# CONCEPT OF ORDER PARAMETER FOR RANDOM MATRIX ENSEMBLE

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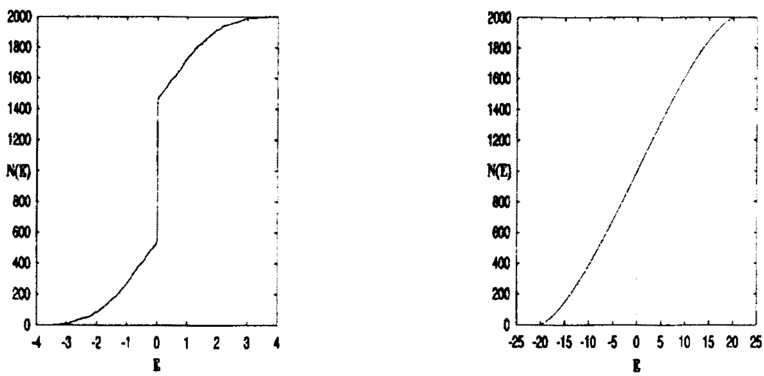
(*Received January 14, 1997; revised version received April 17, 1997*)

For quantum systems exhibiting transition from chaotic properties to integrable ones while the control parameter is varying the order parameter is introduced. The proposed concept is based on division of the random matrix ensemble into equivalence classes of equal matrix ranks. The order parameter is proportional to the number of equivalence classes and is expressed by the jump of the cumulative distribution function at energy $E = 0$.

PACS numbers: 05.30. Ch, 02.10. Sp, 05.45. +b, 05.70. Fh

There are quantum systems exhibiting transition from an integrable quantum system to a chaotic one [1–6] while the control parameter is varying. Numerical evidence suggests [6] that this transition is analogous to the order-disorder phase transition in thermodynamic systems. Quantity describing ordering in the thermodynamic systems is order parameter. It has been shown by the computer simulations [6] that quantum systems described by the Sparse Random Matrix Ensemble have an order parameter being a natural measure of integrability of the system and behaving like thermodynamic order parameter while the control parameter is varying. In this paper we introduce the formal definition of the order parameter for the Random Matrix Ensemble. Let us start from $GOE(\mathcal{N})$ where the average density of states $\langle\rho(E)\rangle$ obeys a simple semicircle law [7]. Let $H$ be a member of $GOE(\mathcal{N})$ and $\rho(E)$ be density of states of $H$. Then $\mathcal{N}(0) = \mathcal{N}\rho(0)$ is multiplicity (degeneration) of the energy level $E = 0$ and $rank\,(H) = \mathcal{N} - \mathcal{N}(0)$ is rank of $H$. It is important for further considerations that the matrix rank is invariant with respect to similarity transformation: $rank\,(H) = rank\,(THT^{-1})$. Going to the sparse random matrix ensemble by diluting the $GOE(\mathcal{N})$ we break the orthogonal symmetry and obtain the $SRME(p, \mathcal{N})$ which is characterized by a finite mean number $p$ of randomly placed nonzero elements per matrix row [2, 3]. In our approach [6] we characterize diluted $GOE(\mathcal{N})$ by fraction $x$ of matrix elements not equal to zero and we consider $SGRSE(x,\mathcal{N})$ (Sparse Gaussian Random Symmetric Ensemble). Breaking of symmetry increases population of the eigenvalue $E = 0$. This effect is reflected in central spike of the density of states [1, 2] and in jump of the cumulative distribution function for $E = 0$, Fig. 1(a). Increasing population of the eigenvalue $E = 0$ is a result of decreasing ranks of some matrices.




(a) (b)

Fig. 1. (a) — The cumulative distribution function $N(E)$ for $x = 0.0005$ ($\mathcal{N} = 1000$). (b) — The cumulative distribution function $N(E)$ for $x = 0.05$ ($\mathcal{N} = 2000$).

Let us divide considered sparse ensemble into equivalence classes by the rank of matrices:

$$SGRSE^{*}(x,\mathcal{N}) = SGRSE(x,\mathcal{N})/RANK\,, \tag{1}$$

where $RANK$ is equivalence relation of equal rank of matrices. Each class of quotient ensemble $SGRSE^{*}$ contains matrices of equal rank. There does not exist any transformation $T$ from class to class. Time evolution of the quantum system in the Heisenberg picture can be expressed by the time evolution operator defined by $H$:

$$M(t) = T(t)MT^{-1}(t)\,, \tag{2}$$

where $M$ is an arbitrary element of $SGRSE$, $T(t) = \exp(iHt)$ and $H$ is a matrix of $SGRSE$ playing the role of Hamiltonian. Both $M(t)$ and $M$ belong to the same equivalence class which means that the time evolution does not move matrix out of its class. Therefore, the class's label $K$ is a constant of motion and it is possible to construct a time invariant operator in every class [9, 10]:

$$I_K = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \exp(-iHt) M_K(0) \exp(iHt) dt . \qquad (3)$$

Since there is no transformation between classes, invariants constructed within different classes are independent. Thus we have derived the following conclusion: the number of independent invariants of motion is equal to the number of equivalence classes of $SGRSE^*$. Having in mind that an integrable system of $\mathcal{N}$ degrees of freedom has $\mathcal{N}$ independent invariants of motion we define the following measure of integrability (order parameter):

$$\mu(x) = \frac{n(x)}{\mathcal{N}}, \qquad (4)$$

where $n(x)$ is the number of $SGRSE^*(x, \mathcal{N})$ classes and $\mathcal{N}$ is the dimension of matrices. $\mu(x)$ is equal to 1 for an integrable system and goes to zero in the limit $\mathcal{N} \to \infty$ for chaotic one. Therefore, $\mu(x)$ plays the role of integrability measure for $SGRSE(x, \mathcal{N})$ and can be used as an order parameter for description of the phase transition from the integrable properties to chaotic ones. By the same way one can introduce the order parameter for the ensembles diluted from $GUE(\mathcal{N})$ and $GSE(\mathcal{N})$. The order parameter (4) will be useful if one can calculate the number of equivalence classes, at least approximately.

Transforming Wigner's semicircle density of states into the cumulative distribution function $N(E)$ one obtains everywhere smooth function of energy, in particular for $E = 0$. Below certain threshold $x < x_c$ the density of states gets spike for $E = 0$ which is a sign that a new class of the lower rank matrices appears [2]. Corresponding cumulative distribution function gets jump for $E = 0$ (Fig. 1(a) and (b)):

$$\Delta n = N(0^+) - N(0^-) . \qquad (5)$$

$\Delta n$ increase when $x$ decreases and is a measure of the highest multiplicity of the eigenvalue $E = 0$. Since each equivalence class contains an infinite number of matrices one can make the following assumption: it is very probable that when a new equivalence class appears as a result of decreasing $x$ than all classes existing before remain. By this assumption we relate $\Delta n$ to an approximation of the number of equivalence classes and finally to the order parameter:

$$\mu(x) = \Delta N(x) = \frac{\Delta n(x)}{\mathcal{N}} . \qquad (6)$$

As an example we present the cumulative distribution function $N(E)$ that represents the spectra of the matrix $2000 \times 2000$ for the fractions $x = 0.0005$

and $x = 0.05$ plotted in Fig. 1(a) and (b), respectively. An example of dependence of $\Delta N$ versus $\log(x)$ for dimension $\mathcal{N} = 100$ is presented in Fig. 2. The dependence is typical for the continuous phase transitions. Formula (6) has been successfully used for calculation of phase diagram for $SGRSE$ in [6].

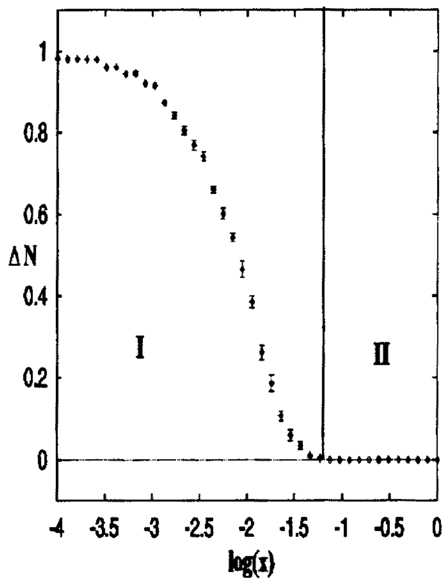



Fig. 2. The order parameter as a function of logarithm of $x$ together with numerical errors ($\mathcal{N} = 100$).

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