TENSOR HARMONIC ANALYSIS ON HOMOGENEOUS SPACES

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The Hilbert space of tensor functions on a homogeneous space with the compact stability group is considered. The functions are decomposed onto a sum of tensor plane waves (defined in the text), components of which are transformed by irreducible representations of the appropriate transformation group. The orthogonality relation and the completeness relation for tensor plane waves are found. The decomposition constitutes a unitary transformation, which allows to obtain the Parseval equality. The Fourier components can be calculated by means of the Fourier transformation, the form of which is given explicitly.

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1. Introduction

The harmonic analysis of a scalar field on a homogeneous space $X = G/K$ is well developed. There is a general theory and many particular cases are completely solved. For details see, *e.g.*, [1]. However, the harmonic analysis of a tensor field on a manifold might be useful in various physical applications. The manifold is often a homogeneous space. In the case of a tensor field new problems arise because, apart from the theory of group representations, differential geometry has to be involved. In terms of the group theory new (irreducible) representations of the transformation group G may appear in the Fourier decomposition. Actually, vector bundles on homogeneous spaces can be used to investigate some classes of unitary representations, see *e.g.*, [2]. We are interested in the decomposition rather than in the creation of representations. Until now several homogeneous spaces have been studied in such a way on different occasions. Quantization of a massive particle leads to spinor fields on the three-dimensional Lobatchevsky space of four-momenta [3]. Another example is the vector of the electomagnetic field on the two-dimensional sphere, which was considered in [4].

The present paper gives a general and efficient method of producing the harmonic analysis for a wide range of homogeneous spaces whose stability groups K are compact, provided that the harmonic analysis on the transformation group is known, namely, the Plancherel measure and plane waves (eigenfunctions of invariant operators in the enveloping algebra) on the group G are given. As far as the plane waves are considered, one can construct the maximal set of algebraically independent invariant operators using, *e.g.*, the generators of the one-parameter subgroups of G. Then the eigenfunctions can be calculated as the solutions of linear differential equations. The Plancherel measure is of crucial importance since its explicit form is still not known for all Lie groups. However, most of the interesting cases are completed. For further details see, *e.g.*, [1].

We shall use the term "Fourier" or "harmonic analysis" precisely for what follows:

1. "Tensor plane waves" on the homogeneous space $X = G/K$. These tensor functions are supposed to satisfy the completness relation and the orthogonality relation.

2. The spectral synthesis formula – the decomposition of a tensor function onto a sum (an integral) of the plane waves. The components have to be transformed by means of irreducible representations of the group G during a transformation of the function on X.

3. The Fourier transform, which allows to calculate the Fourier components integrating the original function with the appropriate plane wave over X.

4. The Parseval equality, which expresses the unitarity of the Fourier transformation.

2. Formulation of the problem

Let $X = G/K$ be a homogeneous space and its transformation group G be a unimodular Lie group. The action of $g \in G$ on $x \in X$ is denoted in the following way

$$g : X \ni x \to gx \in X.$$

Formally speaking, g is a function of coordinates on X.

Let $K \subset G$ be the stability subgroup of G for a fixed point $x_0 \in X$. K is required to be compact. We identify elements from X and G/K: $X \ni x \simeq [g] \equiv gK \in G/K$, where $x = gx_0$.

One can show that such a homogeneous space admits:

1. a positive definite metric η, which is G-invariant;

2. a G-invariant measure $dx = d(gK)$ given by the equality [5]:

$$\int_G dg\, f(g) = \int_X d(gK) \int_K dk\, f(gk),$$

where dg, dk are the Haar measures on G and K, respectively.

Consider the Hilbert space of square integrable (according to the measure dx) tensor functions on X. This Hilbert space is equipped with the scalar product, for $u, v \in H$ we have

$$\langle u|v\rangle_H = \int_X dx\, \langle v(x)|u(x)\rangle_{T_x},$$

where (for a fixed point $x \in X$) $\langle u(x)|v(x)\rangle_{T_x} = u^\dagger(x)\eta(x)v(x)$ is the scalar product in the tangent space T_x and † denotes the hermitian conjugation. We shall use the same symbol for any tensor power of a matrix (here η). The multiplicity of the product is always evident from the context. The functions from H can be understood as the components (in the coordinate basis) of a contravariant tensor field. A (passive) transformation $g \in G$ of a tensor field defines a unitary representation U of the group G on H

$$U(g)u(x) = \hat{g}(g^{-1}x)u(g^{-1}x),$$

where the matrix element $[\hat{g}(x)]^i{}_j = \frac{\partial g^i}{\partial x^j}(x)$. We shall find the decomposition of a function from H onto irreducible (Fourier) components. Our method is based on the well developed harmonic analysis on the group G. In accordance with the standard procedure (slightly modified) from the theory of induced representations one can move functions from X onto G. It is easy to show that H is unitarily isomorphic to the Hilbert space $\tilde{H}$ of square integrable functions with domain in G and range in $T \simeq T_x$ which fulfil the following condition

$$\forall k \in K, g \in G, \tilde{u} \in \tilde{H} : \tilde{u}(g) = L(k)\tilde{u}(gk), \tag{1}$$

where

$$L(k) = \eta^{1/2}(x_0)\hat{k}(x_0)\eta^{-1/2}(x_0).$$

The scalar product in $\tilde{H}$ has the form

$$\langle \tilde{u}|\tilde{v}\rangle_{\tilde{H}} = \int_G dg\, \langle \tilde{u}(g)|\tilde{v}(g)\rangle_T,$$

where $\langle|\rangle_T$ denotes the scalar product in the space T with the unit bilinear form

$$\langle \tilde{u}(g)|\tilde{v}(g)\rangle_T = u^\dagger(g)v(g).$$

The isomorphism $H \to \tilde{H}$ is given by the formula

$$\tilde{u}(g) = \eta^{1/2}(x_0)\widehat{g^{-1}}(gx_0)u(gx_0).$$

We define on $\tilde{H}$ an (equivalent to U) representation $\tilde{U}$ constructing the commutative diagram

$$\begin{array}{ccc} u & \overset{U}{\to} & U(g)u \\ \downarrow^{\sim} & & \downarrow^{\sim} \\ \tilde{u} & \overset{\tilde{U}}{\to} & \tilde{U}(g)\tilde{u}. \end{array}$$

One should make a few comments.

1. The unitarity of U ($\tilde{U}$) is the consequence of the fact that the metric η is G-invariant. Namely,

$$\eta(x) = \hat{g}^T(x)\eta(gx)\hat{g}(x).$$

2. The representation $\tilde{U}$ acts independently (!) on each component of $\tilde{u}$ as the left regular representation. In fact,

$$\tilde{U}(g)\tilde{u}(h) = \tilde{u}(g^{-1}h).$$

It means that $\tilde{H}$ is a subspace of $\bigoplus_{i=1}^{d} H^i_{\mathcal{L}}$ and $\tilde{U} \simeq d \times U_{\mathcal{L}}|_{\tilde{H}}$, where $d = \dim T$; $H^i_{\mathcal{L}} \simeq \mathcal{L}^2(G, dg)$; $U_{\mathcal{L}}$ denotes the left regular representation of G, and the restriction $|_{\tilde{H}}$ corresponds to the condition (1)[1]. Hence to carry out the decomposition of $\tilde{U}$ one can treat each of its components separately choosing the Fourier components so that the condition (1) holds.

3. L can be understood as a unitary (according to the scalar product $\langle | \rangle_T$) representation of the group K acting on T. The unitarity is due to the K-invariance of η.

4. $\tilde{U}$ is the representation induced on G by the representation L.

5. We assume, without loss of generality, that L is an irreducible representation of the group K. If $L = \bigoplus_\sigma L_\sigma$, where L_σ denotes an irreducible representation, we have $\tilde{U} = \bigoplus_\sigma \tilde{U}_\sigma$, $\tilde{U}_\sigma$ being the representation induced on G by L_σ. Then in what follows, we focus on one component $\tilde{U}_\sigma$. In that case one should add an extra summation over σ.

[1] We shall use the same letter for equivalent representations. It makes the notation simpler and emphasizes their equivalence. On the other hand, for isomorphic carrier spaces we shall use different symbols, which helps to differentiate spanning bases.

3. Analysis on the transformation group

From the generalized Peter–Weyl theorem [1] one knows that, for $\tilde{u}^i \in H^i_{\mathcal{L}}$,

$$U_{\mathcal{L}} \simeq \oplus\int_{\check{G}} d\mu(\lambda) \bigoplus_q U_q(\lambda),$$

$$H^i_{\mathcal{L}} = \oplus\int_{\check{G}} d\mu(\lambda) \bigoplus_q H^i_q(\lambda),$$

$$\tilde{u}^i(g) = \int_{\check{G}} d\mu(\lambda) \sum_{pq} e_{pq}(\lambda, g)\check{\tilde{u}}^i_{pq}(\lambda).$$

Let us clarify the notation. $\check{G}$ is the dual (the space of all irreducible representations) of G. $d\mu$ is the Plancherel measure on $\check{G}$. λ, p, q are multiindices which enumerate the eigenvalues of two-sided-, right-, and left-invariant, respectively, operators on $\mathcal{L}^2(G, dg)$. The exact assumptions about these operators are given, *e.g.*, in Ref. [1]. For the sake of simplicity we assume that the left- and right-invariant operators have purely discrete spectra. $H^i_q(\lambda)(\simeq H^{i'}_{q'}(\lambda))$ denotes a G-irreducible subspace of $H^i_{\mathcal{L}}$ and forms the carrier space of the representation $U_q(\lambda)(\simeq U_{q'}(\lambda)) \overset{\text{df}}{\simeq} U(\lambda)$, which is a G-irreducible subrepresentation of $U_{\mathcal{L}}$. $e_{pq}(\lambda, \cdot)$ is a plane wave on G and simultaneously a matrix element of the representation $U_{\mathcal{L}}$ (as well as of the right regular representation). It means that for a certain basis $\{e^i_{pq}(\lambda)\}_p$ in the space $H^i_q(\lambda)$ this function can be represented as follows

$$\langle e^i_{pq}(\lambda)|U_{\mathcal{L}}(g)e^i_{rq}(\lambda)\rangle_{H^i_q(\lambda)} = e_{pr}(\lambda, g).$$

The right hand side of the above formula depends neither on i nor on q. One can say that the bases in the spaces $H^i_q(\lambda)$ for various i and q are "adjusted" to one another or, formally speaking, there exists an intertwining operator from $H^i_q(\lambda)$ into $H^{i'}_{q'}(\lambda)$ which transforms the basic elements $H^i_q(\lambda) \ni e^i_{pq}(\lambda) \to e^{i'}_{pq'} \in H^{i'}_{q'}(\lambda)$. Finally, $\check{\tilde{u}}^i_{pq}(\lambda)$ are components constituting the Fourier transform of $\tilde{u}^i$. They are transformed (by an element $g \in G$) according to the transformation law

$$\tilde{U}(g)\check{\tilde{u}}^i_{pq}(\lambda) \overset{\text{df}}{=} (\tilde{U}(g)\tilde{u})\check{}^i_{pq}(\lambda) = \sum_r \overline{e_{pr}(\lambda, g)}\check{\tilde{u}}^i_{rq}(\lambda).$$

Let us split every G-irreducible space $H^i_q(\lambda)$ onto K-irreducible components. We have

$$U(\lambda)|_K \simeq \bigoplus_{l\in\check{K}} n(l,\lambda)\times U_l,$$

$$H_q^i(\lambda) = \bigoplus_{l\in\check{K}} \bigoplus_{\alpha=1}^{n(l,\lambda)} H^i_{(l\alpha)q}(\lambda),$$

where $\check{K}$ is the dual of the group K; U_l is a K-irreducible subrepresentation of $U(\lambda)|_K$; $H^i_{(l\alpha)q}(\lambda)$, being a K-irreducible subspace of the space $H_q^i(\lambda)$, is the carrier space of U_l; $l\in\check{K}$ classifies (differentiates) non–equivalent irreducible representations of K. $n(l,\lambda)\geq 0$ is the multiplicity of U_l in $U(\lambda)|_K$; and the index α enumerates isomorphic (in the sence of the action of $U(\lambda)|_K$) subspaces $H^i_{(l\alpha)q}(\lambda)$ of the space $H_q^i(\lambda)$.

We choose the base $\{e^i_{pq}(\lambda)\}_p$ in $H_q^i(\lambda)$ demanding from each vector $e^i_{pq}(\lambda)$ to be completely included in one of the spaces $H^i_{(l\alpha)q}(\lambda)$. Then the index p (as well as q) can be represented in the form[2] $p\equiv(l\alpha m)$, where $m=1,\ldots,\dim H^i_{(l\alpha)q}(\lambda)<\infty$ enumerates the orthogonal directions in $H^i_{(l\alpha)q}(\lambda)$. Moreover, from the viewpoint of the representation $U_{\mathcal{L}}|_K$ the spaces $H^i_{(l\alpha)q}(\lambda)$ with different α or λ (not only i or q as it is for the whole representation $U_{\mathcal{L}}$) but the same l are undistinguishable. Thus we can "adjust" the bases in every space $H^i_{(l\alpha)q}(\lambda)$ (for different α or λ) so as to obtain

$$\begin{aligned}\forall k\in K: \langle e^i_{(l\alpha m)q}(\lambda)|U_{\mathcal{L}}(k)e^i_{(l'\alpha' m')q}(\lambda)\rangle_{H_q^i(\lambda)} &\equiv e_{(l\alpha m)(l'\alpha' m')}(\lambda,k)\\ &= \delta_{ll'}\delta_{\alpha\alpha'}t^l_{mm'}(k),\end{aligned}$$

where $t^l_{mm'}(k)$ denotes a matrix element of the l-representation of the group K. We insert this formula into the decomposition of $\tilde{u}^i$ using the Makey's decomposition of an element $g=sk\in G$, where $s\in S$ is uniquely determined by g for a certain Borel set $S\subset G$. We obtain

$$\tilde{u}^i(g) = \int_{\check{G}} d\mu(\lambda) \sum_{pl\alpha mn} e_{p(l\alpha m)}(\lambda,s)t^l_{mn}(k)\check{\tilde{u}}^i_{p(l\alpha n)}(\lambda).$$

Let us choose an orthonormal basis $\{e_i\}_i$ in the space T upon which we impose the following condition

$$\langle e_i|L(k)e_j\rangle_T = \overline{t^{l_0}_{ij}(k)},$$

[2] The features of the indices p, q are determined by the set of left- and right-invariant operators in the enveloping algebra. Such a representation is possible if we include K-invariant operators into the set of left- and right-G-invariant operators, which is natural.

where $l_0 \in \check{K}$ is the identifier of the representation L. Such a property associates the basis $\{e_j\}_j$ in T with the basis $\{e^i_{(l_0\alpha m)q}(\lambda)\}_m$ in $H^i_{(l_0\alpha)q}(\lambda)$ (for any i, q, α, or λ) making them conjugate (up to a unitary isomorphism of Hilbert spaces) to each other. Now, we introduce new Fourier components basing on the vectors e_i

$$\check{\check{u}}^i_{pq}(\lambda) = \check{u}^j_{pq}(\lambda)e^i_j.$$

Let us make a few notes.

1. Only when L is irreducible, as in our case, is the condition on $\{e_i\}_i$ consistent.

2. During a transformation of the coordinate system on X the basis $\{e_i\}_i$ is transformed by means of an orthogonal matrix. Indeed, since the matrix element $e_{pq}(\lambda, g)$ does not depend on coordinates on X neither does the element $t^{l_0}_{ji}(k)$. Whereas the matrix of the representation L does depend on coordinates and in new ones $x' = x'(x)$ it is expressed by

$$L'(k) = \eta'^{1/2}(x'_0)\hat{k}'(x'_0)\eta'^{-1/2}(x'_0),$$

where $k' = x' \circ k \circ x$, which yields that

$$\hat{k}'(x'_0) = \hat{x}'(x_0)\hat{k}(x_0)\hat{x}(x'_0)$$

and, by definition, $[\hat{x}'(x)]^i{}_j = \frac{\partial x'^i}{\partial x^j}(x)$, $x'_0 = x'(x_0)$. On the other hand, the transformation law of η

$$\hat{x}'^T(x_0)\eta'(x'_0)\hat{x}'(x_0) = \eta(x_0)$$

implies

$$\eta'^{1/2}(x'_0)\hat{x}'(x_0)\eta^{-1/2}(x_0) = O,$$

where O is an orthogonal matrix. Consequently,

$$L'(k) = OL(k)O^{-1}, e'_i = Oe_i$$

and $\langle e'_i|L'(k)e'_j\rangle_T = \langle e_i|L(k)e_j\rangle_T$.

3. A "rotation" of the Fourier transform in the space T does not influence the form of the transformation law (by an element $g \in G$) of the Fourier components. One readily verifies that the transformation law for the new components remains unchanged:

$$\tilde{U}(g)\check{\check{u}}^i_{pq}(\lambda) = \sum_r \overline{e_{pr}(\lambda, g)}\check{\check{u}}^i_{rq}(\lambda).$$

Therefore, the new components are "good" Fourier components meaning they are transformed (as the previous ones) by irreducible representations of G.

The essential condition (1) can be rewritten now as follows

$$\forall k \in K : \check{u}^i_{p(l\alpha m)}(\lambda) = \sum_{jn} t^l_{mn}(k)\overline{t^{l_0}_{ij}(k)}\check{u}^j_{p(l\alpha n)}(\lambda).$$

It can be fulfilled if and only if

$$\check{u}^i_{p(l\alpha m)}(\lambda) = d^{-1/2}\delta_{ll_0}\delta_{im}\check{u}_{p\alpha}(\lambda)$$

for a certain (normalized) $\check{u}_{p\alpha}(\lambda)$. Thus

$$\tilde{u}(g) = d^{-1/2}\int_{\check{X}} d\mu(\lambda) \sum_{p\alpha im} e_{p(l_0\alpha m)}(\lambda, s)t^{l_0}_{mi}(k)\check{u}_{p\alpha}(\lambda)e_i,$$

where

$$\check{X} = \left\{\lambda \in \check{G} : n(\lambda) \overset{\text{df}}{=} n(l_0, \lambda) \geq 1\right\}.$$

One should note what follows.

1. The above formula is an interesting connection (through δ_{im}) between the geometrical structure (the index i) and the algebraic structure (the index m).

2. Only if $l = l_0$ does the formula make sense. It is because, exlusively in this case, the indices i and m enumerate directions in isomorphic spaces.

3. The components $\check{u}_{p\alpha}(\lambda)$ reveal the interpretation of the condition (1). This condition zeroes the components lying in all G-irreducible subspaces $H^i_q(\lambda) \equiv H^i_{(l\alpha m)}(\lambda)$ for which $l \neq l_0$ or $i \neq m$. What is more, it demands of components from subspaces with various i $(= m)$ to have the same values.

4. The components $\check{u}_{p\alpha}(\lambda)$ are independent (in the sense of the condition (1)).

5. Due to the isomorphism $H \to \tilde{H}$, the transformation law of $\check{u}_{p\alpha}(\lambda)$ is universal both for a transformation of $\tilde{u}$ and of u. Moreover, the representation $\tilde{U}$ mixes neither components with different indices i nor those with different q. Hence during a transformation of $\tilde{u}$ by an element $g \in G$ the independent components are transformed according to the already known law:

$$U(g)\check{u}_{p\alpha}(\lambda) = \tilde{U}(g)\check{u}_{p\alpha}(\lambda) = \sum_r \overline{e_{pr}(\lambda, g)}\check{u}_{r\alpha}(\lambda)$$

and in this sense they constitute "good" Fourier components.

6. The decomposition of the representation $\tilde{U}$ is visible from the decomposition of $\tilde{u}$. We have

$$\tilde{U} \simeq \oplus\int_{\check{G}} d\mu(\lambda)\, n(\lambda) \times U(\lambda).$$

7. The above decomposition of $\tilde{U}$ is in perfect agreement with the result straightly obtainable from Mackey's generalization of the Frobenius Reciprocity Theorem [6].

4. Analysis on homogeneous space

Let us inverse the isomorphism of functions on X and G and utilize it in the decomposition of $\tilde{u}$. The dependence on k, as one could expect, vanishes after taking into account the action of the matrix $L(k)$ on the vectors e_i. In this way we obtain

$$u(x) = d^{-1/2} \int_{\check{X}} d\mu(\lambda) \sum_{p\alpha m} e_{p(l_0 \alpha m)}(\lambda, s)\hat{s}(x_0)\eta^{-1/2}(x_0)e_m \check{u}_{p\alpha}(\lambda),$$

where $x = sx_0$.

We define a tensor plane wave on X:

$$e_{p\alpha}(\lambda, x) = d^{-1/2} \sum_m e_{p(l_0 \alpha m)}(\lambda, s)\hat{s}(x_0)\eta^{-1/2}(x_0)e_m.$$

Then we have

$$u(x) = \int_{\check{X}} d\mu(\lambda) \sum_{p\alpha} e_{p\alpha}(\lambda, x)\check{u}_{p\alpha}(\lambda).$$

In many applications the group of matrices $\hat{k}(x_0)$, $k \in K$, consists of the totality of matrices preserving the form $\eta(x_0)$. In that case the general solution of the G-invariance equation in the point x_0 is the matrix

$$\hat{g}(x_0) = \eta^{-1/2}(gx_0)\eta^{1/2}(x_0)\hat{k}(x_0)$$

for a certain $k \in K$. Then one can choose the set $S \subset G$ so that for a given $g = sk \in G$

$$\hat{s}(x_0) = \eta^{-1/2}(sx_0)\eta^{1/2}(x_0)$$

and a plane wave takes an elegant form:

$$e_{p\alpha}(\lambda, x) = d^{-1/2} \sum_m e_{p(l_0 \alpha m)}(\lambda, s)\eta^{-1/2}(x)e_m.$$

We have achieved the main goal of this paper – the spectral synthesis formula. Now, the Parseval equality is easy to obtain from the analogous formula for the group G called the Plancherel equality [1], which in our notation has the following form

$$\int_G dg\, \tilde{u}^\dagger(g)\tilde{v}(g) = \int_{\check{G}} d\mu(\lambda) \sum_{pqi} \overline{\check{\tilde{u}}^i_{pq}(\lambda)}\check{\tilde{v}}^i_{pq}(\lambda).$$

Let us replace the functions on G and their Fourier transforms for the appropriate quantities on X. Integrating over K (as the dependence on k vanishes) leads directly to the Parseval equality:

$$\int_X dx\, u^\dagger(x)\eta(x)v(x) = \int_{\check{X}} d\mu(\lambda) \sum_{p\alpha} \overline{\check{u}_{p\alpha}(\lambda)}\check{v}_{p\alpha}(\lambda).$$

The Plancherel equality is equivalent to the orthogonality relation of plane waves if the Plancherel measure $d\mu$ is an absolutely continuous function of the Lebesgue measure $d\lambda$ on the set $\check{G}$ [1]. Namely, $d\mu(\lambda) = \mu(\lambda)d\lambda$ and one has

$$\int_G dg\, \overline{e_{pq}(\lambda, g)}e_{p'q'}(\lambda', g) = \mu^{-1}(\lambda)\delta(\lambda - \lambda')\delta_{pp'}\delta_{qq'}.$$

One can prove easily, by virtue of the G-invariance of η, that the above relation implies the orthogonality relation for tensor plane waves:

$$\int_X dx\, e^\dagger_{p\alpha}(\lambda, x)\eta(x)e_{p'\alpha'}(\lambda', x) = \mu^{-1}(\lambda)\delta(\lambda - \lambda')\delta_{pp'}\delta_{\alpha\alpha'}.$$

If the set of invariant operators contains an elliptic operator, which is typical of many applications, one can invert the spectral synthesis formula [1]. For the group G one has

$$\check{\tilde{u}}^i_{pq}(\lambda) = \int_G dg\, \overline{e_{pq}(\lambda, g)}\tilde{u}^i(g).$$

After a few obvious steps we get

$$\check{u}_{p\alpha}(\lambda) = \int_X dx \sum_m \overline{e_{p(l_0\alpha m)}(\lambda, s)}e^\dagger_m \eta^{1/2}(x_0)\widehat{s^{-1}}(x)u(x),$$

where $x = sx_0$. Finally, utilizing the G-invariance equation for η one finds

$$\check{u}_{p\alpha}(\lambda) = \int_X dx\, e^\dagger_{p\alpha}(\lambda, x)\eta(x)u(x).$$

The possibility of inversion allows to formulate the completeness relation. An elementary substitution of the spectral synthesis formula into the Fourier transform implies

$$\int_{\check{X}} d\mu(\lambda) \sum_{p\alpha} \epsilon_{p\alpha}(\lambda, x)\epsilon^{\dagger}_{p\alpha}(\lambda, x') = \delta(x - x')\eta^{-1}(x),$$

where δ is the Dirac distribution according to the measure dx.

5. Summary

We have constructed a set of functions on a homogeneous space called tensor wave planes. They are transformed characteristicaly both for tensor objects and for basic elements of irreducible representations of the transformation group. The algebraic and geometrical structures are closely related to each other. Despite the crucial restriction which is the requirement of compactness of the stability group. the obtained result is remarkable for its generality. The compactness is an essential assumption in our approach and the author can not see any easy modification of the construction which would allow to abandon it.

Our method omits the necessity of finding the invariant operators on $\mathcal{L}^2(X, dx)$ which come from the outside of the center of the enveloping algebra. This problem has to be dealt with in the standard approach to the (scalar) harmonic analysis on homogeneous spaces [1]. Such additional operators may appear when the stability group K is "small" and they introduce their own summation indices to the spectral synthesis formula. In our case the role of those indices is played by the multiindex α.

To sum up. we recall the most important formulae. The spectral synthesis formula

$$u(x) = \int_{\check{X}} d\mu(\lambda) \sum_{p\alpha} \epsilon_{p\alpha}(\lambda, x)\check{u}_{p\alpha}(\lambda).$$

The Fourier transform

$$\check{u}_{p\alpha}(\lambda) = \int_{X} dx\, e^{\dagger}_{p\alpha}(\lambda, x)\eta(x)u(x).$$

The Parseval equality

$$\int_{X} dx\, u^{\dagger}(x)\eta(x)v(x) = \int_{\check{X}} d\mu(\lambda) \sum_{p\alpha} \overline{\check{u}_{p\alpha}(\lambda)}\check{v}_{p\alpha}(\lambda).$$

The orthogonality relation

$$\int\limits_X dx\, e^{\dagger}_{p\alpha}(\lambda, x)\eta(x)e_{p'\alpha'}(\lambda', x) = \mu^{-1}(\lambda)\delta(\lambda - \lambda')\delta_{pp'}\delta_{\alpha\alpha'},$$

and the completness relation

$$\int\limits_{\check{X}} d\mu(\lambda) \sum_{p\alpha} e_{p\alpha}(\lambda, x)e^{\dagger}_{p\alpha}(\lambda, x') = \delta(x - x')\eta^{-1}(x).$$

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