LIE BIALGEBRA STRUCTURES ON TWO–DIMENSIONAL GALILEI ALGEBRA AND THEIR LIE–POISSON COUNTERPARTS*

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All bialgebra structures on two-dimensional Galilei algebra are classified. The corresponding Lie–Poisson structures on Galilei group are found.

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1. Introduction

Much interest has been recently attracted to the problem of deformations of space-time symmetries (see [1–5] and references contained therein). Most papers are devoted to the relativistic case. There is, however, a number of papers dealing with Galilean symmetries also. These are the papers of Firenze group [6–8] as well as some others [9–13].

The present paper is the first of the series devoted to systematic study of quantum Galilei groups. We are going to study the following problems:

- the classification of all bialgebra structures (resp. Lie–Poisson structures) on two-dimensional Galilei algebra (resp. group);

- the classification of such structures in the fourdimensional case;

- quantization of the resulting Lie bialgebras and Poisson–Lie groups; their structure and duality relations;

- nonrelativistic quantum space–times;

- differential calculi on quantum Galilei groups and corresponding space-times;

- representations, central extensions;

- physical applications;

In this first paper we classify the bialgebra structures on two-dimensional Galilei algebra. To this end we find the most general 1–cocycle δ. However, two such δ's should be viewed as equivalent if they are related under the automorphism of the algebra. We find the action of automorphism group in the space of 1–cocycles and classify the orbits. This allows to find all nonequivalent Lie–Poisson structures on Galilei group. The whole procedure follows quite closely the one presented in Ref. [14] for E(2) group.

As a result we find nine nonequivalent bialgebra structures–four families parametrized by nonnegative dimensionless parameter and five discrete "points". In the general case δ is a sum of few terms which makes necessary to introduce dimensionful parameters to provide the proper dimensions to all terms. They are arbitrary but fixed and nonvanishing; varying them we obtain equivalent bialgebras. The number of dimensionful parameters can be in general reduced by ascribing proper dimension to the Poisson brackets.

It is remarkable that only one discrete case corresponds to coboundary δ. This is in sharp contrast with the semisimple case [15] as well as fourdimensional Poincare group [16, 17]. The bialgebra (or Lie–Poisson) structures appearing in the literature fit into the scheme. For example the one from Ref. [13] corresponds to first case in Table I with $\varepsilon = 0$; the bialgebra structure inferred from Ref. [6] with vanishing central element M lies on the orbit described in fourth row of Table I with $\varepsilon = 1$.

Actually, our results to large extent coincide with those of Ref. [18] where the Heisenberg–Weyl group was considered; this is due to the fact that both groups are isomorphic. However, our aim was to give a precise classification of all structures which are nonequivalent with respect to the action of the group of automorphisms.

2. Two-dimensional Galilei group, algebra and their automorphisms

Two-dimensional Galilei group is defined as the group of transformations of two-dimensional space-time.

$$\left. \begin{array}{ccccc} x & \to & x' & = & x + vt + a \\ t & \to & t' & = & t + \tau \end{array} \right\} .$$ (1)

Accordingly, the composition law reads

$$(\tau, v, a) \star (\tau', v', a') = (\tau + \tau', v + v', a + a' + v\tau') .$$ (2)

There exists the following global exponential parametrization of group elements

$$g = e^{-i\tau H}\, e^{iaP}\, e^{ivK} \tag{3}$$

which defines the generators of time translations (V), space translations (P) and boots (K); their dimensions, respectively, are: $[H] = (\text{time})^{-1}$, $[P] = (\text{distance})^{-1}$, $[K] = (\text{velocity})^{-1}$.

The resulting Lie algebra reads

$$[K, H] = iP, \quad [K, P] = 0, \quad [H, P] = 0. \tag{4}$$

This algebra can be realized in terms of left–invariant fields to be calculated according to the standard rules from the composition law (2):

$$
\begin{aligned}
H &= i\left(\frac{\partial}{\partial \tau} + v\frac{\partial}{\partial a}\right) \equiv X_\tau^L, \\
P &= -i\frac{\partial}{\partial a} \equiv X_a^L, \\
K &= -i\frac{\partial}{\partial v} \equiv X_v^L.
\end{aligned}
\tag{5}
$$

One can also compute the right–invariant fields

$$
\begin{aligned}
X_\tau^R &= -i\frac{\partial}{\partial \tau}, \\
X_a^R &= i\frac{\partial}{\partial a}, \\
X_v^R &= i\left(\frac{\partial}{\partial v} + \tau\frac{\partial}{\partial a}\right)
\end{aligned}
\tag{6}
$$

which obey the same commutation rules.

The following observation will be important below. If we redefine $H \to H/\tau_0$, $P \to P/v_0\tau_0$, $K \to K/v_0$, where τ_0 and v_0 are arbitrary constants of dimension time and velocity, respectively, we obtain the same algebra (4), the generators H, P, K being now, however, dimensionless.

It is very easy to describe all automorphisms of the algebra (4) (taken in dimensionless form).
Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R}), \quad \vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \tag{7}$$

then the general automorphism H can be written in form

$$\begin{pmatrix} K \\ H \\ P \end{pmatrix} \to \begin{pmatrix} K' \\ H' \\ P' \end{pmatrix} = \begin{pmatrix} A & \vec{X} \\ 0 & \det A \end{pmatrix}\begin{pmatrix} K \\ H \\ P \end{pmatrix}. \tag{8}$$

The group of automorphisms is therefore sixdimensional. The composition law reads

$$(A, \vec{X}) \star (\vec{A'}, \vec{X'}) = (A \cdot A', A\vec{X'} + (\det A')\vec{X}). \qquad (9)$$

3. The bialgebra structures on two-dimensional Galilei algebra

In this Section we find all bialgebra structures on two-dimensional Galilei algebra. As it is well known given any Lie algebra L the bialgebra (L, δ) is defined by a skewsymmetric cocommutator $\delta : L \to L \otimes L$ such that:

(i) δ is a 1–cocycle, *i.e.*

$$\begin{aligned}
\delta([X, Y]) &= [\delta(X), 1 \otimes Y + Y \otimes 1] \\
&\quad + [1 \otimes X + X \otimes 1, \delta(Y)] \quad X, Y \in L,
\end{aligned}$$

(ii) the dual map $\delta^\star : L^\star \otimes L^\star \to L^\star$ defines a Lie bracket on $L^\star$.

Our aim is to find all bialgebra structures on the algebra (4). The general form of δ obeying *(i)* is

$$\begin{aligned}
\delta(P) &= sK \wedge P - rH \wedge P, \\
\delta(H) &= sK \wedge H + \alpha K \wedge P - \beta H \wedge P, \\
\delta(K) &= rK \wedge H + \gamma K \wedge P - \rho H \wedge P, \qquad (10)
\end{aligned}$$

$s, r, \alpha, \beta, \gamma, \rho$ being arbitrary real parameters. The condition *(ii)* adds two further constraints

$$\begin{aligned}
2s\rho - r(\beta + \gamma) &= 0, \\
2r\alpha - s(\beta + \gamma) &= 0. \qquad (11)
\end{aligned}$$

Eqs (10)–(11) define all bialgebra structures on two-dimensional Galilei algebra. However, it is not the end of the story. Two δ's which can be transformed into each other by automorphism (8) are considered as equivalent. Therefore, we are interested in classification of nonequivalent bialgebra structures. To this end we find first the transformation rules for the parameters. They read

$$\begin{aligned}
\alpha' &= \frac{1}{\Delta^2}(d^2\alpha + cd(\beta + \gamma) + c^2\rho)], \\
\beta' &= \frac{1}{\Delta^2}[(bd\alpha + ad\beta + bc\gamma + ac\rho) - (cr + ds)x_1 + (ar + bs)x_2], \\
\gamma' &= \frac{1}{\Delta^2}[(bd\alpha + bc\beta + ad\gamma + ac\rho) + (cr + ds)x_1 - (ar + bs)x_2],
\end{aligned}$$

$$\rho' = \frac{1}{\Delta^2}(b^2\alpha + ab(\beta + \gamma) + a^2\rho)\,,$$

$$r' = (ar + bs)\frac{1}{\Delta}\,,$$

$$s' = (cr + ds)\frac{1}{\Delta}\,, \quad \Delta \equiv \det A \equiv ad - bc\,. \tag{12}$$

We see that, apart from Δ–factors, r, s transform according to the defining representation of $GL(2,\mathbb{R})$, $\alpha, \beta + \gamma, \rho$ form an irreducible triplet from symmetric product of two defining representations while the transformation rule for $\beta - \gamma$ reads

$$\beta' - \gamma' = \frac{1}{\Delta}(\beta - \gamma) + \frac{2}{\Delta^2}(-(cr + ds)x_1 + (ar + bs)x_2)\,. \tag{13}$$

In order to classify the inequivalent δ's we observe first that, as far as r and s are concerned, there are two orbits: $r = s = 0$ or $|r| + |s| \neq 0$. Take $r = s = 0$; denoting $T_{11} = \rho, T_{22} = \alpha, T_{12} = (1/2)(\beta + \gamma)$ one gets

$$T' = \frac{1}{(\det A)^2}ATA^T\,, \tag{14}$$

$$\beta' - \gamma' = \frac{1}{\Delta}(\beta - \gamma)\,. \tag{15}$$

Moreover, there are no further constraints on α, β, γ and ρ because, Eqs (11) (which are invariant under the automorphisms (12)) are obeyed automatically.

Let us diagonalize T by choosing on appropriate orthogonal A. According to the Sylvester theorem there are the following inequivalent possibilities for eigenvalues λ_1, λ_2 of T:

(a) $\lambda_1 > 0, \lambda_2 > 0\,,$

(b) $\lambda_1 < 0, \lambda_2 < 0\,,$

(c) $\lambda_1 > 0, \lambda_2 < 0\,,$

(d) $\lambda_1 > 0, \lambda_2 = 0\,,$

(e) $\lambda_1 < 0, \lambda_2 = 0\,,$

(f) $\lambda_1 = 0, \lambda_2 = 0\,,$

Taking further

$$A = \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_1 \end{pmatrix}\,, \quad \mu_1\mu_2 \neq 0\,,$$

we get $\lambda_i \to \lambda_i/\mu_i^2$. For $\lambda_1\lambda_2 \neq 0$, *i.e.* for (a)–(c) cases we take $\mu_i = \sqrt{|\lambda_i|}$ enforcing $|\lambda_i| = 1$. For (a), (b) cases T is further invariant under $0(2)$

transformations while for the case (c) — under $0(1,1)$ ones. Indeed, taking the determinant of both sides of equation

$$T = \frac{1}{(\det A)^2} A T A^T \,,$$

we get

$$(\det A)^2 = 1 \,, \quad i.e. \quad \det A = \pm 1$$

and

$$T = A T A^T \,.$$

By choosing an appropriate sign of $\det A$ one can achieve, according to Eq. (15), $\beta' - \gamma' \geq 0$.

If (d) or (e) holds, μ_2 can be arbitrary while μ_1 is chosen to enforce $|\lambda_1| = 1$. Therefore, according to Eq. (15) one can take μ_2 such that $\beta' - \gamma' = 0$, respectively $\beta' - \gamma' = 2$ depending on whether $\beta - \gamma = 0$, respectively $\beta - \gamma \neq 0$; the same holds for (f).

Summarizing, for $r = s = 0$ there are the following inequivalent possibilities:

$$\alpha = 1 \,, \quad \rho = 1 \,, \quad \beta = -\gamma = \varepsilon \geq 0 \,,$$
$$\alpha = -1 \,, \quad \rho = -1 \,, \quad \beta = -\gamma = \varepsilon \geq 0 \,,$$
$$\alpha = -1 \,, \quad \rho = 1 \,, \quad \beta = -\gamma = \varepsilon \geq 0 \,,$$
$$\alpha = 0 \,, \quad \rho = 1 \,, \quad \beta = \gamma = 0 \,,$$
$$\alpha = 0 \,, \quad \rho = 1 \,, \quad \beta = -\gamma = 1 \,,$$
$$\alpha = 0 \,, \quad \rho = -1 \,, \quad \beta = \gamma = 0 \,,$$
$$\alpha = 0 \,, \quad \rho = -1 \,, \quad \beta = -\gamma = 1 \,,$$
$$\alpha = 0 \,, \quad \rho = 0 \,, \quad \beta = -\gamma = 1 \,.$$

Let us now consider the case $|r| + |s| \neq 0$. There is now one orbit for the (r, s)-dublet. Therefore we can choose $r' = 1$, $s' = 0$, i.e.

$$\begin{cases} ar + bs & = & \Delta \\ cr + ds & = & 0 \end{cases} \tag{16}$$

Solving Eqs (16) one obtains

$$c = -s \,, \quad d = r \,. \tag{17}$$

This, together, with Eqs (11) and (12) implies

$$\alpha' = 0 \,, \quad \beta' + \gamma' = 0 \,.$$

On the other hand Eq. (13) gives

$$\beta' - \gamma' = \frac{1}{\Delta}((\beta - \gamma) + 2x_2).$$ (18)

Therefore, taking $x_2 = -(\beta - \gamma)/2$ and c, d as determined by Eq. (17) we arrive at $\alpha' = \beta' = \gamma' = s' = 0$, $r' = 1$; it remains to find the possible values of ρ'. Now, Eqs (11) imply also

$$r((\beta + \gamma)^2 - 4\alpha\rho) = 0, \ s((\beta + \gamma)^2 - 4\alpha\rho) = 0 \ ,$$

or, due to $|r| + |s| \neq 0$,

$$(\beta + \gamma)^2 = 4\alpha\rho.$$ (19)

The quadratic form

$$\alpha b^2 + \rho a^2 + (\beta + \gamma)ab$$

entering the expression for ρ', is semidefinite. It vanishes provided

$$2\rho a + (\beta + \gamma)b = 0$$ (20)

which has no common solution with $ra + sb = \Delta$ $(2\rho s - r(\beta + \gamma) = 0!)$ unless $\rho = (\beta + \gamma) = 0$. If the latter holds we can take $\rho' = 0$. If not, $\rho' > 0$, resp. $\rho' < 0$ if $\rho > 0$, resp. $\rho < 0$. Nothing more can be done because if we start with $\alpha = \beta + \gamma = s = 0$, $r = 1$, then, according to the Eqs (12) and (17), $\Delta = a$ and, consequently, $\rho = \rho'$.

Accordingly, for $|r| + |s| \neq 0$ we have the following canonical position: $r = 1, s = 0, \alpha = \beta = \gamma = 0, \rho \in \mathbb{R}$ –arbitrary. The results obtained so far are summarized in Table I.

TABLE I

	α	β	γ	ρ	r	s	Remarks
1	0	0	0	ε	1	0	$\varepsilon \in \mathbb{R}$
2	1	ε	$-\varepsilon$	1	0	0	$\varepsilon \geq 0$
3	-1	ε	$-\varepsilon$	-1	0	0	$\varepsilon \geq 0$
4	-1	ε	$-\varepsilon$	1	0	0	$\varepsilon \geq 0$
5	0	0	0	1	0	0	
6	0	1	-1	1	0	0	
7	0	0	0	-1	0	0	
8	0	1	-1	-1	0	0	
9	0	1	-1	0	0	0	coboundary

We have checked explicitly that all above bialgebra structures are consistent and inequivalent. It remains to show that only the last structure is a coboundary. Let us remind that a cocommutator δ given by

$$\delta(X) = i[1 \otimes X + X \otimes 1, \, r], \; r \in L \wedge L, \; X \in L \tag{21}$$

defines a coboundary Lie bialgebra if and only if r fulfills the modified classical Yang–Baxter equation

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, \, [[r, r]]] = 0, \quad X \in L, \tag{22}$$

where $[[r, r]]$ is the Schouten bracket

$$[[r, r]] \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}];$$

here $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$ *etc.* In our case

$$r = aP \wedge H + bP \wedge K + cH \wedge K \tag{23}$$

and

$$[[r, r]] = 3ic^2 P \wedge K \wedge H \tag{24}$$

and (22) holds for any values a, b, c; on the other hand classical Yang–Baxter equation implies $c = 0$. Eqs (21) and (23) give now

$$\begin{aligned} \delta(P) &= 0, \\ \delta(H) &= cH \wedge P, \\ \delta(K) &= cK \wedge P. \end{aligned} \tag{25}$$

Under the redefinition $H \to H$, $K \to K/c$, $P \to -P/c$ the above structure is converted to the one given in the last line of Table I. Finally, let us note that the classical r-matrix obeying CYBE defines here trivial cocommutator.

4. The Lie–Poisson structures on two-dimensional Galilei group

Let G be a Lie group, L–its Lie algebra, $\{X_i^R\}$ — the set of rightinvariant fields on G. Let

$$\eta(g) = \eta^{ij}(g) X_i \otimes X_j \tag{26}$$

be the map $G \to \wedge^2 L$. Then

$$\{\Psi, \Phi\} \equiv \eta^{ij}(g) X_i^R \Psi X_j^R \Phi \tag{27}$$

provides G with a Poisson–Lie group structure if and only if

$$(i) \quad \eta^{il} X_l^R \eta^{jk} + \eta^{kl} X_l^R \eta^{ij} + \eta^{jl} X_l^R \eta_{ki} - c_{lp}^j \eta^{il} \eta^{pk}$$
$$-c_{lp}^i \eta^{kl} \eta^{pj} - c_{lp}^k \eta^{jl} \eta^{pi} = 0 \,, \tag{28a}$$

$$(ii) \quad \eta(gh) = \eta(g) + Ad_g \eta(h) \,, \tag{28b}$$

η defines the bialgebra structure on L through

$$\delta(x) = \left. \frac{d\eta(e^{itx})}{dt} \right|_{t=0} \tag{29}$$

Our aim here is to find the Lie–Poisson structures which correspond, via (29), to the bialgebra structures found in previous Section. To this end we use Eq. (28b) to find η. Then the Poisson bracket is calculated according to Eq. (26) and Jacobi identities checked. In such a way, following Ref. [14] we avoid an explicit use of Eqs (28a). Define

$$\eta(a, v, \tau) = \lambda(a, v, \tau) P \wedge H + \mu(a, v, \tau) P \wedge K + \nu(a, v, \tau) H \wedge K \,. \tag{30}$$

Eq. (28b) gives

$$\begin{aligned}
\eta(a \;+\; a' + v\tau', v + v', \tau + \tau') &= (\lambda(a, v, \tau) + \lambda(a', v', \tau') \\
&+ \tau\nu(a', v', \tau')) P \wedge H \\
&+ (\mu(a, v, \tau) + \mu(a', v', \tau') - v\nu(a', v', \tau')) P \wedge K \\
&+ (\nu(a, v, \tau) + \nu(a', v', \tau')) H \wedge K \,. \tag{31}
\end{aligned}$$

Consequently, one obtains the following set of equations determining λ, μ, ν

$$\begin{aligned}
\lambda(a + a' + v\tau', v + v', \tau + \tau') &= \lambda(a, v, \tau) + \lambda(a', v', \tau') + \tau\nu(a', v', \tau') \,, \\
\mu(a + a' + v\tau', v + v', \tau + \tau') &= \mu(a, v, \tau) + \mu(a', v', \tau') - v\nu(a', v', \tau') \,, \\
\nu(a + a' + v\tau', v + v', \tau + \tau') &= \nu(a, v, \tau) + \nu(a', v', \tau') \,. \tag{32}
\end{aligned}$$

Due to

$$(a, v, \tau) = (0, 0, \tau) \star (a, 0, 0) \star (0, v, 0) \,,$$

it is sufficient to find η for one–parameter subgroups generated by H, P and K. We get, respectively:

$$\lambda(0, 0, \tau + \tau') = \lambda(0, 0, \tau) + \lambda(0, 0, \tau') + \tau\nu(0, 0, \tau'),$$
$$\mu(0, 0, \tau + \tau') = \mu(o, o, \tau) + \mu(0, 0, \tau'),$$
$$\nu(0, 0, \tau + \tau') = \nu(0, 0, \tau) + \nu(0, 0, \tau') \tag{33a}$$

with

$$\lambda(0, 0, \tau) = \tfrac{b}{2}\tau^2 + c\tau,$$
$$\mu(0, 0, \tau) = k\tau,$$
$$\nu(0, 0, \tau) = b\tau \tag{34a}$$

and

$$\lambda(0, v + v', 0) = \lambda(0, v, 0) + \lambda(0, v', 0),$$
$$\mu(0, v + v', 0) = \mu(0, v, 0) + \mu(0, v', 0) - v\nu(0, v', 0),$$
$$\nu(0, v + v', 0) = \nu(0, v, 0) + \nu(0, v', 0) \tag{33b}$$

with

$$\lambda(0, v, 0) = ev,$$
$$\mu(0, v, 0) = fv - \tfrac{d}{2}v^2,$$
$$\nu(0, v, 0) = dv \tag{34b}$$

as well as

$$\lambda(a + a', 0, 0) = \lambda(a, 0, 0) + \lambda(a', 0, 0),$$
$$\mu(a + a', 0, 0) = \mu(a, 0, 0) + \mu(a', 0, 0),$$
$$\nu(a + a', 0, 0) = \nu(a, 0, 0) + \nu(a', 0, 0) \tag{33c}$$

with

$$\lambda(a, 0, 0) = ga,$$
$$\mu(a, 0, 0) = ha,$$
$$\nu(a, 0, 0) = ja. \tag{34c}$$

Let us now use Eqs (34) to construct the general form of λ, μ and ν. We write

$$(a, v, \tau) = ((0, 0, \tau) \star (a, 0, 0)) \star (0, v, 0) \tag{35}$$

and use Eqs (32) to get

$$\lambda(a, v, \tau) = \frac{b}{2}\tau^2 + c\tau + d(v\tau - a) + ev,$$

$$\mu(a, v, \tau) = k\tau - ba + fv - \frac{d}{2}v^2,$$

$$\nu(a, v, \tau) = b\tau + dv. \tag{36}$$

Note that the number of free parameters in Eqs (36) is smaller than in Eqs (34). This is due to the fact that once the general forms of λ, μ, ν is calculated from Eqs (32) and (35) they have to be reinserted back in Eqs (32) which provide further constraints. These constraints arise because we have used a specific order of factors on the right-hand side of Eq. (35) so that the associativity has to be still imposed.

The general form of η is given by Eqs (30), (36). However, given an automorphism of Galilei group (which results also in some automorphism of its algebra) we can easily find its action on η. We call two η's equivalent if they are related by such an automorphism. Our next aim is to classify nonequivalent η's. The simplest way to do this is to find all maps η giving rise, via Eq. (29), to cocomutators δ classified in previous Section. Simple calculation gives

$$\delta(H) = -cP \wedge H - kP \wedge K - bH \wedge K,$$
$$\delta(P) = -dP \wedge H - bP \wedge K,$$
$$\delta(K) = eP \wedge H + fP \wedge K + dH \wedge K. \tag{37}$$

By comparing Eqs (10) and (37) we get

$$d = -r, \quad k = \alpha, \quad f = -\gamma,$$
$$b = s, \quad c = -\beta, \quad e = \rho. \tag{38}$$

Eqs (11) impose now further constraints on parameters entering η. This is because Eqs (28a) haven't been used yet.

Table I of Section 3 can be now used to generate all nonequivalent Lie-Poisson structures on two-dimensional Galilei group. However, one should keep in mind that the classification in Sec. 3 was given for dimensionless form of Lie algebra while here rather the generators with proper dimensions are necessary (of Eqs (5), (6)). This can be cured as described in Sec. 3.

Let us first write out the general form of Poisson bracket following from Eqs (6), (27) and (30); it reads

$$\{f, g\} = \lambda \left(\frac{\partial f}{\partial a} \frac{\partial g}{\partial \tau} - \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial a} \right) + \mu \left(\frac{\partial f}{\partial v} \frac{\partial g}{\partial a} - \frac{\partial f}{\partial a} \frac{\partial g}{\partial v} \right)$$
$$+ \nu \left(\frac{\partial f}{\partial \tau} \left(\frac{\partial g}{\partial v} + \tau \frac{\partial g}{\partial a} \right) - \left(\frac{\partial f}{\partial v} + \tau \frac{\partial f}{\partial a} \right) \frac{\partial g}{\partial \tau} \right) . \tag{39}$$

In particular

$$\{a, v\} = -\mu = -k\tau + ba - fv + \frac{d}{2}v^2 ,$$

$$\{a, \tau\} = \lambda - \nu\tau = -\frac{b}{2}\tau^2 + c\tau - da + ev ,$$

$$\{v, \tau\} = -\nu = -b\tau - dv .$$

Taking care about proper dimension we get finally the set of Lie–Poisson structures on two-dimensional Galilei group corresponding to the bialgebra structures classified in Sec. 3. It is given in Table II

TABLE II

	$\{a, v\}$	$\{a, \tau\}$	$\{v, \tau\}$	Remarks
1	$-\frac{\tau_0 v^2}{2}$	$\tau_0 a + \varepsilon\tau_0^2 v$	$\tau_0 v$	$\varepsilon \in \mathbb{R}$
2	$-v_0^2\tau - \varepsilon v_0\tau_0 v$	$-\varepsilon v_0\tau_0\tau + \tau_0^2 v$	0	$\varepsilon \geq 0$
3	$v_0^2\tau - \varepsilon v_0\tau_0 v$	$-\varepsilon v_0\tau_0\tau - \tau_0^2 v$	0	$\varepsilon \geq 0$
4	$v_0^2\tau - \varepsilon v_0\tau_0 v$	$-\varepsilon v_0\tau_0\tau + \tau_0^2 v$	0	$\varepsilon \geq 0$
5	0	$\tau_0^2 v$	0	
6	$-v_0\tau_0 v$	$-v_0\tau_0\tau + \tau_0^2 v$	0	
7	0	$-\tau_0^2 v$	0	
8	$-v_0\tau_0 v$	$-v_0\tau_0\tau - \tau_0^2 v$	0	
9	$-v_0\tau_0 v$	$-\tau_0 v_0\tau$	0	

here v_0, τ_0 are arbitrary but <u>fixed</u> nonzero constants while ε–dimensionless parameter. We have checked explicitly the Jacobi identities as well as Poisson–Lie property for all the above cases.

5. Conclusions

We have classified above all bialgebra structures on two-dimensional Galilei algebra and found the corresponding Lie–Poisson structures on Galilei group.

It is worthwhile to stress the following point. In order to impose a Lie–Poisson structure on Galilei group one needs two dimensionful constants. They can attain arbitrary nonzero values, different choices being related by automorphisms. The only relevant free parameter is dimensionless parameter ε; different values of ε correspond to nonequivalent Lie–Poisson structures.

One should also note that the relatively rich family of nonequivalent Lie–Poisson structures considered here contains only one coboundary; this is in sharp contrast with semisimple case [15] as well as the case of fourdimensional Poincare group [16, 17].

The more challenging problem is the classification of all Lie–Poisson structures (resp. bialgebra structures) on fourdimensional Galilei group (resp. algebra). This problem will be considered in forthcoming paper.

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