SOLVABLE POTENTIALS GENERATED BY THE HEISENBERG ALGEBRA

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The operators a and a^+ realizing the Heisenberg algebra are used to solve the spectral problems with potentials different from the harmonic oscillator. By reducing the stationary Schrödinger equation to the form of the Whittaker equation (or hypergeometric equation), and next by comparison with the a^+a product a family of the solvable potentials is obtained. The solutions of the pseudo-eigenvalue problem ($\mathrm{H}(E_n)\psi_n = E_n\psi_n$) with the new potentials are given. It is also shown how to construct such pseudo-eigenvalue equations with eigenfunctions related to the Whittaker function. The results are used to build up Hamiltonians which are expressible by the a^+a product.

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1. Introduction

Resolving the problems of non-relativistic quantum mechanics by means of algebraic techniques have always attracted much attention. The introduction of a few new concepts had an exceptionally strong impact on these studies and helped to view this field from a new angle. The famous supersymmetry (SUSY) algebra [1] has been successfully utilized to achieve a SUSY generalization [2–4] of the harmonic oscillator raising and lowering operators for SUSY shape-invariant potentials. Although shape-invariance is not a general feature of solvable potentials [5], this has made the SUSY method a powerful algebraic technique for the spectral resolution of a variety of such potentials of physical interest [5–10]. In recent times a new class of spectral problems has been discovered [12–16] with the help of the algebraic technique. It occupies an intermediate position between the exactly-solvable problems (like the harmonic oscillator which is reducible to the diagonal form with the aid of an algebraic procedure), and the exactly-nonsolvable problems (in those cases the algebraic methods would not allow to diagonalize the Hamiltonians). This class called quasi- exactly-solvable is distinguished by the fact that only a part of the eigenvalues and eigenfunctions, and not the whole spectrum, can be found. It occurs when the Hamiltonian matrix has a special block structure, what of course is not a feature of every Hamiltonian. However there is a method ensuring the block structure for specially chosen cases. This gives a possibility to have a part of solutions of the spectral problem with the new potentials.

In this paper the harmonic oscillator plays a special role, as it is the simplest quantum system both from a physical and a mathematical points of view. This statement is due to the fact that it is completely solvable by means of the linear differential operators satisfying Heisenberg algebra. The Hamiltonian of the harmonic oscillator written in the formalism of creation and annihilation operators can be treated as a pattern for other solved differential equations (the Whittaker equation and the hypergeometric equation). In order to make them equivalent it is necessary to accept the dependence between the equations parameters and the natural numbers. Having such the conditions the next step seems to be obvious: an appropriate change of variable in the stationary Schrödinger equation reduces its form to the known differential equation, and by comparison gives a family of solvable potentials. The generalization of this problem is also possible by taking the initial Schrödinger equation with a fixed eigenvalue equal to zero. This gives an ability to solve the so called pseudo-eigenvalue problem $\mathrm{H}(E_n)\psi_n=E_n\psi_n$ (it means that the energy enters into the Hamiltonian as a parameter) with the potentials in more general form than those, for which the exact solutions of the Schrödinger equation are known.

2. The Heisenberg algebra

In this Section we deal only with a few aspects of the Heisenberg algebra which are useful for our purpose. Let $\{\psi_n\}$ be the functions defined by:

$$a^+ a\psi_n(x) = \bar{\lambda}\psi_n(x)\,, \qquad x \geq 0\,, \tag{2.1}$$

with the scalar product

$$(\psi_n(x), \psi_m(x)) = \int_0^\infty \psi_n^*(x)\psi_m(x)dx = \delta_{mn}\,, \qquad n, m = 0,\, 1,\, 2, \ldots\,, \tag{2.2}$$

where the linear differential operators a, a^+ satisfy the conditions:

$$[a, a^+] = 1\,, \tag{2.3}$$

$$a\psi_0 = 0\,. \tag{2.4}$$

The change of variable

$$x = f(q)\,, \qquad q \in [q_0, q_1]\,, \tag{2.5}$$

where f is a given function of q, implies that the scalar product is transformed into

$$(\psi_n(x), \psi_m(x)) = \int_{q_0}^{q_1} \psi_n^*(x)\psi_m(x) f'(q) dq \tag{2.6}$$

and $'$ denotes differentiation with respect to q. The general form for the operator a is taken to be

$$a = g(q)\frac{d}{dq} + W(q)\,, \tag{2.7}$$

where g and W are two arbitrary functions of q. Thus, the operator a^+ is determined by (2.6) leading to the form:

$$a^+ = -g(q)\frac{d}{dq} - \frac{[f'(q)g(q)]'}{f'} + W(q)\,, \tag{2.8}$$

where the condition

$$f'(q)g(q)\psi_n^*(q)\psi_m(q)\Big|_{q=q_0}^{q=q_1} = 0 \tag{2.9}$$

has to be fulfilled. From (2.3) we get

$$W(q) = \frac{1}{2}\int^q \frac{dz}{g(z)} + \frac{1}{2}\frac{[f'(q)g(q)]'}{f'} + b\,, \qquad b \in \boldsymbol{R}\,, \tag{2.10}$$

and the operators a, a^+ become

$$a = g(q)\frac{d}{dq} + \frac{1}{2}\left[\int \frac{dq}{g(q)} + 2b\right] + \frac{1}{2}\frac{(f'g)'}{f'}\,, \tag{2.11}$$

$$a^+ = -g(q)\frac{d}{dq} + \frac{1}{2}\left[\int \frac{dq}{g(q)} + 2b\right] - \frac{1}{2}\frac{(f'g)'}{f'}\,. \tag{2.12}$$

Combining (2.4), (2.9), (2.11) and (2.12) with (2.1), we finally obtain the solution (without normalization constant)

$$\psi_{2n+1}(q) = (f'g)^{-\frac{1}{2}} H_{2n+1}\left[\frac{1}{\sqrt{2}}\left(\int \frac{dq}{g(q)} + 2b\right)\right] \times e^{-1/4[\int \frac{dq}{g(q)}+2b)]^2}\,, \qquad n = 0, 1, 2, 3 \ldots\,, \tag{2.13}$$

where

$$\int \frac{dq}{g(q)} + 2b \geq 0 \tag{2.14}$$

and

$$\bar{\lambda} = 2n + 1\,. \tag{2.15}$$

(Even eigenvalues and the even order Hermite polynomials H_{2n} are excluded because of (2.9).)

Now from (2.10)–(2.15) it follows that the general form of equation (2.1) is given by

$$\begin{aligned} a^{+}a\psi_{2n+1} &= \left(-g^2\frac{d^2}{dq^2} - g\left(g' + \frac{(f'g)'}{f'}\right)\frac{d}{dq}\right. \\ &\quad \left. + \frac{1}{4}\left[\int \frac{dq}{g(q)} + 2b\right]^2 - \frac{1}{4}\left[\frac{(f'g)'}{f'}\right]^2 - \frac{1}{2}g\left[\frac{(f'g)'}{f'}\right]'\right)\psi_{2n+1} \\ &= (2n+1)\psi_{2n+1}\,. \end{aligned} \tag{2.16}$$

This plays an important role in connection with the Whittaker equation.

3. Whittaker's differential equation

We now come to the central problem of this work. The equation (2.1) is written in the model form (2.16) with all the above mentioned results. We suppose that this model can be compared with other solvable differential equations. The questions are: which differential equations should we choose and what kind of dependence such comparison creates? As we shall see in this Section it can be done with Whittaker's differential equation in the easiest way. Thus we start from the Whittaker equation

$$\left\{\frac{d^2}{d\hat{x}^2} + \frac{\lambda}{\hat{x}} + \frac{\frac{1}{4} - \mu^2}{\hat{x}^2} - \frac{1}{4}\right\} y(\hat{x}) = 0\,, \tag{3.1}$$

where two linearly independent solutions are given by

$$M_{\lambda,\mu}(\hat{x}) = \hat{x}^{\mu+\frac{1}{2}} e^{-\frac{1}{2}\hat{x}} {}_1F_1(\mu - \lambda + \tfrac{1}{2}, 2\mu + 1; \hat{x})\,, \tag{3.2}$$

$$M_{\lambda,-\mu}(\hat{x}) = \hat{x}^{-\mu+\frac{1}{2}} e^{-\frac{1}{2}\hat{x}} {}_1F_1(-\mu - \lambda + \tfrac{1}{2}, -2\mu + 1; \hat{x})\,, \tag{3.3}$$

and ${}_1F_1$ is the hypergeometric function. The change of variable

$$\hat{x} = 2\mu\left[\int \frac{dq}{g(q)} + 2b\right]^2, \quad q \in [q_0, q_1]\,, \quad \mu > 0\,, \tag{3.4}$$

together with

$$y(q) = (f'g)^{\frac{1}{2}}\left[\int \frac{dq}{g(q)} + 2b\right]^{\frac{1}{2}} \hat{\phi}(q)\,, \tag{3.5}$$

gives the Whittaker equation in the form

$$\begin{aligned}\frac{1}{2}\Bigg\{ & -\frac{1}{4\mu}g^2\frac{d^2}{dq^2} - \frac{1}{4\mu}g\left(g' + \frac{(f'g)'}{f}\right)\frac{d}{dq} + \mu\left[\int \frac{dq}{g} + 2b\right]^2 \\ & + \frac{1}{4}\left(4\mu - \frac{1}{4\mu}\right)\left[\int \frac{dq}{g} + 2b\right]^{-2} \\ & - \frac{1}{4}\frac{1}{4\mu}\left(\frac{(f'g)'}{f'}\right)^2 - \frac{1}{2}\frac{1}{4\mu}g\left(\frac{(f'g)'}{f'}\right)' - 2\mu\Bigg\}\hat{\phi}(q) = (\lambda - \mu)\hat{\phi}(q)\,,\end{aligned} \tag{3.6}$$

what can be symbolically written

$$\text{l.h.s.}\,(\mu) = \text{r.h.s.}\,(\lambda - \mu)\,. \tag{3.7}$$

The displacement of parameters

$$\mu \to \mu - \kappa = \tfrac{1}{4}\,, \qquad \lambda \to \lambda - \kappa\,, \tag{3.8}$$

implies

$$\lambda - \mu \to \lambda - \kappa - \mu + \kappa = \lambda - \mu\,, \tag{3.9}$$

and by comparison of (3.6) with (2.16) we get

$$\lambda - \mu = \tfrac{1}{2}(2n + 1) = n + \tfrac{1}{2}\,, \qquad n = 0, 1, 2, \ldots\,. \tag{3.10}$$

Hence, the normalizable solutions of (3.6) are given, up to the normalization constant, by

$$\begin{aligned}\hat{\phi}(q) \;=\; & (f'g)^{-\frac{1}{2}}\left[\int \frac{dq}{g} + 2b\right]^{2\mu+\frac{1}{2}} e^{-\mu[\int \frac{dq}{g}+2b]^2} \\ & \times {}_1F_1\left(-n, 2\mu + 1, 2\mu\left[\int \frac{dq}{g} + 2b\right]^2\right).\end{aligned} \tag{3.11}$$

The results obtained in this Section can be immediately used to investigate the Schrödinger equation.

4. The Schrödinger equation

Now we turn to the stationary Schrödinger equation in the form given by:

$$\left\{-\frac{d^2}{dr^2} + U(r) - \hat{E}\right\}\Psi(r) = 0\,, \tag{4.1}$$

where in general $r \in \boldsymbol{R}$. The change of variable and function

$$r = \bar{f}(q) \qquad \Psi(q) = (\bar{f}'f'g^2)^{\frac{1}{2}}\phi(q)\,, \tag{4.2}$$

implies that equation (4.1) is transformed into

$$\begin{aligned}\Bigg\{ &-g^2\frac{d^2}{dq^2} - g\left(g' + \frac{(f'g)'}{f'}\right)\frac{d}{dq} - \frac{1}{2}gg'' + \frac{1}{4}(g')^2 - \frac{1}{2}g\left[\frac{(f'g)'}{f'}\right]' \\ &-\frac{1}{4}\left[\frac{(f'g)'}{f'}\right]^2 + \frac{1}{4}g^2\left(\frac{\bar{f}''}{\bar{f}'}\right)^2 - \frac{1}{2}g^2\left(\frac{\bar{f}''}{\bar{f}'}\right)' \\ &+(\bar{f}'g)^2[U(\bar{f}(q)) - \hat{E}]\Bigg\}\phi(q) = 0\,. \end{aligned} \tag{4.3}$$

To simplify the future calculations, without loosing generality, we can take

$$\frac{(f'g)'}{f'} = -g' \tag{4.4}$$

and additionally

$$\frac{\bar{f}''}{\bar{f}'} = \text{const.}\,, \tag{4.5}$$

$$(g')^2 = gg''\,, \tag{4.6}$$

what means that we get the simpler forms of equations (3.6) and (4.3). Now we are in a position to equate them. This gives

$$(\bar{f}'g)^2[U(\bar{f}) - \hat{E}] \simeq \left[\int\frac{dq}{g} + 2b\right]^2 + \left[\int\frac{dq}{g} + 2b\right]^{-2} + \text{const.} \tag{4.7}$$

and the general form of the wave function

$$\begin{aligned}\Psi(q) &= (\bar{f}'g)^{\frac{1}{2}}\left[\int\frac{dq}{g} + 2b\right]^{2\mu+\frac{1}{2}} \\ &\times e^{-\mu(\int\frac{dq}{g}+2b)^2}{}_1F_1\left(-n, 2\mu+1, 2\mu\left(\int\frac{dq}{g} + 2b\right)^2\right)\end{aligned} \tag{4.8}$$

up to a multiplication constant. Thus we get three independent equations

$$(\bar{f}'g)^2 = \text{const.}, \tag{4.9}$$

$$(\bar{f}'g)^2 = \text{const.}\left[\int \frac{dq}{g} + 2b\right]^2, \tag{4.10}$$

$$(\bar{f}'g)^2 = \text{const.}\left[\int \frac{dq}{g} + 2b\right]^{-2}, \tag{4.11}$$

which lead us to the potential of the three-dimensional harmonic oscillator, the hydrogen atom potential and the one-dimensional Morse potential respectively.

5. Generalization with $b = 0$

The conclusions emerging from (4.7) can be simply generalized if we take

$$\hat{E} = 0\,. \tag{5.1}$$

Naturally, it means consideration of a Schrödinger equation with energy equal zero, but taking (3.10) into account we can get the dependence between values of the potential parameters when the solution of a Schrödinger equation (with a fixed eigenvalue) is in the form (4.8). This generalization must also include Eqs (4.9)–(4.11) as special cases. To simplify the calculations in this Section and the next one we take

$$b = 0\,. \tag{5.2}$$

Choosing generalization in the form

$$(\bar{f}'g)^2 \simeq \left(\int \frac{dq}{g}\right)^{2\sigma}, \quad \sigma \neq -1\,, \tag{5.3}$$

means that the equation (4.11) is excluded from our considerations. (This case will be discussed separately in Section 8.)

From (2.14), (4.7), (5.1), (5.2) and (5.3) we get

$$\bar{f} \simeq \left[\int \frac{dq}{g}\right]^{\sigma+1}, \quad \bar{f} \geq 0\,, \tag{5.4}$$

and

$$U(\bar{f}) \simeq \bar{f}^{2\frac{1-\sigma}{1+\sigma}} + \bar{f}^{-2} + \bar{f}^{-\frac{2\sigma}{1+\sigma}}\,. \tag{5.5}$$

Thus we have the Schrödinger equation in the form

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+Ar^{2\frac{1-\sigma}{1+\sigma}}+Br^{-2}-Cr^{-\frac{2\sigma}{1+\sigma}}\right\}\Psi(r)=0\,, \tag{5.6}$$

which can be reduced to the transformed Whittaker equation (3.6) taking

$$r=\alpha e^{(\sigma+1)q}\,,\qquad \Psi(q)=e^{\frac{1}{2}(\sigma+1)q}\phi(q)\,,\qquad \alpha=\text{const.}\,,\qquad q\in\boldsymbol{R}\,. \tag{5.7}$$

Performing calculations we get $A>0$, $B\geq 0$, $C>0$, and for

$$B=\frac{\hbar^2}{2m}l(l+1)\,, \tag{5.8}$$

we get

$$\alpha^{\frac{4}{1+\sigma}}=\frac{\hbar^2}{2m}\frac{\left(l+\frac{1}{2}\right)^2}{A}\,. \tag{5.9}$$

From (3.10) it follows that

$$\frac{\sigma+1}{4}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}\frac{C}{A^{\frac{1}{2}}}=n+\frac{1}{2}+\mu\,, \tag{5.10}$$

where we have two conditions:

$$\mu=\frac{(\sigma+1)(l+\frac{1}{2})}{2}\quad \text{for}\quad \sigma+1>0\,, \tag{5.11}$$

$$\mu=-\frac{(\sigma+1)(l+\frac{1}{2})}{2}\quad \text{for}\quad \sigma+1<0\,. \tag{5.12}$$

Each of them leads us to the different types of potentials, however the condition (5.12) will not be considered in this paper. Thus the form of (5.10) is given by

$$\frac{\sigma+1}{4}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}\frac{C}{A^{\frac{1}{2}}}=n+\frac{1}{2}+\frac{1}{2}(\sigma+1)(l+\frac{1}{2})\,,\ \sigma+1>0\,, \tag{5.13}$$

what enables us to specify two cases:

(a) for $\sigma=0$ we get from (5.6) and (5.13)

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+Ar^2+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}-C\right\}\Psi(r)=0\,, \tag{5.14}$$

and

$$C = E = \hbar\omega(2n + l + \tfrac{3}{2}), \tag{5.15}$$

where

$$A = \tfrac{1}{2}m\omega^2; \tag{5.16}$$

(b) for $\sigma = 1$ we have

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{C}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} + A\right\}\Psi(r) = 0, \tag{5.17}$$

and

$$-A = E = -\frac{mC^2}{2\hbar^2(n+l+1)^2}. \tag{5.18}$$

As we see the separation of constant in equation (5.6) gives the energy spectrum. The same situation takes place also for the other values of σ, but this time it leads us to the pseudo-eigenvalue problems. For

$$-\frac{2\sigma}{1+\sigma} = k, \quad k \in \boldsymbol{N}, \sigma + 1 > 0, \tag{5.19}$$

we get

$$U(r) = Ar^{2(k+1)} - Cr^k + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}. \tag{5.20}$$

Making the displacement of variable

$$r \to \hat{r} - \gamma, \quad \hat{r} \geq \gamma, \tag{5.21}$$

we obtain the Schrödinger equation (dropping the caret over r) in the form

$$\begin{aligned}\Bigg\{ & -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + A\left[\sum_{j/1}^{2(k+1)} (-1)^{2(k+1)-j}\begin{pmatrix} 2(k+1) \\ j \end{pmatrix}\gamma^{2(k+1)-j}r^j\right] \\ & -C\left[\sum_{i/1}^{k}(-1)^{k-i}\begin{pmatrix} k \\ i \end{pmatrix}\gamma^{k-i}r^i\right] + \frac{\hbar^2}{2m}\frac{l(l+1)}{(r-\gamma)^2}\Bigg\}\Psi(r) = E\Psi(r),\end{aligned} \tag{5.22}$$

where

$$-E = A(-1)^{2(k+1)}\gamma^{2(k+1)} - C(-1)^k\gamma^k, \tag{5.23}$$

with the quantum condition

$$\frac{1}{2}\frac{1}{k+2}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}\frac{C}{A^{\frac{1}{2}}} = n + \frac{1}{2} + \frac{1}{k+2}\left(l + \tfrac{1}{2}\right). \tag{5.24}$$

The displacement parameter can be expressed by the potential coefficients

$$\gamma = \delta \left(\frac{C}{A}\right)^{\frac{1}{k+2}}, \tag{5.25}$$

where δ is a dimensionless constant. Combining (5.23) with (5.25) we have

$$E = (-\delta^{2(k+1)} + (-1)^k \delta^k) A^{-\frac{k}{k+2}} C^{\frac{2(k+1)}{k+2}} \tag{5.26}$$

or, using (5.24),

$$E \sim \left(n + \tfrac{1}{2}\right)^{2\frac{k+1}{k+2}} A^{\frac{1}{k+2}}. \tag{5.27}$$

The same result is obtained by means of virial theorem and WKB method considering the potential given by

$$V(r) \sim Ar^p. \tag{5.28}$$

Then the dependence between the energy spectrum and the quantum number can be expressed in the form

$$E \sim (n + \tfrac{1}{2})^{\frac{2p}{p+2}} A^{\frac{2}{2+p}}. \tag{5.29}$$

In our case $p = 2(k+1)$ what changes (5.29) into (5.27).

In framework of this method we have another chance to construct the solvable pseudo-eigenvalue problem. For

$$2\frac{1-\sigma}{1+\sigma} = k, \qquad k \in \boldsymbol{N} \tag{5.30}$$

the potential is

$$U(r) = Ar^k - Cr^{\frac{1}{2}(k-2)} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}. \tag{5.31}$$

We proceed in a similar way as for the previous case. Thus we get

$$\left\{ -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + A\left[\sum_{i/1}^{k}(-1)^{k-i}\binom{k}{i}\gamma^{k-i}r^i\right] - C(r-\gamma)^{\frac{k-2}{2}} \right.$$
$$\left. + \frac{\hbar^2}{2m}\frac{l(l+1)}{(r-\gamma)^2}\right\}\Psi(r) = E\Psi(r), \tag{5.32}$$

where $r \geq \gamma$

$$-E = (-1)^k \gamma^k A, \tag{5.33}$$

and the quantum condition is given by

$$\frac{1}{k+2}\left(\frac{2m}{\hbar^2}\right)^2 \frac{C}{A^{\frac{1}{2}}} = n + \tfrac{1}{2} + \frac{2}{k+2}(l + \tfrac{1}{2}) \,. \tag{5.34}$$

Expressing γ by the potential coefficients in the following form

$$\gamma = \delta\left(\frac{C}{A}\right)^{\frac{2}{k+2}}, \tag{5.35}$$

where δ is a dimensionless constant, we get

$$E = (-1)^{k+1}\delta^k A^{\frac{2-k}{2+k}} C^{\frac{2k}{k+2}}\,, \tag{5.36}$$

what (together with (5.34)) is in accordance with the results obtained by means of virial theorem and WKB method.

6. Schrödinger equation structured with the aid of a^+a product

As we see in Section 3, we are in a position to reduce the transformed Whittaker equation (3.6) into the form (2.16) taking

$$\mu = \tfrac{1}{4}\,. \tag{6.1}$$

By substitution

$$\tfrac{1}{4} = \tfrac{1}{2}(\sigma + 1)(l + \tfrac{1}{2}) \tag{6.2}$$

we get

$$\sigma = -\frac{2l}{2l+1}\,, \qquad \sigma + 1 > 0 \quad \text{for} \quad l \geq 0\,. \tag{6.3}$$

As a consequence we obtain the Schrödinger equation in the form

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + Ar^{2(4l+1)} - Cr^{4l} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right\}\Psi(r) = 0 \tag{6.4}$$

with the quantum condition (5.13) given by

$$\frac{C}{A^{\frac{1}{2}}} = \left(\frac{\hbar^2}{2m}\right)^{\frac{1}{2}}(2l+1)(4n+3)\,. \tag{6.5}$$

This form leads us to one special case. For

$$l = 0\,, \qquad A = \tfrac{1}{2}m\omega^2 \tag{6.6}$$

we have

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{1}{2}m\omega^2r^2-C\right\}\Psi(r)=0\,, \tag{6.7}$$

where

$$C=E=\hbar\omega\left(2n+\tfrac{3}{2}\right). \tag{6.8}$$

We can also proceed as in the previous Section. Making the displacement of variable in equation (6.4) we get

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+A\left[\sum_{i/1}^{2(4l+1)}\binom{2(4l+1)}{i}(-\gamma)^{2(4l+1)-i}r^i\right]\right.$$
$$\left.-C\left[\sum_{j/1}^{4l}\binom{4l}{j}(-\gamma)^{4l-j}r^j\right]+\frac{\hbar^2}{2m}\frac{l(l+1)}{(r-\gamma)^2}\right\}\Psi(r)=E\Psi(r)\,, \tag{6.9}$$

where

$$r\geq\gamma\,,\qquad E=-A\gamma^{2(4l+1)}+C\gamma^{4l}. \tag{6.10}$$

Taking

$$\gamma=\delta\left(\frac{C}{A}\right)^{\frac{1}{4l+2}}, \tag{6.11}$$

where δ is a dimensionless constant, and using Eq. (6.5) we have

$$E=(-\delta^{2(4l+1)}+\delta^{4l})\left(\frac{\hbar^2}{2m}\right)^{\frac{2l+\frac{1}{2}}{2l+1}}(2l+1)^{\frac{4l+1}{2l+1}}(4n+3)^{\frac{4l+1}{2l+1}}(A)^{\frac{1}{2(2l+1)}}\,, \tag{6.12}$$

what also stays in accordance with the results given by virial theorem and WKB method.

7. Generalization with $b\neq 0$

The results of previous Sections can be easily generalized by taking $b\neq 0$. It means that we have to take one more term in (4.7) into account, and although the calculation method is the same as previously, it leads us to new solvable problems. We start with

$$\hat{E}=0 \tag{7.1}$$

and

$$(\bar{f}'g)^2\simeq\left(\int\frac{dq}{g}\right)^{2\sigma}, \tag{7.2}$$

where

$$\bar{f} \geq 0\,, \qquad \sigma \neq -1\,, \qquad \int \frac{dq}{g} + 2b \geq 0\,; \tag{7.3}$$

thus we get

$$\int \frac{dq}{g} \simeq \left[\bar{f} + (-2b)^{\sigma+1}\right]^{\frac{1}{\sigma+1}}\,, \qquad b < 0\,, \tag{7.4}$$

and from (4.7)

$$\begin{aligned} U(\bar{f}) \simeq & \left\{\left[\bar{f} + (-2b)^{\sigma+1}\right]^{\frac{1}{\sigma+1}} + 2b\right\}^{-2}\left[\bar{f} + (-2b)^{\sigma+1}\right]^{-\frac{2\sigma}{\sigma+1}} \\ & + \left[\bar{f} + (-2b)^{\sigma+1}\right]^{2\frac{1-\sigma}{\sigma+1}} + \left[\bar{f} + (-2b)^{\sigma+1}\right]^{\frac{1-2\sigma}{\sigma+1}} + \left[\bar{f} + (-2b)^{\sigma+1}\right]^{-2} \\ & + \left[\bar{f} + (-2b)^{\sigma+1}\right]^{-\frac{2\sigma}{\sigma+1}}\,. \end{aligned} \tag{7.5}$$

Hence the exact form of Schrödinger equation arising from (3.6) is given by

$$\begin{aligned} \Bigg\{ & -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + A\left[r + \left(\frac{B}{2A}\right)^{\sigma+1}\right]^{2\frac{1-\sigma}{1+\sigma}} - B\left[r + \left(\frac{B}{2A}\right)^{\sigma+1}\right]^{\frac{1-2\sigma}{1+\sigma}} \\ & - C\left[r + \left(\frac{B}{2A}\right)^{\sigma+1}\right]^{\frac{-2\sigma}{1+\sigma}} + V\left(\frac{B}{2A}, \sigma; r\right)\Bigg\}\Psi(r) = 0 \end{aligned} \tag{7.6}$$

with $r \geq 0\,, \quad A > 0\,, \quad B \geq 0\,, \quad C > 0$, where

$$\begin{aligned} V\left(\frac{B}{2A}, \sigma; r\right) = & \frac{\hbar^2}{2m}\frac{l(l+1)}{\left\{\left[r + \left(\frac{B}{2A}\right)^{\sigma+1}\right]^{\frac{1}{1+\sigma}} - \left(\frac{B}{2A}\right)\right\}^2\left[r + \left(\frac{B}{2A}\right)^{\sigma+1}\right]^{\frac{2\sigma}{\sigma+1}}} \\ & + \frac{\hbar^2}{2m}\left[\frac{1}{4} - \frac{1}{4(\sigma+1)^2}\right]\left\{\frac{1}{\left\{\left[r + \left(\frac{B}{2A}\right)^{\sigma+1}\right]^{\frac{1}{1+\sigma}} - \left(\frac{B}{2A}\right)\right\}^2\left[r + \left(\frac{B}{2A}\right)^{\sigma+1}\right]^{\frac{2\sigma}{\sigma+1}}}\right. \\ & \left. - \frac{1}{\left[r + \left(\frac{B}{2A}\right)^{\sigma+1}\right]^2}\right\}\,, \end{aligned} \tag{7.7}$$

and

$$V\left(\frac{B}{2A}, 0; r\right) = V(0, \sigma; r) = \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\,, \tag{7.8}$$

$$V\left(\frac{B}{2A}, -\frac{2l}{2l+1}; r\right) = \frac{\hbar^2}{2m}\frac{l(l+1)}{\left[r + \left(\frac{B}{2A}\right)^{\frac{1}{2l+1}}\right]^2}. \tag{7.9}$$

We also get the new quantum condition in the form

$$\frac{1}{16}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}(\sigma+1)\frac{B^2}{A^{\frac{3}{2}}} + \frac{1}{4}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}(\sigma+1)\frac{C}{A^{\frac{1}{2}}} = n + \tfrac{1}{2} + \mu, \tag{7.10}$$

where

$$\mu = \tfrac{1}{2}(\sigma+1)(l+\tfrac{1}{2}) \tag{7.11}$$

is greater than zero what means that

$$\sigma + 1 > 0. \tag{7.12}$$

Now we can see the possibility to separate the energy spectrum for three cases:

(a) $\sigma=0$,

$$U(r) = Ar^2 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - \frac{B^2}{4A} - C, \tag{7.13}$$

where

$$\frac{B^2}{4A} + C = E = \hbar\omega\left(2n + l + \tfrac{3}{2}\right), \tag{7.14}$$

$$A = \tfrac{1}{2}m\omega^2; \tag{7.15}$$

(b) $\sigma=1$

$$U(r) = A - B\left[r + \left(\frac{B}{2A}\right)^2\right]^{-\frac{1}{2}} - C\left[r + \left(\frac{B}{2A}\right)^2\right]^{-1} + V\left(\frac{B}{2A}, 1, r\right), \tag{7.16}$$

where

$$A = -E, \tag{7.17}$$

and

$$\frac{1}{8}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}\frac{B^2}{A^{3/2}} + \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}\frac{C}{A^{\frac{1}{2}}} = n + l + 1; \tag{7.18}$$

(c) $\sigma = \frac{1}{2}$,

$$U(r) = A\left[r + \left(\frac{B}{2A}\right)^{\frac{3}{2}}\right]^{\frac{2}{3}} - C\left[r + \left(\frac{B}{2A}\right)^{\frac{3}{2}}\right]^{-\frac{2}{3}} + V\left(\frac{B}{2A}, \frac{1}{2}; r\right) - B, \tag{7.19}$$

where

$$B = E = \left[-4AC + \frac{32}{3}\left(\frac{\hbar^2}{2m}\right)^{\frac{1}{2}} A^{3/2}\left[n + \frac{1}{2} + \frac{3}{4}\left(l + \frac{1}{2}\right)\right]\right]^{\frac{1}{2}} \tag{7.20}$$

with

$$n + \frac{3}{4}\left(l + \frac{1}{2}\right) > \frac{3}{8}\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} A^{-\frac{1}{2}}C - \frac{1}{2}\,. \tag{7.21}$$

Naturally, we can also construct pseudo-eigenvalue problems using a displacement in variable, but this method has been described in Section 5. We only mention about the equations structured by means of the a^+a product. As in Section 6 we take

$$\mu = \tfrac{1}{4}\,, \tag{7.22}$$

thus $\sigma = -\frac{2l}{2l+1}$, $\sigma + 1 > 0$. Substitution σ into (7.6) we get the Schrödinger equation in the form given by

$$\Bigg\{ -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + A\left[r + \left(\frac{B}{2A}\right)^{\frac{1}{2l+1}}\right]^{2(4l+1)} - \left[r + \left(\frac{B}{2A}\right)^{\frac{1}{2l+1}}\right]^{6l+1}$$
$$-C\left[r + \left(\frac{B}{2A}\right)^{\frac{1}{2l+1}}\right]^{4l} + \frac{\hbar^2}{2m}\frac{l(l+1)}{\left[r + \left(\frac{B}{2A}\right)^{\frac{1}{2l+1}}\right]^2}\Bigg\}\Psi(r) = 0\,. \tag{7.23}$$

Taking

$$l = 0 \tag{7.24}$$

implies the form

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + Ar^2 - \frac{B^2}{4A} - C\right\}\Psi(r) = 0\,, \tag{7.25}$$

where from Eq. (7.10) we have

$$\frac{B^2}{4A} + C = E = \hbar\omega\left(2n + \frac{3}{2}\right), \tag{7.26}$$

what is the expected result.

8. Morse potential

According to equation (4.11) there is one more case to be considered. This separate treatment is connected with the different dependence between $\bar{f}$ and $\int \frac{dq}{g}$, than that generated by Eq. (5.3) what in consequence leads us to the quite new types of the solvable pseudo-eigenvalue problems. Naturally, the method of obtaining the potentials does not change, what means that using Eq. (4.7) we are looking for a Schrödinger equation which is reducible to the form (3.6). Let us start with

$$\hat{E} = 0\,, \qquad b \neq 0\,. \tag{8.1}$$

From (4.11) we get

$$\bar{f} \simeq \ln\left(\int \frac{dq}{g}\right), \tag{8.2}$$

where

$$\int \frac{dq}{g} + 2b \geq 0\,. \tag{8.3}$$

We shall do the generalization in a way which preserves the logarithmic character of (8.2). Taking

$$\bar{f}^{\sigma} \simeq \ln\left(\int \frac{dq}{g}\right) \qquad \sigma = \text{const.}, \tag{8.4}$$

and additionally, to simplify the calculations $g = e^{-q}$ we obtain from Eq. 4.7)

$$U(\bar{f}) \simeq \bar{f}^{2(\sigma-1)} + \bar{f}^{2(\sigma-1)} e^{2\bar{f}^{\sigma}} [e^{\bar{f}^{\sigma}} + 2b]^{-2} + \bar{f}^{2(\sigma-1)} (e^{4\bar{f}^{\sigma}} + e^{3\bar{f}^{\sigma}} + e^{2\bar{f}^{\sigma}})\,, \tag{8.5}$$

where

$$\bar{f} \geq [\ln(-2b) + \text{const.}]^{\frac{1}{\sigma}}\,, \qquad b < 0\,. \tag{8.6}$$

The exact form of the Schrödinger equation connected with this potential is given by

$$\begin{aligned} \Bigg\{ & -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + A\left(\frac{r}{a}\right)^{4l} e^{4(\frac{r}{a})^{2l+1}} - B\left(\frac{r}{a}\right)^{4l} e^{3(\frac{r}{a})^{2l+1}} \\ & -C\left(\frac{r}{a}\right)^{4l} e^{2(\frac{r}{a})^{2l+1}} + D\left(\frac{r}{a}\right)^{4l} e^{2(\frac{r}{a})^{2l+1}} \\ & \times \left[e^{(\frac{r}{a})^{2l+1}} - \frac{B}{2A}\right]^{-2} + \frac{\hbar^2}{8ma^2}(2l+1)^2\left(\frac{r}{a}\right)^{4l} \\ & + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\Bigg\}\Psi(r) = 0\,, \end{aligned} \tag{8.7}$$

where

$$A > 0\,, \qquad B \geq 0\,, \qquad C > 0\,, \qquad D \geq 0\,, \tag{8.8}$$

and

$$\frac{r}{a} \geq \left(\ln\left(\frac{B}{2A}\right)\right)^{\frac{1}{\sigma}}, \qquad \sigma = 2l + 1\,. \tag{8.9}$$

The choice of σ as an odd number unrestricts the variable. We also get the quantum condition arises from (3.10)in the form

$$\frac{1}{4}\left(\frac{2ma^2}{\hbar^2}\right)^{\frac{1}{2}} \frac{1}{2l+1}\left[\frac{C}{A^{\frac{1}{2}}} + \frac{B^2}{4A^{3/2}}\right] = n + \frac{1}{2} + \mu \tag{8.10}$$

with μ given by

$$\mu = \left[\frac{1}{16} + \frac{1}{4}\frac{2m}{\hbar^2}\left(\frac{a}{2l+1}\right)^2 D\right]^{\frac{1}{2}}. \tag{8.11}$$

As we see from Eqs (8.2), (8.4) and (8.7) we can go back to the Schrödinger equation with Morse potential putting

$$l = 0\,, \qquad B = 0 \qquad , A = V_0\,, \qquad C = 2V_0\,, \qquad a \to -2a\,. \tag{8.12}$$

Thus

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V_0 e^{-2\frac{r}{a}} - 2V_0 e^{-\frac{r}{a}} + D + \frac{1}{16}\frac{\hbar^2}{2ma^2}\right\}\Psi(r) = 0\,, \tag{8.13}$$

where $r \in \boldsymbol{R}$

$$-\left(D + \frac{1}{16}\frac{\hbar^2}{2ma^2}\right) = E = -\frac{\hbar^2}{2ma^2}\left[\left(\frac{2ma^2V_0}{\hbar^2}\right)^{\frac{1}{2}} - \left(n + \frac{1}{2}\right)\right]^2. \tag{8.14}$$

Remembering that $\mu > 0$ we also have

$$n + \frac{1}{2} < \left(\frac{2ma^2V_0}{\hbar^2}\right)^{\frac{1}{2}}, \tag{8.15}$$

what limits the number of the bound states.

To the end of this Section we will only mention the new possibilities coming from comparison of the Schrödinger equation with the Whittaker equation. If we do not take Eqs. (4.5) and (4.6) into account, then we are in a position to get another condition binding $\bar{f}$ and g. Thus, for

$$b = 0\,, \qquad \hat{E} \neq 0\,, \tag{8.16}$$

we have

$$(\bar{f}'g)^2 \simeq (g')^2\,, \tag{8.17}$$

what implies

$$\bar{f} \simeq \ln(g)\,, \qquad g > 0 \quad \text{for} \quad q \in [q_0, q_1] \tag{8.18}$$

and $\int \frac{dq}{g} \geq 0$.

Comparing Eqs. (3.6) and (4.3) we determine

$$U(q) \simeq \left(\frac{g}{g'}\right)^2 \left(\frac{g''}{g'}\right)' + \left(\frac{gg''}{(g')^2}\right)^2 + \frac{gg''}{(g')^2} + (g')^{-2} + (g')^{-2}\left[\left(\int \frac{dq}{g}\right)^2 + \left(\int \frac{dq}{g}\right)^{-2}\right]. \tag{8.19}$$

The variety of potentials arising from this form and entering into the pseudo-eigenvalue problems is depending on our choice of g. If we take $g = e^{-q}$ for example, then we get Morse potential, as the forms of (8.2) and (8.18) are the same in this case.

9. Conclusions

As we have seen in this paper the results are obtained by the chain of reductions, from the Schrödinger equation to the Whittaker equation and next to the harmonic oscillator differential equation. The construction like this enables us to replace the Whittaker equation by other solvable differential equation, of course only this one, which is reducible to the form of (2.16). For instance we can take the hypergeometric equation in the form given by

$$\left\{y(1-y)\frac{d^2}{dy^2} + [\gamma - (\alpha+\beta+1)y]\frac{d}{dy} - \alpha\beta\right\}u(y) = 0\,. \tag{9.1}$$

The change of variable

$$y = \left[\sin\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^2, \quad \sigma > 0\,, \quad \int\frac{dq}{g} \in \left[0, \frac{\pi}{\sqrt{\sigma}}\right], \tag{9.2}$$

together with the change of function

$$u(q) = (f'g)^{1/2}\left[\sin\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^{-\lambda_1}\left[\cos\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^{-\lambda_2} v(q)\,, \tag{9.3}$$

where

$$\lambda_1 = \gamma - \tfrac{1}{2}, \quad \lambda_2 = \alpha + \beta + \tfrac{1}{2} - \gamma, \quad \frac{(f'g)'}{f'} = -g', \tag{9.4}$$

imply that Eq. (9.1) is transformed into

$$\left\{ -g^2\frac{d^2}{dq^2} + \frac{1}{2}gg'' - \frac{1}{4}(g')^2 + \frac{\lambda_1(\lambda_1-1)}{4}\frac{\sigma}{[\sin(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g})]^2} + \frac{\lambda_2(\lambda_2-1)}{4}\frac{\sigma}{[\cos(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g})]^2} - \frac{(\alpha+\beta)^2}{4}\sigma \right\} v(q) = -\alpha\beta\sigma v(q) . \tag{9.5}$$

Now putting

$$\sigma = \frac{k}{\beta}, \quad k > 0, \tag{9.6}$$

we are in a position to reduce Eq. (9.5) to the form of (2.16) ($\lambda_1 = 1$ and $\beta \to \infty$), or to the form of (3.6) ($\lambda_1 > 1$ and $\beta \to \infty$). As a consequence we get

$$v(q) = g^{-\frac{1}{2}}\left[\sin\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^{\lambda_1}\left[\cos\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^{\lambda_2} {}_2F_1 \times\left(-n, \beta, \gamma; \left[\sin\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^2\right) \tag{9.7}$$

with

$$\lambda_1 \geq 1, \quad \lambda_2 > 1, \quad \sigma = \frac{2(2\lambda_1 - 1)}{\beta}, \tag{9.8}$$

and

$$\alpha = -n . \tag{9.9}$$

By comparison of (9.5) with (4.3) we obtain equation in the form

$$(\bar{f}'g)^2[U(\bar{f}) - E] \simeq \left[\sin\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^{-2} + \left[\cos\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^{-2} + \text{const.} \tag{9.10}$$

which is analogous to (4.7), but more general (here $b = 0$ and $\hat{E} \neq 0$ are taken). Three independent equations:

$$(\bar{f}'g)^2 = \text{const.}, \tag{9.11}$$

$$(\bar{f}'g)^2 \simeq \left[\sin\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^{-2}, \tag{9.12}$$

$$(\bar{f}'g)^2 \simeq \left[\cos\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^{-2}, \tag{9.13}$$

coming from (9.10) are not unique, as we can change them with the aid of trigonometric unit. Taking for example

$$(\bar{f}'g)^2 = \text{const.}, \tag{9.14}$$

we get from (9.10) the Pöschl–Teller potential

$$U(\bar{f}) \simeq [\sin(\bar{f})]^{-2} + [\cos(\bar{f})]^{-2}, \quad \bar{f} \in \left[0, \frac{\pi}{2}\right]. \tag{9.15}$$

or, choosing

$$(\bar{f}'g)^2 = \left[\tan\left(\frac{\sqrt{\sigma}}{2}\int\frac{dq}{g}\right)\right]^2, \tag{9.16}$$

we obtain the Hulthén type potential in the form given by

$$U(\bar{f}) \simeq \left(\frac{e^{-\bar{f}}}{1-e^{-\bar{f}}}\right)^2 + \frac{e^{-\bar{f}}}{1-e^{-\bar{f}}}, \quad \bar{f} \geq 0. \tag{9.17}$$

This manner of doing leads us to numerous solvable potentials but to catch this method it is enough to consider only few of them.

We should stress significance of the unitary transformations in generating new solvable pseudo-eigenvalue problems what is also noticeable in this paper (the form of Eq. (2.16) can be obtained with the aid of unitary transformation acting on the same equation with $b = 0$). Joining those transformations into this method gives new results which will be presented soon.

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REFERENCES

[1] E. Witten, *Nucl. Phys.* **B185**, 513 (1981).
[2] L. Gendenshtein, *JETP Lett.* **38**, 356 (1983).
[3] R. Dutt, A. Khare, U.P. Sukhatme, *Ann. Phys.* **56**, 163 (1988).
[4] R. Dutt, A. Khare, U.P. Sukhatme, *Phys. Lett.* **181B**, 295 (1986).
[5] F. Cooper, N. Ginocchio, A. Khare, *Phys. Rev.* **D36**, 2458 (1987).
[6] F. Cooper, N. Ginocchio, A. Wipf, *Phys. Lett.* **129A**, 145 (1988).
[7] G. Levai, *J. Phys. A: Math. Gen.* **22**, 689 (1989).
[8] C.V. Sukumar, *J. Phys. A: Math. Gen.* **18**, L57, L697 (1985).
[9] C.V. Sukumar, *J. Phys. A: Math. Gen.* **18**, 2917, 2937 (1985).

[10] A.A. Andrianov, N.V. Borisov, M.I. Eides, M.V. Ioffe, *Phys.Lett.* **109A**, 143 (1985).

[11] M. Abramowitz, I. Stegun, *A Handbook of Math. Functions*, Dover 1972.

[12] A.V. Turbiner, A.G. Ushveridze, *Phys. Lett.* **126A**, 181 (1987).

[13] A.V. Turbiner, *Commun. Math. Phys.* **118**, 467 (1988).

[14] A.V. Turbiner, *Zh. Eksp. Teor. Fiz.* **94**, 33 (1988).

[15] A.G. Ushveridze, *Kratk. Soob. po Fiz.* **37**, 44 (1988), in Russian.

[16] M.A. Shifman, *Inter. J. Mod. Phys.* **A4**, 2897 (1989).

[17] M. Dineykhan, G.V. Efimov, G. Gandbold, S.N. Nedelko, *Oscillator Representation in Quantum Physics*, Springer 1995.