BURGERS VELOCITY FIELDS AND ELECTROMAGNETIC FORCING IN DIFFUSIVE (MARKOVIAN) MATTER TRANSPORT*

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We explore a connection of the unforced and deterministically forced Burgers equation for local velocity fields with probabilistic solutions (here, Markovian diffusion processes) of the so-called Schrödinger boundary data problem. An issue of deducing the most likely interpolating dynamics from the given initial and terminal probability density data is investigated to give account of the perturbation by external electromagnetic fields. A suitable modification of the Hopf–Cole logarithmic transformation extends the standard framework, both in the Burgers and Schrödinger's interpolation cases, to non-gradient drift fields and forces.

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1. The Burgers equation in Schrödinger's interpolation problem

The Schrödinger problem of deducing the detailed microscopic dynamics from the given input-output statistics data is known to admit a particular class of solutions in terms of Markov diffusion processes, [1–8]. That especially pertains to an explicit modelling of any unknown in detail physical process solely on the basis of the available statistics (conditional probabilities and averages, invariant measures, time-dependent probability densities, density boundary-data) presumed to refer to random motions with a given finite time of duration.

At this point, let us invoke a probabilistic problem, originally due to Schrödinger: given two strictly positive (usually on an open space-interval) boundary probability densities $\rho_0(x), \rho_T(x)$ for a process with the time of duration $T \geq 0$. Can we uniquely identify the stochastic process interpolating between them?

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The answer is known to be affirmative, if we assume the interpolating process to be Markovian. In particular, we can get here a unique Markovian diffusion process which is specified by the joint probability distribution

$$m_T(A,B) = \int_A d^3x \int_B d^3y\, m_T(\vec{x},\vec{y})\,, \tag{1}$$

$$\int d^3y\, m_T(\vec{x},\vec{y}) = \rho_0(\vec{x})\,,$$

$$\int d^3x\, m_T(\vec{x},\vec{y}) = \rho_T(y)\,,$$

where

$$m_T(\vec{x},\vec{y}) = u_0(\vec{x})\, k(x,0,y,T)\, v_T(\vec{y}) \tag{2}$$

and the two unknown functions $u_0(\vec{x})$, $v_T(\vec{y})$ come out as solutions of *the same sign* of the integral identities (1). Provided, we have at our disposal a continuous bounded strictly positive (ways to relax this assumption were discussed in Ref. [4]) integral kernel $k(\vec{x},s,\vec{y},t), 0 \le s < t \le T$.

We shall confine further attention to cases governed by the familiar Feynman–Kac kernels. Then, the solution of the Schrödinger boundary-data problem in terms of the interpolating Markovian diffusion process is found to rely on the adjoint pairs of parabolic equations. In case of gradient forward drift fields, the process can be determined by checking (this imposes limitations on the admissible potential) whether the Feynman–Kac kernel

$$k(\vec{y},s,\vec{x},t) = \int \exp\left[-\int_s^t c(\vec{\omega}(\tau),\tau)d\tau\right] d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\omega) \tag{3}$$

is positive and continuous in the open space-time area of interest (then, additional limitations on the path measure need to be introduced, [3]), and whether it gives rise to positive solutions of the adjoint pair of generalized heat equations:

$$\begin{aligned}\partial_t u(\vec{x},t) &= \nu\triangle u(\vec{x},t) - c(\vec{x},t)u(\vec{x},t)\,,\\ \partial_t v(\vec{x},t) &= -\nu\triangle v(\vec{x},t) + c(\vec{x},t)v(\vec{x},t)\,.\end{aligned} \tag{4}$$

Here, a function $c(\vec{x},t)$ is restricted only by the positivity and continuity demand for the kernel (3), see *e.g.* [2]. In the above, $d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\omega)$ is the conditional Wiener measure over sample paths of the standard Brownian motion.

Solutions of (4), upon suitable normalization give rise to the Markovian diffusion process with the *factorized* probability density $\rho(\vec{x},t) = u(\vec{x},t)v(\vec{x},t)$

which, while evolving in time, interpolates between the boundary density data $\rho(\vec{x},0)$ and $\rho(\vec{x},T)$. The interpolation admits a realization in terms of Markovian diffusion processes with the respective forward and backward drifts defined as follows:

$$\vec{b}(\vec{x},t) = 2\nu\frac{\nabla v(\vec{x},t)}{v(\vec{x},t)}\,,$$
$$\vec{b}_*(\vec{x},t) = -2\nu\frac{\nabla u(\vec{x},t)}{u(\vec{x},t)} \tag{5}$$

in the prescribed time interval $[0,T]$.

The related transport equations for the densities easily follow. For the forward interpolation, the familiar Fokker–Planck equation holds true:

$$\partial_t\rho(\vec{x},t) = \nu\triangle\rho(\vec{x},t) - \nabla[\vec{b}(\vec{x},t)\rho(\vec{x},t)]\,, \tag{6}$$

while for the backward interpolation we have:

$$\partial_t\rho(\vec{x},t) = -\nu\triangle\rho(\vec{x},t) - \nabla[\vec{b}_*(\vec{x},t)\rho(\vec{x},t)]\,. \tag{7}$$

We have assumed that drifts are gradient fields, $\operatorname{curl}\vec{b}=0$. As a consequence, those that are allowed by the prescribed choice of $c(\vec{x},t)$ *must* fulfill the compatibility condition

$$c(\vec{x},t) = \partial_t\Phi + \tfrac{1}{2}\left(\frac{b^2}{2\nu} + \nabla b\right) \tag{8}$$

which establishes the Girsanov-type connection of the forward drift $\vec{b}(\vec{x},t) = 2\nu\nabla\Phi(\vec{x},t)$ with the Feynman–Kac, *c.f.* [2, 3], potential $c(\vec{x},t)$. In the considered Schrödinger's interpolation framework, the forward and backward drift fields are connected by the identity $\vec{b}_* = \vec{b} - 2\nu\nabla\ln\rho$.

One of the distinctive features of Markovian diffusion processes with the positive density $\rho(\vec{x},t)$ is that, given the transition probability density of the (forward) process, the notion of the *backward* transition probability density $p_*(\vec{y},s,\vec{x},t)$ can be consistently introduced on each finite time interval, say $0\le s<t\le T$:

$$\rho(\vec{x},t)p_*(\vec{y},s,\vec{x},t) = p(\vec{y},s,\vec{x},t)\rho(\vec{y},s)\,, \tag{9}$$

so that

$$\int \rho(\vec{y},s)p(\vec{y},s,\vec{x},t)d^3y = \rho(\vec{x},t)$$

and

$$\rho(\vec{y},s) = \int p_*(\vec{y},s,\vec{x},t)\rho(\vec{x},t)d^3x\,.$$

The transport (density evolution) equations (6) and (7) refer to processes running in opposite directions in a fixed, common for both, time-duration period. The forward one, (6), executes an interpolation from the Borel set A to B, while the backward one, (7), executes an interpolation from B to A, compare *e.g.* the defining identities (1).

The knowledge of the Feynman–Kac kernel (3) implies that the transition probability density of the forward process reads:

$$p(\vec{y}, s, \vec{x}, t) = k(\vec{y}, s, \vec{x}, t)\frac{v(\vec{x}, t)}{v(\vec{y}, s)}\,. \tag{10}$$

while the corresponding (derivable from (10), since $\rho(\vec{x}, t)$ is given) transition probability density of the backward process has the form:

$$p_*(\vec{y}, s, \vec{x}, t) = k(\vec{y}, s, \vec{x}, t)\frac{u(\vec{y}, s)}{u(\vec{x}, t)}\,. \tag{11}$$

Obviously, [2, 6], in the time interval $0 \leq s < t \leq T$ there holds:

$$\begin{aligned} u(\vec{x}, t) &= \int u_0(\vec{y})k(\vec{y}, s, \vec{x}, t)d^3y\,, \\ v(\vec{y}, s) &= \int k(\vec{y}, s, \vec{x}, T)v_T(\vec{x})d^3x\,. \end{aligned} \tag{12}$$

Now, we are at the point, where a connection of the previous probabilistic formalism with an issue of the Burgers velocity-driven matter transport, [5], can be disclosed.

The prototype nonlinear field equation named the Burgers or "nonlinear diffusion" equation (typically without, [9, 10], the forcing term $\vec{F}(\vec{x}, t)$):

$$\partial_t \vec{v}_{\rm B} + (\vec{v}_{\rm B}\nabla)\vec{v}_{\rm B} = \nu\triangle\vec{v}_B + \vec{F}(\vec{x}, t) \tag{13}$$

recently has acquired a considerable popularity in the variety of physical contexts, [5].

Burgers velocity fields can be analysed on their own with different (including random) choices of the initial data and/or force fields. However, we are interested in the possible diffusive matter transport that is locally governed by Burgers flows, *c.f.* [5]. In this particular connection, let us point out a conspicuous hesitation that could have been observed in attempts to establish the most appropriate matter transport rule, if any diffusion-type microscopic dynamics assumption is adopted to underlie the "nonlinear diffusion" (13).

Depending on the particular phenomenological departure point, one either adopts the standard continuity equation, [11, 12], that is certainly valid to a high degree of accuracy in the so-called low viscosity limit $\nu \downarrow 0$, but incorrect on mathematical grounds *if* there is a genuine Markovian diffusion process involved *and* simultaneously a solution of (13) stands for the respective *current* velocity of the flow: $\partial_t \rho(\vec{x}, t) = -\nabla[\vec{v}(\vec{x}, t)\rho(\vec{x}, t)]$.

Alternatively, following the white noise calculus tradition telling that the stochastic integral

$$\vec{X}(t) = \int_0^t \vec{v}_B(\vec{X}(s), s)ds + \int_0^t \vec{\eta}(s)ds$$

necessarily implies the Fokker–Planck equation, one is tempted to adopt: $\partial_t \rho(\vec{x}, t) = \nu \triangle \rho(\vec{x}, t) - \nabla[\vec{v}_B(\vec{x}, t)\rho(\vec{x}, t)]$ which is clearly problematic in view of the classic Mc Kean's discussion of the propagation of chaos for the Burgers equation, [13–15] and the derivation of the stochastic "Burgers process" in this context: "the fun begins in trying to describe this Burgers motion as the path of a tagged molecule in an infinite bath of like molecules", [13].

To put things on the solid ground, let us consider a Markovian diffusion process, which is characterized by the transition probability density (generally inhomogeneous in space and time law of random displacements) $p(\vec{y}, s, \vec{x}, t)$, $0 \leq s < t \leq T$, and the probability density $\rho(\vec{x}, t)$ of its random variable $\vec{X}(t)$, $0 \leq t \leq T$. The process is completely determined by these data. For clarity of discussion, we do not impose any spatial boundary restrictions, nor fix any concrete limiting value of T which, in principle, can be moved to infinity.

Let us confine attention to processes defined by the standard backward diffusion equation. Under suitable restrictions (boundedness of involved functions, their continuous differentiability) the function:

$$g(\vec{x}, s) = E\{g(\vec{X}(T))|\vec{X}(s) = \vec{x}; s \leq T\} = \int p(\vec{x}, s, \vec{y}, T)g(\vec{y}, T)d^3y\,, \quad (14)$$

satisfies the equation

$$-\partial_s g(\vec{x}, s) = \nu \triangle g(\vec{x}, s) + [\vec{b}(\vec{x}, s)\nabla]g(\vec{x}, s)\,. \quad (15)$$

Let us point out that the validity of (14) is known to be a *necessary* condition for the existence of a Markov diffusion process, whose probability density $\rho(\vec{x}, t)$ is to obey the Fokker–Planck equation (the forward drift $\vec{b}(\vec{x}, t)$ replaces the previously utilized Burgers velocity $\vec{v}_B(\vec{x}, t)$).

The case of particular interest, in the traditional nonequilibrium statistical physics literature, appears when $p(\vec{y}, s, \vec{x}, t)$ is a *fundamental solution* of (15) with respect to variables $\vec{y}, s$, [16–18], see however [2] for an analysis of alternative situations. Then, the transition probability density satisfies *also* the second Kolmogorov (*e.g.* the Fokker–Planck) equation in the remaining $\vec{x}, t$ pair of variables. Let us emphasize that these two equations form an adjoint pair of partial differential equations, referring to the slightly counter-intuitive for physicists, though transparent for mathematicians, [6,7,19–22], issue of time reversal of diffusions.

We can consistently introduce the random variable of the process in the form

$$\vec{X}(t) = \int_0^t \vec{b}(\vec{X}(s), s)\, ds + \sqrt{2\nu}\vec{W}(t)\,.$$

Then, in view of the standard rules of the Itô stochastic calculus, [6,7,22,23], we realize that for any smooth function $f(\vec{x}, t)$ of the random variable $\vec{X}(t)$ the conditional expectation value:

$$\lim_{\triangle t \downarrow 0} \frac{1}{\triangle t} \left[\int p(\vec{x}, t, \vec{y}, t + \triangle t) f(\vec{y}, t + \triangle t) d^3y - f(\vec{x}, t)\right]$$
$$= (D_+ f)(\vec{X}(t), t) = (\partial_t + \vec{b}\nabla + \nu\triangle) f(\vec{x}, t)\,, \tag{16}$$

where $\vec{X}(t) = \vec{x}$, determines the forward drift $\vec{b}(\vec{x}, t)$ of the process (if we set components of $\vec{X}$ instead of f) and, moreover, allows to introduce the local field of (forward) accelerations associated with the diffusion process, which we *constrain* by demanding (see *e.g.* Refs [6,7,22,23] for prototypes of such dynamical constraints):

$$(D_+^2\vec{X})(t) = (D_+\vec{b})(\vec{X}(t), t) = (\partial_t\vec{b} + (\vec{b}\nabla)\vec{b} + \nu\triangle\vec{b})(\vec{x}, t) = \vec{F}(\vec{x}, t)\,, \tag{17}$$

where $\vec{X}(t) = \vec{x}$ and, at the moment arbitrary, function $\vec{F}(\vec{x}, t)$ may be interpreted as an external forcing applied to the diffusing system, [3].

By invoking (9), we can also define the backward derivative of the process in the conditional mean (*c.f.* [3,24,25] for a discussion of these concepts in case of the most traditional Brownian motion and Smoluchowski-type diffusion processes)

$$\lim_{\triangle t \downarrow 0} \frac{1}{\triangle t} \left[\vec{x} - \int p_*(\vec{y}, t - \triangle t, \vec{x}, t) \vec{y} d^3y\right] = (D_-\vec{X})(t) = \vec{b}_*(\vec{X}(t), t) \tag{18}$$

$$(D_- f)(\vec{X}(t), t) = (\partial_t + \vec{b}_*\nabla - \nu\triangle) f(\vec{X}(t), t)\,.$$

Accordingly, the backward version of the acceleration field reads

$$(D_-^2\vec{X})(t) = (D_+^2\vec{X})(t) = \vec{F}(\vec{X}(t), t)\,, \tag{19}$$

where in view of $\vec{b}_* = \vec{b} - 2\nu\nabla \ln \rho$ we have explicitly fulfilled the *forced Burgers equation*:

$$\partial_t \vec{b}_* + (\vec{b}_* \nabla)\vec{b}_* - \nu \triangle \vec{b}_* = \vec{F} \tag{20}$$

and, [3,6,22], under the gradient-drift field assumption, curl $\vec{b}_* = 0$, we deal with $\vec{F}(\vec{x},t) = 2\nu\nabla c(\vec{x},t)$ where the Feynman–Kac potential (3) is explicitly involved.

Let us notice that the familiar (linearization of the nonlinear problem) Hopf–Cole transformation, [10,26], of the Burgers equation into the generalized diffusion equation (yielding explicit solutions in the unforced case) has been explicitly used before (the second formula (4)) in the framework of the Schrödinger interpolation problem. In fact, by defining $\Phi_* = \log u$, we immediately recover the traditional form of the Hopf–Cole transformation for Burgers velocity fields: $\vec{b}_* = -2\nu\nabla\Phi_*$. In the standard considerations that allows to map a nonlinear (unforced Burgers) equation into a linear, heat, equation. In the special case of the standard free Brownian motion, there holds $\vec{b}(\vec{x},t) = 0$ while $\vec{b}_*(\vec{x},t) = -2\nu\nabla \log \rho(\vec{x},t)$.

2. The problem of electromagnetic forcing in the Schrödinger interpolating dynamics

It turns out the crucial point of our previous discussion lies in a *proper* choice of the strictly positive and continuous, in an open space-time area, function $k(\vec{y},s,\vec{x},t)$ which, if we wish to construct a Markov process, has to satisfy the Chapman–Kolmogorov (semigroup composition) equation. It has led us to consider a pair of adjoint parabolic differential equations, as an alternative to more familiar Fookker–Planck and backward diffusion equations.

In the quantally oriented literature dealing with Schrödinger operators and their spectral properties, [27–29], the potential $c(x,t)$ is usually assumed to be a continuous and bounded from below function, but these restrictions can be substantially relaxed (unbounded functions are allowed in principle) if we wish to consider general Markovian diffusion processes and disregard an issue of the bound state spectrum and this of the ground state of the (self-adjoint) semigroup generator, [16,17]. Actually, what we need is merely that properties of $c(\vec{x},t)$ allow for the kernel k which is positive and continuous function. By taking for granted that suitable conditions are fulfilled, [2,27], we can immediately associate with equations (4) an integral kernel of the time-dependent semigroup (the exponential operator should be understood as time-ordered expression, since in general $H(\tau)$ may not commute with

$H(\tau')$ for $\tau \neq \tau'$):

$$k(\vec{y}, s, \vec{x}, t) = \left[\exp(-\int_s^t H(\tau) d\tau) \right] (\vec{y}, \vec{x}) \, , \qquad (21)$$

where $H(\tau) = -\nu\triangle + c(\tau)$ is the pertinent semigroup generator. Then, by the Feynman–Kac formula, [30], we get a standard path integral expression (3) for the kernel, which in turn yields (5)–(8), see *e.g.* [2]. The above formalism is known, [3], to encompass the standard Smoluchowski-type diffusions in conservative force fields.

Strikingly, an investigation of electromagnetically forced diffusions has not been much pursued in the literature, although an issue of deriving the Smoluchowski–Kramers equation (and possibly its large friction limit) from the Langevin-type equation for the charged Brownian particle in the general electromagnetic field has been relegated in Ref. [31], Chap. 6.1 to the status of the innocent-looking exercise. On the other hand, the diffusion of realistic charges in dilute ionic solutions creates a number of additional difficulties due to the apparent Hall mobility in terms of mean currents induced by the electric field (once assumed to act upon the system), see *e.g.* [32,33] and [34].

In connection with the electromagnetic forcing of diffusing charges, the gradient field assumption imposes a severe limitation if we account for typical (nonzero circulation) features of the classical motion due to the Lorentz force, with or without the random perturbation component. The purely electric forcing is simpler to handle, since it has a definite gradient field realization, see *e.g.* [35] for a recent discussion of related issues. The major obstacle with respect to our previous (Section 1) discussion is that, if we wish to regard either the force $\vec{F}$, or drifts $\vec{b}$, $\vec{b}_*$ to have an electromagnetic provenience, then necessarily we need to pass from conservative to non-conservative fields. This subject matter has not been significantly exploited so far in the nonequilibrium statistical physics literature.

Usually, the selfadjoint semigroup generators attract the attention of physicists in connection with the Feynman–Kac formula. A typical route towards incorporating electromagnetism comes from quantal motivations via the minimal electromagnetic coupling recipe which preserves the selfadjointness of the generator (Hamiltonian of the system). As such, it constitutes a part of the general theory of Schrödinger operators. A rigorous study of operators of the form $-\triangle + V$ has become a well developed mathematical discipline, [27]. The study of Schrödinger operators with magnetic fields, typically of the form $-(\nabla - i\vec{A})^2 + V$, is less advanced, although specialized chapters on the magnetic field issue can be found in monographs devoted to functional integration methods, [27, 36], mostly in reference to seminal papers [37, 38].

From the mathematical point of view, it is desirable to deal with magnetic fields that go to zero at infinity, which is certainly acceptable on physical grounds as well. The constant magnetic field does not meet this requirement, and its notorious usage in the literature makes us (at the moment) to decline the asymptotic assumption and inevitably fall into a number of serious complications.

One obvious obstacle can be seen immediately by taking advantage of the existing results, [37]. Namely, an explicit expression for the Feynman–Kac kernel in a constant magnetic field, introduced through the minimal electromagnetic coupling recipe $H(\vec{A}) = -\frac{1}{2}(\nabla - i\vec{A})^2$, is available (up to irrelevant dimensional constants):

$$\exp\left[-tH(\vec{A})\right](\vec{x},\vec{y}) = \left[\frac{B}{4\pi\sinh(\frac{1}{2}Bt)}\right]\left(\frac{1}{2\pi t}\right)^{1/2}$$
$$\times \exp\left\{-\frac{1}{2t}(x_3-y_3)^2 - \frac{B}{4}\coth\left(\frac{B}{2}t\right)[(x_2-y_2)^2+(x_1-y_1)^2] - i\frac{B}{2}(x_1y_2-x_2y_1)\right\}. \tag{22}$$

Clearly, it is *not* real (hence *non-positive* and directly at variance with the major demand in the Schrödinger interpolation problem, as outlined in Section 1), except for directions $\vec{y}$ that are parallel to a chosen $\vec{x}$.

Consequently, a bulk of the well developed mathematical theory is *of no use* for our purposes and new techniques must be developed for a consistent description of the electromagnetically forced diffusion processes along the lines of Section 1, *i.e.* within the framework of Schrödinger's interpolation problem.

3. Forcing via Feynman–Kac semigroups

The conditional Wiener measure $d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\vec{\omega})$, appearing in the Feynman–Kac kernel definition (3), if unweighed (set $c(\vec{\omega}(\tau),\tau) = 0$) gives rise to the familiar heat kernel. This, in turn, induces the Wiener measure P_W of the set of all sample paths,which originate from $\vec{y}$ at time s and terminate (can be located) in the Borel set $A \in R^3$ after time $t - s$:

$$P_W[A] = \int_A d^3x \int d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\vec{\omega}) = \int_A d\mu\,,$$

where, for simplicity of notation, the $(\vec{y}, t - s)$ labels are omitted and $\mu_{(\vec{x},t)}^{(\vec{y},s)}$ stands for the heat kernel.

Having defined an Itô diffusion

$$\vec{X}(t) = \int_0^t \vec{b}(\vec{x}, u)du + \sqrt{2\nu}\vec{W}(t)\,,$$

we are interested in the analogous path measure:

$$P_{\vec{X}}[A] = \int_A dx \int d\mu^{(\vec{y},s)}_{(\vec{x},t)}(\vec{\omega}_{\vec{X}}) = \int_A d\mu(\vec{X})\,.$$

Under suitable (stochastic, [3]) integrability conditions imposed on the forward drift, we have granted the absolute continuity $P_X \ll P_W$ of measures, which implies the existence of a strictly positive Radon–Nikodym density. Its canonical Cameron–Martin–Girsanov form, [3, 27], reads:

$$\frac{d\mu(\vec{X})}{d\mu}(\vec{y}, s, \vec{x}, t) = \exp\frac{1}{2\nu}\left[\int_s^t \vec{b}(\vec{X}(u), u)d\vec{X}(u) - \frac{1}{2}\int_s^t [\vec{b}(\vec{X}(u), u)]^2 du\right]\,. \tag{23}$$

If we assume that drifts are gradient fields, $\mathrm{curl}\,\vec{b} = 0$, then the Itô formula allows to reduce, otherwise troublesome, stochastic integration in the exponent of (23), [27, 36], to ordinary Lebesgue integrals:

$$\begin{aligned}\frac{1}{2\nu}\int_s^t \vec{b}(\vec{X}(u), u)d\vec{X}(u) &= \Phi(\vec{X}(t), t) - \Phi(\vec{X}(s), s)\\ &- \int_s^t du\,[\partial_t\Phi + \frac{1}{2}\nabla\vec{b}](\vec{X}(u), u)\,. \end{aligned}\tag{24}$$

After inserting (24) to (23) and next integrating with respect to the conditional Wiener measure, on account of (10) we arrive at the standard form of the Feynman–Kac kernel (3). Notice that (24) establishes a probabilistic basis for logarithmic transformations (5) of forward and backward drifts: $b = 2\nu\nabla \ln v = 2\nu\nabla\Phi$, $b_* = -2\nu\nabla \ln u = -2\nu\nabla\Phi_*$. The forward version is commonly used in connection with the transformation of the Fokker–Planck equation into the generalized heat equation, [3,39,40]. The backward version is just the Hopf–Cole transformation, mentioned in Section 1, used to map the Burgers equation into the very same generalized heat equation, [10].

However, presently we are interested in non-conservative drift fields, $\mathrm{curl}\,\vec{b} \neq 0$, and in that case the stochastic integral in (23) is the major source of computational difficulties, [22, 27, 36], for nontrivial vector potential field configurations. It explains the virtual absence of magnetically forced diffusion problems in the nonequilibrium statistical physics literature.

At this point, some steps of the analysis performed in Ref. [41] in the context of the "Euclidean quantum mechanics", *c.f.* also [7], are extremely useful. Let us emphasize that electromagnetic fields we utilize, are always meant to be ordinary Maxwell fields with *no* Euclidean connotations (see *e.g.* Chap. 9 of Ref. [36] for the Euclidean version of Maxwell theory).

Let us consider a gradient drift-field diffusion problem according to Section 1, with (2), (24) involved and thus an adjoint pair (4) of parabolic equations completely defining the Markovian diffusion process. Furthermore, let $\vec{A}(\vec{x})$ be the time-independent vector potential for the Maxwellian magnetic field $\vec{B} = \mathrm{curl}\,\vec{A}$. We pass from the gradient realization of drifts to the new one, generalizing (5), for which the following decomposition into the gradient and nonconservative part is valid:

$$\vec{b}(\vec{x},t) = 2\nu\nabla\,\Phi(\vec{x},t) - \vec{A}(\vec{x})\,, \tag{25}$$

We denote $\theta(\vec{x},t) \doteq \exp[\Phi(\vec{x},t)]$ and admit that (25) is a forward drift of an Itô diffusion process with a stochastic differential

$$d\vec{X}(t) = \left[2\nu\frac{\nabla\theta}{\theta} - \vec{A}\right]dt + \sqrt{2\nu}d\vec{W}(t)\,.$$

On purely formal grounds, we deal here with an example of the Cameron–Martin–Girsanov transformation of the forward drift of a given Markovian diffusion process and we are entitled to ask for a corresponding measure transformation, (23).

To this end, let us furthermore *assume* that $\theta(\vec{x},t) = \theta$ solves a partial differential equation

$$\partial_t\theta = -\nu\left[\nabla - \frac{1}{2\nu}\vec{A}(\vec{x})\right]^2\theta + c(\vec{x},t)\theta \tag{26}$$

with the notation $c(\vec{x},t)$ patterned after (8). Then, by using the Itô calculus and (25), (26) on the way, see *e.g.* Ref. [41], we can rewrite (23) as follows:

$$\begin{aligned}
&\frac{d\mu(\vec{X})}{d\mu}(\vec{y},s,\vec{x},t)\\
&= \exp\frac{1}{2\nu}\left[\int_s^t(2\nu\frac{\nabla\theta}{\theta} - \vec{A})(\vec{X}(u),u)d\vec{X}(u) - \frac{1}{2}\int_s^t(2\nu\frac{\nabla\theta}{\theta} - \vec{A})^2(\vec{X}(u),u)\,du\right]\\
&= \frac{\theta(\vec{X}(t),t)}{\theta(\vec{X}(s),s)}\exp\left[-\frac{1}{2\nu}\int_s^t[\vec{A}(u)d\vec{X}(u) + \nu(\nabla\vec{A})(\vec{X}(u))du + \Omega(\vec{X}(u),u)du]\right],
\end{aligned} \tag{27}$$

where $\vec{X}(s) = \vec{y}, \vec{X}(t) = \vec{x}$ and $\Omega(\vec{x},t) = 2\nu c(\vec{x},t)$.

More significant observation is that the Radon–Nikodym density (27), if integrated with respect to the conditional Wiener measure, gives rise to the Feynman–Kac kernel (21) of the *non-selfadjoint* semigroup (suitable integrability conditions need to be respected here as well, [41]), with the generator $H_{\vec{A}} = -\nu[\nabla - \frac{1}{2\nu}\vec{A}(\vec{x})]^2 + c(\vec{x},t)$ defined by the right-hand-side of (26):

$$\begin{aligned}\partial_t\theta(\vec{x},t) &= H_{\vec{A}}\theta(\vec{x},t)\\ &= \left[-\nu\triangle + \vec{A}(\vec{x})\nabla + \frac{1}{2}(\nabla\vec{A}(\vec{x})) - \frac{1}{4\nu}[\vec{A}(\vec{x})]^2 + c(\vec{x},t)\right]\theta(\vec{x},t)\\ &= -\nu\triangle\theta(\vec{x},t) + \vec{A}(\vec{x})\nabla\theta(\vec{x},t) + c_{\vec{A}}(\vec{x},t)\theta(\vec{x},t)\,. \qquad (28)\end{aligned}$$

Here:

$$c_A(\vec{x},t) = c(\vec{x},t) + \frac{1}{2}(\nabla\vec{A})(\vec{x}) - \frac{1}{4\nu}[\vec{A}(\vec{x})]^2\,. \qquad (29)$$

An adjoint parabolic partner of (28) reads:

$$\begin{aligned}\partial_t\theta_* &= -H^*_{\vec{A}}\theta_* = \nu\triangle\theta_* + \nabla[\vec{A}(\vec{x})\theta_*] - c_A(\vec{x},t)\theta_*\\ &= \nu\left[\nabla + \frac{1}{2\nu}\vec{A}(\vec{x})\right]^2\theta_* - c(\vec{x},t)\theta_*\,. \qquad (30)\end{aligned}$$

Consequently, our assumptions (25), (26) involve a generalization of the adjoint parabolic system (14) to a new adjoint one comprising (26), (30). Obviously, the original form of (14) is immediately restored by setting $\vec{A} = \vec{0}$, and executing obvious replacements $\theta_* \to u$, $\theta \to v$.

Let us emphasize again, that in contrast to Ref. [41], where the non-Hermitean generator $2\nu H_{\vec{A}}$, (26), has been introduced as "the Euclidean version of the Hamiltonian" $H = -2\nu^2(\nabla - \frac{i}{2\nu}\vec{A})^2 + \Omega$, our electromagnetic fields stand for solutions of the usual Maxwell equations and *are not* Euclidean at all.

As long as the coefficient functions (both additive and multiplicative) of the adjoint parabolic system (28), (30) are not specified, we remain within a general theory of positive solutions for parabolic equations with unbounded coefficients (of particular importance, if we do not impose any asymptotic fall off restrictions), [16,43–45]. The fundamental solutions, if their existence can be granted, usually live on space-time strips, and generally do not admit unbounded time intervals. We shall disregard these issues at the moment, and assume the existence of fundamental solutions without any reservations.

By exploiting the rules of functional (Malliavin, variational) calculus, under an assumption that we deal with a diffusion (in fact, Bernstein) process associated with an adjoint pair (28), (30), it has been shown in Ref. [41]

that *if* the forward conditional derivatives of the process exist, then $(D_+\vec{X})(t) = 2\nu\frac{\nabla\theta}{\theta} - \vec{A} = \vec{b}(\vec{x},t)$, (32) and:

$$(D_+^2\vec{X})(t) = (D_+\vec{X})(t) \times \text{curl}\,\vec{A}(\vec{x}) + \nabla\Omega(\vec{x},t) + \nu\text{curl}\,(\text{curl}\,\vec{A}(\vec{x}))\,, \quad (31)$$

where $\vec{X}(0) = 0$, $\vec{X}(t) = \vec{x}$, $\times$ denotes the vector product in R^3 and $2\nu c = \Omega$.

Since $\vec{B} = \text{curl}\,\vec{A} = \mu_0\vec{H}$, we identify in the above the standard Maxwell equation for $\text{curl}\,\vec{H}$ comprising magnetic effects of electric currents in the system: $\text{curl}\,\vec{B} = \mu_0[\dot{\vec{D}} + \sigma_0\vec{E} + \vec{J}_{\text{ext}}]$ where $\vec{D} = \epsilon_0\vec{E}$ while $\vec{J}_{\text{ext}}$ represents external electric currents. In case of $\vec{E} = \vec{0}$, the external currents only would be relevant. A demand $\text{curl}\,\text{curl}\,\vec{A} = \nabla(\nabla\vec{A}) - \triangle\vec{A} = 0$ corresponds to a total absence of such currents, and the Coulomb gauge choice $\nabla\vec{A} = 0$ would leave us with harmonic functions $\vec{A}(\vec{x})$.

Consequently, a correct expression for the magnetically implemented Lorentz force has appeared on the right-hand-side of the forward acceleration formula (31), with the forward drift (25) replacing the classical particle velocity $\vec{q}$ of the normal classical formula.

The above discussion implicitly involves quite sophisticated mathematics, hence it is instructive to see that we can bypass the apparent complications by directly invoking the universal definitions (16) and (18) of conditional expectation values, that are based on exploitation of the Itô formula only. Obviously, under an assumption that the Markovian diffusion process with well defined transition probability densities $p(\vec{y},s,\vec{x},t)$ and $p_*(\vec{y},s,\vec{x},t)$, does exist.

We shall utilize an obvious generalization of canonical definitions (5) of both forward and backward drifts of the diffusion process defined by the adjoint parabolic pair (4), as suggested by (25) with $\vec{A} = \vec{A}(\vec{x})$:

$$\vec{b} = 2\nu\frac{\nabla\theta}{\theta} - \vec{A}\,, \quad \vec{b}_* = -2\nu\frac{\nabla\theta_*}{\theta_*} - \vec{A}\,. \quad (32)$$

We also demand that the corresponding adjoint equations (28), (30) *are* solved by θ and θ_* respectively.

Taking for granted that identities $(D_+\vec{X})(t) = \vec{b}(\vec{x},t)$, $\vec{X}(t) = \vec{x}$ and $(D_-\vec{X})(t) = \vec{b}_*(\vec{x},t)$ hold true, we can easily evaluate the forward and backward accelerations (substitute (32), and exploit the equations (28), (30)):

$$(D_+\vec{b})(\vec{X}(t),t) = \partial_t\vec{b} + (\vec{b}\nabla)\vec{b} + \nu\triangle\vec{b} = \vec{b}\times\vec{B} + \nu\,\mathrm{curl}\,\vec{B} + \nabla\,\Omega \tag{33}$$

and

$$(D_-\vec{b}_*)(\vec{X}(t),t) = \partial_t\vec{b}_* + (\vec{b}_*\nabla)\vec{b}_* - \nu\triangle\vec{b}_* = \vec{b}_*\times\vec{B} - \nu\,\mathrm{curl}\,\vec{B} + \nabla\,\Omega\,. \tag{34}$$

Let us notice that the forward and backward acceleration formulas *do not* coincide as was the case before (*c.f.* Eq. (17)). There is a definite time-asymmetry in the local description of the diffusion process in the presence of general magnetic fields, unless $\mathrm{curl}\,\vec{B}=0$. The quantity which is explicitly time-reversal invariant can be easily introduced:

$$\vec{v}(\vec{x},t) = \frac{1}{2}(\vec{b}+\vec{b}_*)(\vec{x},t) \Rightarrow \frac{1}{2}(D_+^2 + D_-^2)(\vec{X}(t)) = \vec{v}\times\vec{B} + \nabla\,\Omega\,. \tag{35}$$

As yet there is no trace of Lorentzian electric forces, unless extracted from the term $\nabla\,\Omega(\vec{x},t)$.

For a probability density $\theta_*\theta=\rho$ of the related Markovian diffusion process, [2,6], we would have fulfilled both the Fokker–Planck and the continuity equations: $\partial_t\rho = \nu\triangle\rho - \nabla(\vec{b}\rho) = -\nabla(\vec{v}\rho) = -\nu\triangle\rho - \nabla(\vec{b}_*\rho)$, as before (*c.f.* Section 1).

In the above, the equation (34) can be regarded as the Burgers equation with a general external magnetic (plus other external force contributions if necessary) forcing, and its definition is an outcome of the underlying mathematical structure related to the adjoint pair (26), (30) of parabolic equations. Our construction shows that the solution of the magnetically forced Burgers equation needs to be sought in the form (32).

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REFERENCES

[1] P. Garbaczewski, J.R. Klauder, R. Olkiewicz, *Phys. Rev.* **E51**, 4114 (1995).
[2] P. Garbaczewski, R. Olkiewicz, *J. Math. Phys.* **37**, 732 (1996).
[3] Ph. Blanchard, P.Garbaczewski, *Phys. Rev.* **E49**, 3815 (1994).
[4] P. Garbaczewski, *Acta Phys. Pol.* **B27**, 617 (1996).

[5] P. Garbaczewski, G. Kondrat, *Phys. Rev. Lett.* **77**, 2608 (1996).

[6] J.C. Zambrini, *J. Math. Phys.* **27**, 3207 (1986).

[7] J.C. Zambrini, pp. 393, in: *Chaos-The Interplay Between Stochastic and Deterministic Behaviour*, eds P. Garbaczewski, M. Wolf, A. Weron, LNP vol 457, Springer-Verlag, Berlin 1995.

[8] T. Mikami, *Commun. Math. Phys.* **135**, 19 (1990).

[9] J.M. Burgers, *The Nonlinear Diffusion Equation*, Reidel, Dordrecht 1974.

[10] E. Hopf, *Commun. Pure Appl. Math.* **3**, 201 (1950).

[11] S.F. Shandarin, B.Z. Zeldovich, *Rev. Mod. Phys.* **61**, 185 (1989).

[12] S. Albeverio, A.A. Molchanov, D. Surgailis, *Prob. Theory Relat. Fields* **100**, 457 (1994).

[13] H.P. Mc Kean, pp. 177, in: *Lecture Series in Differential Equations*, vol. II, ed. A.K. Aziz, Van Nostrand, Amsterdam 1969.

[14] P. Calderoni, M. Pulvirenti, *Ann. Inst. Henri Poincaré*, **39**, 85 (1983).

[15] H. Osada, S. Kotani, *J. Math. Soc. Japan*, **37**, 275 (1985).

[16] M. Krzyżański, A. Szybiak, *Lincei-Rend. Sc. Fis. Mat. e Nat.* **28**, 26 (1959).

[17] A. Friedman, *Partial Differential Equations of Parabolic type*, Prentice-Hall, Englewood, NJ 1964.

[18] W. Horsthemke, R. Lefever, *Noise-Induced Transitions*, Springer-Verlag, Berlin 1984.

[19] U.G. Haussmann, E. Pardoux, *Ann. Prob.* **14**, 1188 (1986).

[20] H. Föllmer,, in: *Stochastic Processes-Mathematics and Physics*, eds S. Albeverio, Ph. Blanchard, L. Streit, LNP vol. 1158, Springer-Verlag, Berlin 1985, p. 119.

[21] H. Hasegawa, *Progr. Theor. Phys.* **55**, 90 (1976).

[22] E. Nelson, *Quantum Fluctuations*, Princeton University Press, Princeton 1985.

[23] E. Nelson, *Dynamical Theories of the Brownian Motion*, Princeton University Press, Princeton 1967.

[24] P. Garbaczewski, J.P. Vigier, *Phys. Rev.* **A46**, 4634 (1992).

[25] P. Garbaczewski, R. Olkiewicz, *Phys. Rev.* **A51**, 3445 (1995).

[26] W.H. Fleming, H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, Berlin 1993.

[27] B. Simon, *Functional Integration and Quantum Physics*, Academic Press, New York 1979.

[28] J. Glimm, A. Jaffe, *Quantum Physics-A Functional Integral Point of View*, Springer-Verlag, Berlin 1981.

[29] K.L. Chung, Z. Zhao, *From Brownian Motion to Schrödinger Equation*, Springer-Verlag, Berlin 1995.

[30] M. Freidlin, *Functional Integration and Partial Differential Equations*, Princeton University Press, Princeton 1985.

[31] Z. Schuss, *Theory and Applications of Stochastic Differential Equations*, Wiley, NY 1980.

[32] J.B. Hubbard, P.G. Wolynes, *J. Chem. Phys.* **75**, 3051 (1981).

[33] W. Sung, H.L. Friedman, *J. Chem. Phys.* **87**, 649 (1987).
[34] H. Mori, *Progr. Theor. Phys.* **33**, 243 (1965).
[35] A.F. Izmailov, S. Arnold, S. Holler, A.S. Myerson, *Phys. Rev.* **E52**, 1325 (1995).
[36] G. Roepstorff, *Path Integral Approach to Quantum Physics*, Springer-Verlag, Berlin 1994.
[37] J. Avron, I. Herbst, B. Simon, *Duke Math. Journ.* **45**, 847 (1978).
[38] J. Avron, I. Herbst, B. Simon, *Commun. Math. Phys.* **79**, 529 (1981).
[39] H. Risken, *The Fokker-Planck Equation*, Springer-Verlag, Berlin 1989.
[40] P. Garbaczewski, *Phys. Lett.* **A175**, 7 (1993).
[41] A.B. Cruzeiro, J.C. Zambrini, *J. Funct. Anal.* **96**, 62 (1991), see also [42].
[42] H. Hasegawa, *Phys. Rev.* **D33**, 2508 (1986).
[43] A.M. Ilin, A.S. Kalashnikov, O.A. Oleinik, *Usp. Mat. Nauk* (in Russian), **27**, 65 (1962).
[44] P. Besala, *Ann. Polon. Math.* **29**, 403 (1975).
[45] D.G. Aronson, P. Besala, *Colloq. Math.* **18**, 125 (1967).