THE ISOMORPHISM HOPF *-ALGEBRAS BETWEEN κ-POINCARÉ ALGEBRA IN CASE $g_{00} = 0$ AND "NULL PLANE" QUANTUM POINCARÉ ALGEBRA

K. Przanowski*

Department of Field Theory, University of Łódź
Pomorska 149/153, 90-236 Łódź , Poland

(Received November 11, 1996)

The isomorphism Hopf *-algebras between κ-Poincaré algebra in case $g_{00} = 0$ defined by P. Kosiński and P. Maślanka in *The κ-Weyl Group and Its Algebra* in "From Field Theory to Quantum Groups" volume on 60^{th} anniversary of J. Lukierski, World Scientific, Singapore 1996 and "null plane" quantum Poincaré algebra by A. Ballesteros, F.J. Herranz and M.A. del Olmo *"Null Plane" Quantum Poincaré Algebra, Phys. Lett.* **B351**, 137 (1995) are defined.

PACS numbers: 11.30. Cp, 03.65. Fd

1. Introduction

Recently, considerable interest has been paid to the deformations of group and algebras of space-time symmetries [12]. An interesting deformation of the Poincarè algebra [5, 2] as well as group [7] has been introduced which depend on the dimensional deformation parameter κ; the relevant objects are called κ–Poincarè algebra and κ–Poincarè group, respectively. Their structure was studied in some detail and many of their properties are now well understood.

In the Section 2 I find κ–Poincaré algebra in the new basis. In the Section 3 I describe the "null plane" quantum Poincaré algebra and define isomorphism between these two algebras. At least in the Section 4 I define isomorphism between κ–Poincaré algebra defined below and the "null plane" quantum Poincaré algebra.

Let us remind the definition of κ–Poincarè algebra. The κ–Poincarè algebra $\tilde{\mathcal{P}}_\kappa$ [2] (in the Majid and Ruegg basis [3]) is a quantized universal

* Supported by Łódź University grant no 487.

enveloping algebra in the sense of Drinfeld [4] described by the following relations:
The commutation rules:

$$\begin{aligned}
[M^{ij}, P_0] &= 0,\\
[M^{ij}, P_k] &= i\kappa(\delta^j{}_k g^{0i} - \delta^i{}_k g^{0j})\left(1 - \exp\left(-\frac{P_0}{\kappa}\right)\right) + i(\delta^j{}_k g^{is} - \delta^i{}_k g^{js})P_s,\\
[M^{i0}, P_0] &= i\kappa g^{i0}\left(1 - \exp\left(-\frac{p_0}{\kappa}\right)\right) + ig^{ik}P_k,\\
[M^{i0}, P_k] &= -i\frac{\kappa}{2}g^{00}\delta^i{}_k\left(1 - \exp\left(-2\frac{P_0}{\kappa}\right)\right) - i\delta^i{}_k g^{0s}P_s\exp\left(-\frac{P_0}{\kappa}\right)\\
&\quad + ig^{0i}P_k\left(\exp\left(-\frac{P_0}{\kappa}\right) - 1\right) + \frac{i}{2\kappa}\delta^i{}_k g^{rs}P_rP_s - \frac{i}{\kappa}g^{is}P_sP_k,\\
[P_\mu, P_\nu] &= 0,\\
[M^{\mu\nu}, M^{\lambda\sigma}] &= i(g^{\mu\sigma}M^{\nu\lambda} - g^{\nu\sigma}M^{\mu\lambda} + g^{\nu\lambda}M^{\mu\sigma} - g^{\mu\lambda}M^{\nu\sigma}).
\end{aligned} \tag{1.1}$$

The coproducts, counit and antipode:

$$\begin{aligned}
\Delta P_0 &= I \otimes P_0 + P_0 \otimes I,\\
\Delta P_k &= P_k \otimes \exp\left(-\frac{P_0}{\kappa}\right) + I \otimes P_k,\\
\Delta M^{ij} &= M^{ij} \otimes I + I \otimes M^{ij},\\
\Delta M^{i0} &= I \otimes M^{i0} + M^{i0} \otimes \exp\left(-\frac{P_0}{\kappa}\right) - \frac{1}{\kappa}M^{ij} \otimes P_j,\\
\varepsilon(M^{\mu\nu}) &= 0; \quad \varepsilon(P_\nu) = 0,\\
S(P_0) &= -P_0,\\
S(P_i) &= -\exp\left(\frac{P_0}{\kappa}\right)P_i,\\
S(M^{ij}) &= -M^{ij},\\
S(M^{i0}) &= -\left(M^{i0} + \frac{1}{\kappa}M^{ij}P_j\right)\exp\left(\frac{P_0}{\kappa}\right),
\end{aligned} \tag{1.2}$$

where $i, j, k = 1, 2, 3$ and the metric tensor $g_{\mu\nu}$, $(\mu\nu = 0, 1, \ldots, 3)$ is represented by an arbitrary nondegenerate symmetric 4×4 matrix (not necessary diagonal).

2. The κ–Poincaré algebra in the new basis

Note that in our paper we mark

$$A^iB^i = \sum_{n=1}^{3} A^nB^n, \; A^iB_i = \sum_{n=1}^{3} A^nB_n$$

for any tensors A^{μ}, B^{ν} and $\varepsilon^{123}=-1$.

We put:

$$\begin{aligned} M^i &= \tfrac{1}{2}\varepsilon^{ijk}M^{jk}, \text{ (we have also } M^{ij}=\varepsilon^{ijk}M^k),\\ N^i &= M^{i0}. \end{aligned}$$

We define the isomorphism on generators of $\kappa-$Poincaré algebra [3]:

$$\begin{aligned} \mathcal{P}_0 &= -P_0,\\ \mathcal{P}_i &= -P_i\exp\left(\frac{P_0}{2\kappa}\right),\\ \mathcal{M}^i &= M^i,\\ \mathcal{N}^i &= \left(N^i-\frac{1}{2\kappa}\varepsilon^{ijk}M^jP_k\right)\exp\left(\frac{P_0}{2\kappa}\right)\\ &= N^i\exp\left(\frac{P_0}{2\kappa}\right)+\tfrac{1}{2\kappa}\varepsilon^{ijk}\mathcal{M}^j\mathcal{P}_k\,. \end{aligned}$$

After some calculus we get the following relations $\kappa-$Poincaré algebra in the new basis:

The commutation rules:

$$\begin{aligned} [\mathcal{P}_\mu,\mathcal{P}_\nu] &= 0,\\ [\mathcal{M}^i,\mathcal{P}_0] &= 0,\\ [\mathcal{M}^i,\mathcal{P}_k] &= i\varepsilon^{ijl}\delta^l_k\left(2\kappa g^{oj}\sinh\left(\frac{\mathcal{P}_0}{2\kappa}\right)+g^{js}\mathcal{P}_s\right),\\ [\mathcal{N}^i,\mathcal{P}_0] &= 2i\kappa g^{i0}\sinh\left(\frac{\mathcal{P}_0}{2\kappa}\right)+ig^{ik}\mathcal{P}_k,\\ [\mathcal{N}^i,\mathcal{P}_k] &= -i\kappa g^{00}\delta^i_k\sinh(\frac{\mathcal{P}_0}{\kappa})-i\delta^i_kg^{0s}\mathcal{P}_s\cosh\left(\frac{\mathcal{P}_0}{2\kappa}\right),\\ [\mathcal{M}^i,\mathcal{M}^j] &= -i\varepsilon^{ijk}g^{ks}\mathcal{M}^s,\\ [\mathcal{N}^i,\mathcal{M}^j] &= i\varepsilon^{jrs}g^{ir}\mathcal{N}^s+ig^{i0}\mathcal{M}^j\cosh\left(\frac{\mathcal{P}_0}{2\kappa}\right)-i\delta^{ij}g^{k0}\mathcal{M}^k\cosh\left(\frac{\mathcal{P}_0}{2\kappa}\right),\\ [\mathcal{N}^i,\mathcal{N}^j] &= ig^{j0}\mathcal{N}^i\cosh\left(\frac{\mathcal{P}_0}{2\kappa}\right)-ig^{i0}\mathcal{N}^j\cosh\left(\frac{\mathcal{P}_0}{2\kappa}\right)-\frac{i}{4\kappa^2}\varepsilon^{ijs}g^{kr}\mathcal{M}^r\mathcal{P}_s\mathcal{P}_k\\ &\quad+\frac{i}{2\kappa}\varepsilon^{jrs}g^{i0}\mathcal{M}^r\mathcal{P}_s\sinh\left(\frac{\mathcal{P}_0}{2\kappa}\right)-\frac{i}{2\kappa}\varepsilon^{irs}g^{j0}\mathcal{M}^r\mathcal{P}_s\sinh\left(\frac{\mathcal{P}_0}{2\kappa}\right)\\ &\quad-i\varepsilon^{ijk}g^{00}\mathcal{M}^k\cosh(\frac{\mathcal{P}_0}{\kappa})-\frac{i}{\kappa}\varepsilon^{ijk}\mathcal{M}^kg^{s0}\mathcal{P}_s\sinh\left(\frac{\mathcal{P}_0}{2\kappa}\right). \end{aligned} \tag{2.1}$$

The coproducts, counit and antipode:

$$\Delta\mathcal{P}_0 = \mathcal{P}_0\otimes I+I\otimes\mathcal{P}_0,$$

$$
\begin{aligned}
\Delta\mathcal{P}_i &= \mathcal{P}_i \otimes \exp\left(\frac{\mathcal{P}_0}{2\kappa}\right) + \exp\left(-\frac{\mathcal{P}_0}{2\kappa}\right) \otimes \mathcal{P}_i, \\
\Delta\mathcal{M}^i &= \mathcal{M}^i \otimes I + I \otimes \mathcal{M}^i, \\
\Delta\mathcal{N}^i &= \mathcal{N}^i \otimes \exp\left(\frac{\mathcal{P}_0}{2\kappa}\right) + \exp\left(-\frac{\mathcal{P}_0}{2\kappa}\right) \otimes \mathcal{N}^i \\
&- \frac{1}{2\kappa}\varepsilon^{ijk}\left(\exp\left(-\frac{\mathcal{P}_0}{2\kappa}\right)\mathcal{M}^j \otimes \mathcal{P}_k - \mathcal{P}_k \otimes \mathcal{M}^j \exp\left(\frac{\mathcal{P}_0}{2\kappa}\right)\right), \\
\varepsilon(\mathcal{X}) &= 0, \text{ for } \mathcal{X} = \mathcal{N}^i,\ \mathcal{M}^i,\ \mathcal{P}_\mu, \\
S(\mathcal{X}) &= -\mathcal{X}, \text{ for } \mathcal{X} = \mathcal{M}^i,\ \mathcal{P}_\mu, \\
S(\mathcal{N}^i) &= -\mathcal{N}^i + 3i\left(g^{i0}\sinh\left(\frac{\mathcal{P}_0}{2\kappa}\right) + \frac{1}{2\kappa}g^{ik}\mathcal{P}_k\right), \\
(\text{or } S(\mathcal{X}) &= -\exp\left(\frac{3\mathcal{P}_0}{2\kappa}\right)\mathcal{X}\exp\left(-\frac{3\mathcal{P}_0}{2\kappa}\right), \text{ for } \mathcal{X} = \mathcal{M}^i,\ \mathcal{N}^i,\ \mathcal{P}_\mu). \quad (2.2)
\end{aligned}
$$

Note, if we take diagonal metric tensor $g_{\mu\nu} = \mathrm{Diag}(1,-1,-1,-1)$, we obtain κ–Poincaré algebra considered in [3, 5, 6].

3. The "null plane" quantum Poincaré algebra and isomorphism

The "null plane" quantum Poincaré algebra is a Hopf *-algebra generated by ten elements: P_+, P_-, P_1, P_2, E_1, E_2, F_1, F_2, J_3, K_3 and z-deformation parameter with the following relations:
The commutation rules:

$$
\begin{aligned}
[K_3, P_+] &= \frac{\sinh(zP_+)}{z}, \\
[K_3, P_-] &= -P_- \cosh(zP_+), \\
[K_3, E_i] &= E_i \cosh(zP_+), \\
[K_3, F_1] &= -F_1 \cosh(zP_+) + zE_1P_- \sinh(zP_+) - z^2 P_2 W_+^q, \\
[K_3, F_2] &= -F_2 \cosh(zP_+) + zE_2P_- \sinh(zP_+) - z^2 P_1 W_+^q, \\
[J_3, P_i] &= -\varepsilon_{ij3}P_j, \\
[J_3, E_i] &= -\varepsilon_{ij3}E_j, \\
[J_3, F_i] &= -\varepsilon_{ij3}F_j, \\
[E_i, P_j] &= \delta_{ij}\frac{\sinh(zP_+)}{z}, \\
[F_i, P_j] &= \delta_{ij}P_- \cosh(zP_+), \\
[E_i, F_j] &= \delta_{ij}K_3 + \varepsilon_{ij3}\cosh(zP_+), \\
[P_+, F_i] &= -P_i, \\
[F_1, F_2] &= z^2 P_- W_+^q + zP_- J_3 \sinh(zP_+),
\end{aligned}
$$

$$[P_-, E_i] = -P_i.$$

The coproducts, counit and antipode:

$$\begin{aligned}
\Delta X &= I \otimes X + X \otimes I, \text{ for } X = P_+,\ E_i,\ J_3, \\
\Delta Y &= \mathrm{e}^{-zP_+} \otimes Y + Y \otimes \mathrm{e}^{zP_+}, \text{ for } Y = P_-,\ P_i, \\
\Delta F_1 &= \mathrm{e}^{-zP_+} \otimes F_1 + F_1 \otimes \mathrm{e}^{zP_+} + z\,\mathrm{e}^{-zP_+} E_1 \otimes P_- \\
&\quad - zP_- \otimes E_1\,\mathrm{e}^{zP_+} + z\,\mathrm{e}^{-zP_+} J_3 \otimes P_2 - zP_2 \otimes J_3\,\mathrm{e}^{zP_+}, \\
\Delta F_2 &= \mathrm{e}^{-zP_+} \otimes F_2 + F_2 \otimes \mathrm{e}^{zP_+} + z\,\mathrm{e}^{-zP_+} E_2 \otimes P_- \\
&\quad - zP_- \otimes E_2\,\mathrm{e}^{zP_+} - z\,\mathrm{e}^{-zP_+} J_3 \otimes P_1 + zP_1 \otimes J_3\,\mathrm{e}^{zP_+}, \\
\Delta K_3 &= \mathrm{e}^{-zP_+} \otimes K_3 + K_3 \otimes \mathrm{e}^{zP_+} + z\,\mathrm{e}^{-zP_+} E_1 \otimes P_1 \\
&\quad - zP_1 \otimes E_1\,\mathrm{e}^{zP_+} + z\,\mathrm{e}^{-zP_+} E_2 \otimes P_2 - zP_2 \otimes E_2\,\mathrm{e}^{zP_+}, \\
\varepsilon(X) &= 0,\ S(X) = -\,\mathrm{e}^{3zP_+} X\,\mathrm{e}^{-3zP_+}, \text{ for } X = P_\pm,\ P_i,\ F_i,\ E_i,\ J_3,\ K_3,
\end{aligned}$$

where $W_+^q = E_1 P_2 - E_2 P_1 + J_3 \frac{\sinh(zP_+)}{z}$ and $i, j = 1, 2$.

If we take metric tensor:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.1}$$

in $\kappa-$Poincaré algebra (2.1), (2.2) and we put:

$$\begin{aligned}
P_+ &= \mathcal{P}_0, \\
P_- &= \mathcal{P}_3, \\
P_i &= \mathcal{P}_i, \\
K_3 &= -i\mathcal{N}^3, \\
F_i &= i\mathcal{N}^i, \\
E_1 &= i\mathcal{M}^2, \\
E_2 &= -i\mathcal{M}^1, \\
J_3 &= -i\mathcal{M}^3, \\
z &= \frac{1}{2\kappa},
\end{aligned}$$

from $\kappa-$Poincaré algebra relations (2.1), (2.2), we get that "null plane" quantum Poincaré algebra relations (3.1), (3.1) holds.

4. Summary

We define isomorphism Hopf *-algebras from κ−Poincaré algebra (1.1), (1.2) for metric tensor (3.1) to the "null plane" quantum Poincaré algebra by putting:

$$
\begin{aligned}
P_+ &= -P_0, \\
P_- &= -P_3 \exp\left(\frac{P_0}{2\kappa}\right), \\
P_i &= -P_i \exp\left(\frac{P_0}{2\kappa}\right), \\
K_3 &= -i\left(M^{30} + \frac{1}{2\kappa} M^{3k} P_k\right) e^{\frac{P_0}{2\kappa}}, \\
F_i &= i\left(M^{i0} + \frac{1}{2\kappa} M^{ik} P_k\right) \exp\left(\frac{P_0}{2\kappa}\right), \\
E_1 &= \frac{i}{2} \varepsilon^{2jk} M^{jk}, \\
E_2 &= -\frac{i}{2} \varepsilon^{1jk} M^{jk}, \\
J_3 &= -\frac{i}{2} \varepsilon^{3jk} M^{jk}, \\
z &= \frac{1}{2\kappa},
\end{aligned}
$$

for $i = 1, 2;\ j, k = 1, 2, 3$.

We conclude that the "null plane" quantum Poincaré algebra is a case of the κ−Poincaré algebra in $g_{00} = 0$.

The numerous discussions with Professor J. Lukierski and dr P. Maślanka are kindly acknowledged.

REFERENCES

[1] A. Ballesteros, F.J. Herranz, M.A. del Olmo, *Phys. Lett.* **B 351**, 137 (1995).
[2] P. Kosiński, P. Maślanka, in *From Field Theory to Quantum Groups* vol. on 60 anniversary of J. Lukierski, World Scientific, Singapore 1996 or **Q-ALG**9512018.
[3] S. Majid, H. Ruegg, *Phys. Lett.* **B 334**, 348 (1994).
[4] V.G. Drinfeld, *Proc. Int. Congr. Math.* Berkeley 1986, p.798.
[5] J. Lukierski, A. Nowicki, H. Ruegg, V. Tolstoy, *Phys. Lett.* **B264**, 331 (1991); J. Lukierski, A. Nowicki, H. Ruegg, *Phys. Lett.* **B293**, 344 (1993); S. Giller, P. Kosiński, J. Kunz, M. Majewski, P. Maślanka, *Phys. Lett.* **B 286**, 57 (1992); Ph. Zaugg, preprint MIT–CTP–2353(1994).

[6] P. Maślanka, Deformation map and hermitean representations of κ–Poincaré algebra - preprint 1/93, University of Łódź.

[7] S. Zakrzewski, *J. Phys.* **A27**, (1994).

[8] S. Giller, C. Gonera, P. Kosiński, P. Maślanka, in print *Acta. Phys. Pol.*, **B**.

[9] P. Maślanka, *J. Math. Phys.* **35**, 5047 (1994).

[10] P. Kosiński, P. Maślanka, preprint IMUŁ3/94 or **Q-ALG**9411033.

[11] P. Kosiński, J. Lukierski, P. Maślanka, A. Sitarz paper in preparation.

[12] W.B. Schmidke, J. Weiss, B. Zumino, *Z. Physik* **52**, 472 (1991); U. Carow-Watamura, M. Schliecker, M. Scholl, S. Watamura, *Int. J. Mod. Phys.* **A6**, 3081 (1991); S.L. Woronowicz, *Comm. Math. Phys.* **136**, 399 (1991); O. Ogievetsky, W.B. Schmidke, J. Weiss, B. Zumino, *Comm. Math. Phys.* **150**, 495 (1992); M. Chaichian, A.P. Demichev, Proceedings of the Workshop: Generalized Symmetries in Physics, Clausthal 1993.V. Dobrev, *J. Phys.* **A 26**, 1317 (1993); L. Castellani in *Quantum Groups Proceedings of XXX Karpacz Winter School of Theoretical Physics, Karpacz 1994*, PWN 1995, p. 13.