x) \\ &\quad +\Psi_{AD(BC)M'}{}^{M'}(x)-\Psi_{BC(AD)M'}{}^{M'}(x) \\ &\quad -\Psi_{BD(AC)M'}{}^{M'}(x)]\,.\end{aligned} \tag{5.25}$$

Now, applying the rule (5.22) to Eq. (5.25) and using the same index blocks as before leads to the equation

$$\nabla_A^{[A'}\nabla_B^{B']}\phi_{CD}(x) = \frac{\pi}{3}\varepsilon^{A'B'}[\Psi_{AB(CD)M'}{}^{M'}(x)-2\Psi_{BA(CD)M'}{}^{M'}(x)], \tag{5.26}$$

which is identical to (5.23). The easiest way to obtain the expression we have been considering is to pick up from the beginning the interesting part of the whole second-order derivative structure. We thus have the statements

$$\begin{aligned}\Xi_{ABCD}^{[A'B']}(x) &= \tfrac{1}{2}\varepsilon^{A'B'}\nabla_{M'[A}\nabla_{B]}^{M'}\phi_{CD}(x)=\tfrac{1}{4}\varepsilon^{A'B'}\varepsilon_{AB}\Box\phi_{CD}(x) \\ &= \pi\varepsilon^{A'B'}\Psi_{AB(CD)M'}{}^{M'}(x)\,,\end{aligned} \tag{5.27}$$

which recover Eq. (5.23) once again. It follows that, adding together (5.20) and (5.23) yields

$$\begin{aligned}\nabla_A^{A'}\nabla_B^{B'}\phi_{CD}(x) &= \nabla_{(A}^{(A'}\nabla_B^{B')}\phi_{CD)}(x)-\frac{5\pi}{3}\nabla_{(A}^{(A'}\varepsilon_{B)(C}J_{D)}^{B')}(x) \\ &\quad +\frac{\pi}{3}\nabla_{(C}^{(A'}\varepsilon_{D)(A}J_{B)}^{B')}(x)+\pi\varepsilon^{A'B'}\varepsilon_{AB}\nabla_{D'(C}J_{D)}^{D'}(x)\,,\end{aligned} \tag{5.28}$$

whence the invariant exactness of the Maxwell part of the system can be accomplished by choosing $A(x) = 0$.

Towards deriving the relations for the Dirac fields, we consider the trivial identity

$$\mathcal{D}_B{}^{B'}\psi_C(x) = \mathcal{D}_{(B}{}^{B'}\psi_{C)}(x) - \frac{\mu}{2}\varepsilon_{BC}\chi^{B'}(x). \tag{5.29}$$

For the second-order (covariant) derivative, we have the expression

$$\begin{aligned}\mathcal{D}_A{}^{A'}\mathcal{D}_B{}^{B'}\psi_C(x) &= \mathcal{D}_{(A}{}^{(A'}\mathcal{D}_B{}^{B')}\psi_{C)}(x) + \tfrac{1}{2}\varepsilon^{A'B'}\mathcal{D}_{M'(A}\mathcal{D}_B{}^{M'}\psi_{C)}(x)\\ &-\tfrac{1}{3}\varepsilon_{AB}[\mathcal{D}^{M(A'}\mathcal{D}_{(C}{}^{B')}\psi_{M)}(x) + \tfrac{1}{2}\varepsilon^{A'B'}\mathcal{D}^M_{M'}\mathcal{D}_{(C}{}^{M'}\psi_{M)}(x)]\\ &-\tfrac{1}{3}\varepsilon_{AC}[\mathcal{D}^{M(A'}\mathcal{D}_{(B}{}^{B')}\psi_{M)}(x) + \tfrac{1}{2}\varepsilon^{A'B'}\mathcal{D}^M_{M'}\mathcal{D}_{(B}{}^{M'}\psi_{M)}(x)]\\ &-\tfrac{1}{2}\varepsilon_{BC}[\mathcal{D}^{(A'}_A\mathcal{D}^{B')M}\psi_M(x) + \tfrac{1}{2}\varepsilon^{A'B'}\mathcal{D}_{AM'}\mathcal{D}^{MM'}\psi_M(x)]\,.\end{aligned} \tag{5.30}$$

The second term of the right-hand side of Eq. (5.30) can be simplified by using the following equation [2]

$$\mathcal{D}_{M'(A}\mathcal{D}_B{}^{M'}\psi_{C)}(x) = -ie\phi_{(AB}(x)\psi_{C)}(x)\,. \tag{5.31}$$

In effect, we have the easy computation

$$\begin{aligned}&\mathcal{D}_{M'(A}\mathcal{D}_B{}^{M'}\psi_{C)}(x) = \mathcal{D}_{M'((A}\mathcal{D}_{B)}{}^{M'}\psi_{C)}(x) = -ie\phi_{(AB}(x)\psi_{C)}(x)\\ &= -\frac{ie}{3}[\phi_{A(B}(x)\psi_{C)}(x) + \phi_{B(A}(x)\psi_{C)}(x) + \phi_{C(A}(x)\psi_{B)}(x)]\\ &= -\frac{ie}{3}[3\phi_{A(B}(x)\psi_{C)}(x) + \phi_{C[B}(x)\psi_{A]}(x) + \phi_{B[C}(x)\psi_{A]}(x)]\\ &= -ie[\phi_{A(B}(x)\psi_{C)}(x) + \frac{1}{3}\varepsilon_{A(B}\phi^M_{C)}(x)\psi_M(x)]\,.\end{aligned} \tag{5.32}$$

It is obvious that the kernels of the individual pieces of the third term can be explicitly written as

$$\mathcal{D}^{M(A'}\mathcal{D}_{(C}{}^{B')}\psi_{M)}(x) = ie\theta^{A'B'}(x)\psi_C(x) + \frac{\mu}{2}\mathcal{D}^{(A'}_C\chi^{B')}(x)\,, \tag{5.33}$$

and

$$\begin{aligned}\mathcal{D}^M_{M'}\mathcal{D}_{(C}{}^{M'}\psi_{M)}(x) &= -\tfrac{1}{4}[3\hat{\Delta}\psi_C(x) + 2ie\phi^M_C(x)\psi_M(x)]\\ &= \tfrac{1}{2}[3\mu^2\psi_C(x) - 4ie\phi^M_C(x)\psi_M(x)]\,.\end{aligned} \tag{5.34}$$

The fourth term can be entirely obtained from the third by interchanging the indices B and C, whereas the bracketed piece of the fifth term is expressed simply by

$$\begin{aligned}&\mathcal{D}^{(A'}_A\mathcal{D}^{B')M}\psi_M(x) + \tfrac{1}{2}\varepsilon^{A'B'}\mathcal{D}_{AM'}\mathcal{D}^{MM'}\psi_M(x)\\ &= \mu\mathcal{D}^{(A'}_A\chi^{B')}(x) - \frac{\mu^2}{2}\varepsilon^{A'B'}\psi_A(x)\,.\end{aligned} \tag{5.35}$$

Whence combining the relations (5.30)–(5.35) yields

$$\begin{aligned} \mathcal{D}_A{}^{A'}\mathcal{D}_B{}^{B'}\psi_C(x) = {} & \mathcal{D}_{(A}{}^{(A'}\mathcal{D}_B{}^{B')}\psi_{C)}(x) - \frac{2ie}{3}\varepsilon_{A(B}(x)\psi_{C)}(x)\theta^{A'B'}(x) \\ & -\mu[\tfrac{1}{3}\varepsilon_{A(B}\mathcal{D}_{C)}{}^{(A'}\chi^{B')}(x) + \tfrac{1}{2}\varepsilon_{BC}\mathcal{D}_A^{(A'}\chi^{B')}(x)] \\ & +\tfrac{1}{2}\varepsilon^{A'B'}\{ie[\varepsilon_{A(B}\phi^M_{C)}(x)\psi_M(x) - \phi_{A(B}(x)\psi_{C)}(x)] \\ & \times\mu^2[\tfrac{1}{2}\varepsilon_{BC}\psi_A(x) - \varepsilon_{A(B}(x)\psi_{C)}(x)]\}\,. \end{aligned} \tag{5.36}$$

In Eq. (5.36), the kernel of the μ^2-contribution equals $-\varepsilon_{AB}\psi_C(x)$ while the piece involving outer products between the unprimed fields is given by

$$\varepsilon_{A(B}\phi^M_{C)}(x)\psi_M(x) - \phi_{A(B}(x)\psi_{C)}(x) = \phi_{BC}(x)\psi_A(x) - 2\phi_{A(B}(x)\psi_{C)}(x). \tag{5.37}$$

To compute the μ-term systematically, we define the quantity

$$\omega^{A'B'}_{EFG}(x) = \varepsilon_{EF}\mathcal{D}_G{}^{(A'}\chi^{B')}(x) = \omega^{(A'B')}_{[EF]G}(x), \tag{5.38}$$

and apply the device (5.22) to the (ordered) indices EFG. This procedure facilitates keeping track of the relevant indices when the calculations are actually carried out. We thus have

$$\begin{aligned} & -\tfrac{1}{3}\omega^{A'B'}_{A(BC)}(x) - \tfrac{1}{2}\omega^{A'B'}_{BCA}(x) \\ = {} & -\tfrac{1}{3}[\omega^{A'B'}_{C[BA]}(x) + \omega^{A'B'}_{B[CA]}(x)] - \tfrac{1}{2}\omega^{A'B'}_{BCA}(x) \\ = {} & \tfrac{1}{3}[\omega^{A'B'}_{C[AB]}(x) - \omega^{A'B'}_{C[BA]}(x)] + \tfrac{2}{3}\omega^{A'B'}_{CBA}(x) \\ = {} & \tfrac{1}{3}[\omega^{A'B'}_{CAB}(x) + \omega^{A'B'}_{CBA}(x)] = \tfrac{2}{3}\omega^{A'B'}_{C(AB)}(x) = \tfrac{2}{3}\varepsilon_{C(A}\mathcal{D}_{B)}{}^{(A'}\chi^{B')}(x)\,. \end{aligned} \tag{5.39}$$

Therefore, substituting these latter results into Eq. (5.36) leads to the invariant exactness statements for the Dirac part.

6. Concluding remarks and outlook

The procedures giving rise to the real wave equations on $\mathbf{RM} \times \mathbf{V}^+$ allow one to build up formal integral solutions for the system of fields, which carry Green's functions satisfying prescribed boundary conditions. In this connection, it might seem natural to make use of the method given by Cardoso [3] whereby explicit solutions to complexified wave equations may be constructed upon $\mathbf{CM}^+$. It becomes clear that the usefulness of the symmetry involving the operators $\Box_\xi$ and $\Box_\eta$, which was brought about particularly by the procedure yielding Eqs. (4.7), appears to be related to the applicability of the above-referred method, and thence also to the formal simplicity of the statements (2.15) and (2.16).

We believe that the implementation of this programme can perhaps lead to a physical interpretation of the solutions of the wave equations recalled in Section 2. It is expected that the pertinent prescriptions will provide structures of special interest if the massless limiting case is taken into consideration. These situations will probably be entertained elsewhere.

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REFERENCES

[1] R. Penrose, W. Rindler, *Spinors and Spacetime*, Vol. 1, CUP, Cambridge 1984.
[2] J.G. Cardoso, *Int. Jour. Mod. Phys.* **8**, 3697 1993).
[3] J.G. Cardoso, *Jour. Math. Phys.* **36**, 2169 (1992).
[4] J.G. Cardoso, *Acta Phys. Pol.* **B23**, 1495 (1992).