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SEMI-ANALYTICAL THIRD-ORDER CALCULATIONS OF THE SMALL-ANGLE BHABHA CROSS SECTIONS*

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We integrate analytically the total cross section of the small-angle Bhabha scattering over the complete multiple photon phase space. Some inclusive distributions are also obtained. The differential distributions are taken from the Monte Carlo event generator BHLUMI and correspond to the second-order matrix element with Yennie-Frautschi-Suura (YFS) exponentiation. In the integration we control terms up to leading third-order and sub-leading second-order, in the leading-logarithmic approximation. The analytical results provide a vital cross-check of the correctness of the BHLUMI program. The analytical and Monte Carlo results agree to within 1.7×10^{-4}. On the other hand, the calculation gives us unique insight into the relation between exclusive YFS exponentiation and naive inclusive exponentiation.

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1. Introduction

The Bhabha scattering process $e^+e^- \to e^+e^-$ at LEP energies consists in fact of two distinct processes (especially at the Z peak): one is the

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Small-Angle Bhabha (SABH) process below about 6° in scattering angle, which is dominated by the gamma t-channel exchange and another one, the Large-Angle Bhabha (LABH) process above 6°, which gets important contributions from various s-channel (annihilation) exchanges. The SABH process is employed almost exclusively to determine the luminosity in the e^+e^- colliders, using dedicated luminometer sub-detectors. The LABH provides input data for precision electroweak tests of the Standard Model (SM), in particular the electron partial width Γ_e of the Z boson. In this work we shall concentrate on the SABH process at LEP. At $\sqrt{s} = M_Z$, in the 1°–3° angular range it gives about four times more events than Z decays. It is therefore ideally suited for precise measurements of the luminosity from the point of view of the statistical error. Even more important; it is dominated by "known physics", that is by t-channel exchange of a photon; it is therefore calculable from "first principles", *i.e.* from the Lagrangian of the Quantum Electrodynamics (QED) with methods of the standard Quantum Field Theory such as Feynman diagrams, *etc.* This work presents a major contribution to the problem of a reliable theoretical calculation of the SABH process with a precision of 0.1% or better. Let us now review briefly the main aspects of our work.

1.1. Theoretical error in the luminosity

At present, the luminosity measurement at LEP using the SABH process has a very small statistical and experimental systematic error, typically 0.07-0.11% [1] for the single LEP experiments and 0.05% for combined LEP results [2–4]. The uncertainty of the theoretical calculation of the SABH process has to be combined with this error. It is called the "theoretical error" (the theory uncertainty) of the luminosity. Last year, it was reduced to 0.16% [5] and is now at the level of 0.11% [6, 7]: in spite of the progress, it is still a dominant component of the total luminosity error. This error enters into that of the total cross section measured at LEP. The experimental precision of the so-called invisible width of Z (number of neutrinos) is strongly dependent on the precision of the luminosity measurement. The other quantities used for tests of the SM are also affected. An example of the influence of the luminosity error on the LEP measurable used in the test of the SM is illustrated in Table 20 of Ref. [6]; see also Ref. [7].

Obviously it would be worthwhile to lower the theoretical uncertainty in the calculation of the SABH cross section below the combined experimental precision of the LEP experiments, which is already at the level of 0.05%. From the beginning of the LEP operation both experimental and theoretical components in the error of the luminosity went from the level of 2% to 0.1%. Why was it always difficult to reduce the theoretical error even further? The main obstacles were the need for non-trivial calculations of the higher-order contributions and the complicated Event Selection (ES) in the actual measurement. Due to the complicated ES, the phase-space boundaries in the calculation of the SABH cross section are too complicated for any analytical calculation: the calculation has to be numerical, most desirably in the form of Monte Carlo event generator (MCEG). How does one improve the precision of the theoretical calculation of the SABH cross section? Of course, one has to add higher-order terms in the perturbative expansion. On the other hand, the theoretical calculation would be completely useless if in the calculation of the SABH cross section we did not control its "technical precision", corresponding to *all* possible numerical uncertainties. The control over the technical precision is probably the most difficult and labour-consuming part of the whole enterprise. The present paper is addressing both questions — we demonstrate new methods of determining the technical precision of the theoretical calculation of the SABH cross section and we add certain numerically important higher-order contributions in the perturbative expansion. For an up-to-date review on the precise calculations of the luminosity cross section we refer the reader to Ref. [8].

1.2. The BHLUMI Monte Carlo event generator

In the last five years the LEP collaborations have used the BHLUMI MCEG in order to calculate the SABH cross section for any type of experimental ES. The program, originally written in 1988 [9], was published for the first time in 1992 [10], at that time with the first-order exponentiated QED matrix element, $\mathcal{O}(\alpha^1)_{\rm exp}$ (exponentiation according to the Yennie–Frautschi–Suura theory). BHLUMI provides multiple soft and hard photons in the complete phase-space in all versions. The Monte Carlo multi photon integration over the multi photon phase-space was slightly improved over the years, but its basic Monte Carlo algorithm has remained essentially unchanged since the first version. Gradual improvements concern mainly the matrix element; this was recently upgraded by adding the missing second order in the leading-logarithmic (LL) approximation [11]. The new matrix element is fully described and discussed in this work. The aim of the analytical integrations over the phase-space presented here is to cross-check if the new matrix element has indeed the correct second-order LL behaviour and whether it is correctly implemented in the version 4.04 of the BHLUMI MC program [11]. The BHLUMI package also includes the LL sub-generator LUMLOG, featuring a strictly collinear emission of photons in the initial and final states. The third-order LL analytical results of the present work are also implemented in the newest LUMLOG. More and more cross-checks are built up in order to better determine BHLUMI's technical precision, see [5, 12].

This paper contributes substantially to all above-mentioned cross-checks — in fact these are based on the calculations presented here.

The aim of this work is to consolidate these earlier calculations and to present the complete results in a single, self-contained paper.

1.3. Importance of the various QED corrections

The electron mass is very small and the LL approximation in terms of the big logarithm $L = \ln(|t|/m_e^2) - 1$ is a very useful tool. In Table I we show numerical values of the "canonical coefficients" for various LL and sub-leading QED radiative corrections. As we see from the table, for a precision of order 0.5% it is enough to include the entire first-order $\mathcal{O}(\alpha)$ and the second order leading-log $\mathcal{O}(\alpha^2 L^2)$, while at the present precision, of order 0.05%–0.10%, it is necessary to have control over $\mathcal{O}(\alpha^2 L)$ and $\mathcal{O}(\alpha^3 L^3)$. The contributions of $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha^3 L^2)$ and $\mathcal{O}(\alpha^4 L^4)$, not shown in the table, are definitely below the level of interest of 10^{-4}. In the present version of BHLUMI 4.04 or Ref. [11] we have complete control over $\mathcal{O}(\alpha^3 L^3)$ *photonic* (bremsstrahlung) contributions (thanks to results of this work) while $\mathcal{O}(\alpha^2 L)$ is still incomplete.

TABLE I

The canonical coefficients in units of 10^{-3} indicating the generic magnitude of various leading and sub-leading contributions up to third order. The big-log $L = \ln(|t|/m_e^2) - 1$ is calculated for $\theta_{\min} = 30$ mrad and $\theta_{\min} = 60$ mrad and for two values of the centre-of-mass energy: at LEP1 ($\sqrt{s} = M_Z$), where the corresponding values of $|t| = (s/4)\theta_{\min}^2$ are 1.86 and 7.53 GeV^2, and at a LEP2 energy ($\sqrt{s} = 200$ GeV), where the corresponding values of $|t|$ are 9 and 36 GeV^2, respectively.

		$\theta_{\min} = 30$ mrad		$\theta_{\min} = 60$ mrad	
		LEP1	LEP2	LEP1	LEP2
$\mathcal{O}(\alpha L)$	$\frac{\alpha}{\pi}4L$	137	152	150	165
$\mathcal{O}(\alpha)$	$2\frac{1}{2}\frac{\alpha}{\pi}$	2.3	2.3	2.3	2.3
$\mathcal{O}(\alpha^2 L^2)$	$\frac{1}{2}\left(\frac{\alpha}{\pi}4L\right)^2$	9.4	11	11	14
$\mathcal{O}(\alpha^2 L)$	$\frac{\alpha}{\pi}\left(\frac{\alpha}{\pi}4L\right)$	0.31	0.35	0.35	0.38
$\mathcal{O}(\alpha^3 L^3)$	$\frac{1}{3!}\left(\frac{\alpha}{\pi}4L\right)^3$	0.42	0.58	0.57	0.74

For the purpose of the present paper we shall denote the entire first-order $\mathcal{O}(\alpha)$ plus the second order leading-log $\mathcal{O}(\alpha^2 L^2)$ as the *second order pragmatic* approximation, $\mathcal{O}(\alpha^2)_{\rm prag}$ in short. Adding $\mathcal{O}(\alpha^2 L)$ and $\mathcal{O}(\alpha^3 L^3)$ brings us to the *third-order pragmatic* approximation, $\mathcal{O}(\alpha^3)_{\rm prag}$.

The present work provides all of the ingredients for a definite answer about the importance of an $\mathcal{O}(\alpha^3 L^3)$ contribution of the pure bremsstrahlung type, *i.e.* of the so-called *photonic* type. The Monte Carlo tool (the LUMLOG event generator) for calculating $\mathcal{O}(\alpha^3 L^3)$ corrections for arbitrary ES is included in the version 4.04 of the BHLUMI package [11]. Numerical results obtained using analytical formulas from this work (implemented in LUMLOG) were already shown in Ref. [12], and the question of the importance of the photonic $\mathcal{O}(\alpha^3 L^3)$ corrections seems to be closed. In the matrix element presented in this work and used in BHLUMI, the $\mathcal{O}(\alpha^2 L)$ contributions are still incomplete. The first attempt at its direct numerical evaluation for realistic ES was presented in Ref. [13] and this work is still in progress.

1.4. Why analytical integration?

The main content of this work is the analytical integration of the matrix element, exactly the same as in the last version of BHLUMI, over the phase-space, keeping in the calculation *the exact soft photon behaviour* and all terms up to $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$. The first immediate question is: Since the matrix element has only correct terms[1] up to $\mathcal{O}(\alpha^2 L^2)$, why bother to trace exactly all terms of $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$? There are basically two important reasons: (a) our principal aim is to get in the future version of BHLUMI all these terms completed; (b) we have to keep in mind that ultimately we have to have control over the technical precision of BHLUMI down to 10^{-4}. The comparison of the analytical and Monte Carlo phase-space integrations down to the 10^{-4} precision level is *not possible* without accounting for terms of $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$, even if they are (perturbatively) incomplete! The actual status of terms of $\mathcal{O}(\alpha^3 L^3)$ is even more interesting than described above: the analytical calculation presented in this work can be used to show that the incompleteness of the $\mathcal{O}(\alpha^3 L^3)$ is numerically at the level of 2×10^{-4} [12] and therefore it is not even worth upgrading the BHLUMI matrix element to full $\mathcal{O}(\alpha^3 L^3)$. A similar conclusion for $\mathcal{O}(\alpha^2 L)$ is not yet reached[2].

In spite of its importance and usefulness we want to stress that the analytical integration is not a substitute for the Monte Carlo. As we shall see, it will be limited to one or two examples of the rather unrealistic ESs.

[1] The terms beyond $\mathcal{O}(\alpha^2 L^2)$ in the BHLUMI matrix element are present. The terms of $\mathcal{O}(\alpha^2 L)$ are based on an ansatz and of $\mathcal{O}(\alpha^3 L^3)$ are generated by exponentiation.

[2] It is not yet clear if we shall be able to argue that the missing $\mathcal{O}(\alpha^2 L)$ contribution is negligible for a wide class of ESs, because it seems to depend more strongly on the type of the ESs than does the missing $\mathcal{O}(\alpha^3 L^3)$.

We would like also to give justice to the authors of a classical paper Yennie, Frautschi and Suura [14], as an early precursor of the analytical approach presented here. These authors also integrate analytically over the real single photon phase-space using the LL approximation and taking into account "spectator" soft photons. This is very much in the spirit of the present work. The important difference is that we keep track of two more orders in the LL approximation and we keep account also of NLL terms up to second order. Much as they did, we regard the analytical integration within LL+NLL as a pure technical method[3] of dealing with the phase-space integration. In 1961 the analytical approximate calculation over the multi photon phase-space was *the only* available method — it was unthinkable at the time that such integrals could be evaluated exactly using numerical methods! (The precision requirements were anyway at that time at the level of a few per cent only.) Nowadays, we are in a much more comfortable situation — we can evaluate such integrals without any approximation using Monte Carlo methods[4], and the analytical calculation is only an additional useful tool to test the Monte Carlo program.

1.5. Exclusive YFS exponentiation

The analytical integration presented here sheds light also on the question of the relation between "exclusive YFS exponentiation" and so-called "inclusive naive exponentiation". In the exclusive YFS exponentiation formulated in Ref. [14] and later implemented in several Monte Carlo event generators [9, 16, 17] the summation of infrared singularities to infinite order is done at the level of the differential distributions, before the phase-space integration, including an arbitrary number of hard photons all over the entire phase-space, without any artificial distinction between hard and soft photons. This leads to complicated phase-space integrals, which in 1961 could be dealt with only using an approximate analytical approach. Nowadays we can evaluate these phase-space integrals using MC methods and even provide in this way a MC event generator.

In many works, the analytical partial results from Ref. [14] were used to devise an *ad hoc* method, which we refer to as the "naive inclusive exponentiation". In this method one takes a finite-order (typically $\mathcal{O}(\alpha^1)$) inclusive (partially integrated) distribution, typically one-dimensional, for instance the total energy lost due to photon emission, and combines it by means of extrapolation with the analytical result similar, or identical, to results in

[3] Quite often, casual readers of Ref. [14] get the wrong impression that the YFS exponentiation is limited to the LL approximation because this approximation was used there for the integration over the phase-space.

[4] The first example of such a calculation was presented in Ref. [15].

Ref. [14] for the same distribution, but with an infinite number of photons being soft, see Ref. [18] for more discussion.

The interesting question always was: can this obviously *ad hoc* method, without any systematic generalization to higher-orders, be put on more solid ground? The answer is Yes, provided that we do not take fragments of the results in Ref. [14], but rather the relevant YFS multi photon integral as it stands and calculate the relevant inclusive distribution analytically. It is not a simple task but, as we shall see in this work (and some examples were already given in Ref. [14]), it is possible, provided we integrate over the phase-space using a numerically reasonable approximation. Not surprisingly, the resulting inclusive distribution obtained by the integration over the exclusive YFS fully differential distributions often looks quite similar to the result of the typical "naive inclusive exponentiation". The advantage of our method is, however, that the procedure is unique, well understood, with a definite meaning at every perturbative order. In one word, *the naive inclusive exponentiation gets replaced by the analytical phase-space integration.* The analytical calculation of this paper represents a perfect example of such an approach.

1.6. Outline

In the following Section 2 we shall describe in full detail the new matrix element used in BHLUMI, which will be integrated analytically over the phase-space. In Section 3 we demonstrate the basics of our analytical integration technique using the simple case of the contribution from the so-called $\bar{\beta}_0$ as an example. In Sections 4 and 5 we calculate further components due to the so-called $\bar{\beta}_1$ and $\bar{\beta}_2$; the total result is presented in Section 6. Cross checks of the LL part are done in Section 7 and another variant of the calculation for calorimetric ES is discussed briefly in Section 8. Appendix A proves the correctness of certain basic approximations.

2. Differential distributions

In this Section we define completely and exactly the differential multi photon distributions used in BHLUMI 4.04 and used later in this work as a starting point for the analytical integrations over the phase-space. There are two types (A) and (B) of such a matrix element, which coincide within $\mathcal{O}(\alpha^2)_{\rm prag}$. For the first time they were defined explicitly in Ref. [5]. Choice (A) is the natural extension to second order of the matrix element implemented in the BHLUMI 2.02 version of Ref. [10]. The new choice (B), which is a starting point for this work, looks at first sight more complicated, but it turns out to be integrable analytically more easily, especially beyond

$\mathcal{O}(\alpha^2)_{\rm prag}$. The numerical difference between cross sections obtained with the matrix elements (A) and (B) at the level of $\mathcal{O}(\alpha^2)_{\rm exp}$ is very small, below 0.01%. It is quite sizeable at the $\mathcal{O}(\alpha^1)_{\rm exp}$, up to 0.3%. The relevant MC numerical result will be shown at the end of the paper.

2.1. Second order — no exponentiation

The second-order integrated cross section for the process $e^-(p_1)+e^+(q_1)\rightarrow e^-(p_2)+e^+(q_2)+n\gamma k_j+n'\gamma(k_l)$ reads in terms of the Lorenz phase-space integration over the differential distribution as follows:

$$\sigma^{(r)} = \sum_{0\le n+n'\le r} \frac{1}{n!}\frac{1}{n'!} \int \frac{d^3p_2}{p_2^0} \int \frac{d^3q_2}{q_2^0} \prod_{j=1}^{n} \int_{k_j\notin\Omega_U} \frac{d^3k_j}{k_j^0} \prod_{l=1}^{n'} \int_{k'_l\notin\Omega_L} \frac{d^3k'_l}{k'^0_l}$$
$$\times\, \delta^{(4)}\Big(p_1-p_2+q_1-q_2-\sum_{j=1}^{n} k_j - \sum_{l=1}^{n'} k'_l\Big) D^{(r)}_{[n,n']}(k_1...k_n;k'_1...k'_{n'})\,, \qquad (1)$$

where $\Omega_{U,L}$ are 3-dimensional regions around the infrared soft singularity excluded from the phase-space. Usually, this is done by requiring the photon energy to be above some value $E_{\rm max}$ in a certain reference frame[5]. Virtual contributions and real soft photon contributions below $E_{\rm max}$ (regularized typically with photon mass λ) are combined and are included in the corresponding $D^{(r)}_{[n,n']}$.

Let us show explicitly all the $\mathcal{O}(\alpha^r)$ $r=0,1,2$ distributions $D^{(r)B}_{[n,n']}$ for the new type (B) of matrix element. At the end of this Section we also show the older choice $D^{(r)A}_{[n,n']}$, which in the $\mathcal{O}(\alpha)$ coincides with the matrix element implemented in the BHLUMI 2.x version of Ref. [10]. The older choice (A) is simpler, but its serious disadvantage is that it cannot be integrated analytically beyond $\mathcal{O}(\alpha^2)_{\rm prag}$ using methods presented in this work.

Let us start defining various components of the differential distribution with expressions for the functions $D^{(r)B}_{[0,0]}$, $r=0,1,2$ in the $\mathcal{O}(\alpha^r)_{\rm prag}$; see Sect. 1.3 for the definition of the $\mathcal{O}(\alpha^r)_{\rm prag}$ approximations. It is given simply by

$$D^{(r)B}_{[0,0]} = \frac{4\pi\alpha^2}{t_p t_q}\, b_0\,(1+v^{(r)}),$$
$$b_0 = \frac{1}{2}(1+(1-\xi)^2), \quad \xi=\frac{|t|}{s},$$

[5] The reference frame might be different for upper and lower lines — provided the upper/lower line interference is neglected.

$$
\begin{aligned}
v^{(0)} &= 0, \quad v^{(1)} = 2\gamma \ln \Delta + \frac{3}{2}\gamma - \frac{\alpha}{\pi}, \\
v^{(2)} &= 2\gamma \ln \Delta + 2\gamma^2 \ln^2 \Delta + (1 + 2\gamma \ln \Delta)\left(\frac{3}{2}\gamma - \frac{\alpha}{\pi}\right) + \frac{9}{8}\gamma^2 - \frac{3}{2}\frac{\alpha}{\pi}\gamma,
\end{aligned}
\tag{2}
$$

where

$$
\gamma = 2\frac{\alpha}{\pi}\left(\ln\frac{1}{\delta} - 1\right), \quad \delta = \frac{m_e^2}{|t|}, \quad t = (p_1 - p_2 - \sum k_i)^2. \tag{3}
$$

The case of $r = 0$ with $v^{(0)} = 0$ represents the Born approximation, $v^{(1)}$ represents the exact $\mathcal{O}(\alpha^1)$ from the Feynman rules (up-down interference excluded), while $9\gamma^2/8$ in $v^{(2)}$ stands for the LL approximation in the $\mathcal{O}(\alpha^2)$. At this point we should define the infrared domains $\Omega_{U,L}$ and the infrared cut parameter Δ. Let us postpone their definition to the moment when we define $D^{(r)B}_{[1,0]}$. Let us only remark now that the virtual corrections in Eq. (2) are given for the $\Omega_{U,L}$, which are exactly the same as in the actual Monte Carlo phase-space algorithm — the cut on the photon energy is done in the Breit (rest) frame of the $p_1 + p_2$ or $q_1 + q_2$; the minimum energy of the real photon in such a frame is $\sim \Delta\frac{1}{2}\sqrt{2p_1p_2}$.

Let us now define the $\mathcal{O}(\alpha)_{\rm prag}$ and $\mathcal{O}(\alpha^2)_{\rm prag}$ single real photon emission distributions $D^{(r)}_{[1,0]}(k_1)$ and $D^{(r)}_{[0,1]}(k_1)$, $r = 1, 2$. In the following we explicitly show expressions for the upper line emission part $D^{(r)}_{[1,0]}(k_1)$. The lower line distribution $D^{(r)}_{[0,1]}(k_1)$ is defined in a completely analogous way. The first-order distribution ($r = 1$) has no virtual corrections (tree level) and the second-order distribution ($r = 2$) includes the one-loop virtual photon correction, which is calculated in the LL approximation[6]. The case of the type (B) matrix element reads as follows

$$
\begin{aligned}
D^{(r)B}_{[1,0]}(k_1) &= \frac{4\pi\alpha^2}{t_p t_q}\,\tilde{S}_p(\tilde{\alpha}_1, \tilde{\beta}_1)\left(1 + v^{(r)}_{[1,0]}(\tilde{\alpha}_1, \tilde{\beta}_1)\right) H(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p), \\
v^{(1)}_{[1,0]} &= 0, \\
v^{(2)}_{[1,0]} &= (\gamma_p + \gamma_q)\ln\Delta + \frac{3}{2}\gamma - \frac{\alpha}{\pi} - \frac{3}{4}\gamma\ln(1 - \tilde{\beta}_1) - \frac{1}{4}\gamma\ln(1 - \tilde{\alpha}_1) \\
&\quad + (\gamma_p - \gamma)\left(\frac{1}{4} - \ln(1 - \tilde{\beta}_1)\right), \\
v_1 &= \tilde{\alpha}_1 + \tilde{\beta}_1 - \tilde{\alpha}_1\tilde{\beta}_1,
\end{aligned}
\tag{4}
$$

[6] This is done by convolving twice the non-singlet Altarelli–Parisi kernel with itself, see Ref. [19] for many examples of such a procedure.

$$
\begin{aligned}
H(\tilde\alpha_1,\tilde\beta_1,\psi_p) &= (1+\delta_p^*(\tilde\alpha_1,\tilde\beta_1))\,\frac{1}{4}\Big[1+\Big(1-\frac{\xi}{1-\tilde\alpha_1}\Big)^2\\
&\quad +R(\tilde\alpha_1,\tilde\beta_1,\psi_p)^2+(R(\tilde\alpha_1,\tilde\beta_1,\psi_p)-(1-\tilde\beta_1)\xi)^2\Big],\\
R(\tilde\alpha_1,\tilde\beta_1,\psi_p) &= (1-\tilde\alpha_1)(1-\tilde\beta_1)\\
&\quad +2\cos\psi_p\sqrt{\tilde\alpha_1\tilde\beta_1(1-\tilde\alpha_1)(1-\tilde\beta_1)}+\tilde\alpha_1\tilde\beta_1,\\
\delta_p^*(\tilde\alpha_1,\tilde\beta_1) &= \delta_p\frac{1}{\tilde\alpha_1\tilde\beta_1}\,\frac{(\tilde\alpha_1^2+\tilde\beta_1^2)^2}{(1+(1-\tilde\alpha_1)^2(1-\tilde\beta_1)^2)},\\
\gamma_p &= 2\frac{\alpha}{\pi}\Big(\ln\frac{1}{\delta_p}-1\Big),\ \gamma_q = 2\frac{\alpha}{\pi}\Big(\ln\frac{1}{\delta_q}-1\Big),\\
\delta_p &= m_e^2/|t_p|,\ \delta_q = m_e^2/|t_q|,\ t_p=(p_1-p_2)^2,\ t_q=(q_1-q_2)^2, \qquad (5)
\end{aligned}
$$

where

$$
\tilde S_p(k_1)=\frac{\alpha}{4\pi^2}\left(\frac{2p_1p_2}{(kp_1)(kp_2)}-\frac{m^2}{(k_1p_1)^2}-\frac{m^2}{(k_1p_2)^2}\right) \qquad (6)
$$

is the standard Yennie–Frautschi–Suura soft factor. Note that $H=(s^2+u^2+s_1^2+u_1^2)/(4s^2)$ in the notation of Ref. [10]. The last term in $v^{(2)}_{[1,0]}$ proportional to $(\gamma_p-\gamma)$ is pure sub-leading and it is added *ad hoc* in order to simplify the exponentiated version of the matrix element, see below.

Since the above distributions are given in terms of the normalized Sudakov variables $\tilde\alpha_i$ $\tilde\beta_i$ of paper [9], let us therefore recall their definition. Below we define them in the more general case of n photons emitted from the upper line. In the t-channel Breit (rest) frame QRS$_p$, where $p_1^0=p_2^0=E_p$, $\vec p_1+\vec p_2=0$ and $Q_p=p_2-p_1=(0,0,0,2E_p)$, we define

$$
\begin{aligned}
k_i^0 &= (\alpha_i+\beta_i)E_p\,, \quad k_i^3=(-\alpha_i+\beta_i)E_p\,,\\
k_i^1 &= k_T\cos\phi_i\,, \quad k_i^2=k_T\sin\phi_i\,,\quad k_T=2E_p\sqrt{\alpha_i\beta_i}\,,\\
\alpha_i &= \tilde\alpha_i K_p\,, \quad \beta_i=\tilde\beta_i K_p\,,\\
K_p &= \Big(1-\sum_{j=1}^n\tilde\beta_j\Big)^{-1}=p_1(p_2+\sum_{j=1}^n k_j)/p_1p_2\,. \qquad (7)
\end{aligned}
$$

The angle ψ_p (present implicitly in the formulas of Ref. [9]) is defined in another frame QMS_p, where $Q=(0,0,0,|t|)$ and $p_1=(E_1,0,0,-|p_1|)$. It is an angle between two planes, one spanned with $\vec Q$ and $\vec p_2$ and another spanned with $\vec Q$ and $\vec q_1$.

From the perturbative calculation point of view let us remark that in Eq. (4) the only one-loop $\mathcal{O}(\alpha^2)$ virtual correction $v^{(2)}_{[1,0]}$ comes from the LL ansatz — the rest originates from the tree-level $\mathcal{O}(\alpha^1)$ Feynman diagrams. Two approximations were employed: rejection of the up-down interference and of some mass terms giving rise to $\mathcal{O}(m_e^2/|t|)\sim 10^{-7}$ contributions in the integrated cross section. Let us also stress that in the $v^{(2)}_{[1,0]}$ a term proportional to $\ln\Delta$ is totally constrained by the YFS soft limit derived in Ref. [14]. We obtain the logarithmic terms in $v^{(2)}_{[1,0]}$ from a double convolution of the non-singlet Altarelli–Parisi splitting function and the non-logarithmic term is chosen arbitrarily (such that, later, the expression for $\bar{\beta}_1$ is simpler).

Now, having defined all kinematics, we are able to define the infrared domains

$$\{\Omega_U : \max(\tilde{\alpha}_i,\tilde{\beta}_i) < \Delta\}, \qquad \{\Omega_L : \max(\tilde{\alpha}'_j,\tilde{\beta}'_j) < \Delta\}, \tag{8}$$

which enter in many places throughout our calculation. Needless to say, nothing depends on the actual choice of $\Omega_{U,L}$ and we always witness perfect cancellations of the infrared real and virtual divergences.

The distribution describing double photon emission from the upper line is pure $\mathcal{O}(\alpha^2)$ and it is based on the LL ansatz. The main ingredient in the ansatz is the double convolution of the non-singlet splitting kernel for the longitudinal momenta. Both photons feature non-collinear transverse momentum distributions. It is very important that we require the distribution to have the correct soft limit in the case when both photons are soft and in the case when one of them is soft and the other one is hard. This is fulfilled by taking the product of the $\mathcal{O}(\alpha)$ single photon distribution for the harder photon and the soft factor times the splitting kernel for the softer one. The advantage of the LL ansatz (as compared to exact expressions known in the literature) is that it is quick in the computer evaluation and its LL content is manifest. There is obviously freedom in the choice of the above LL ansatz (up to non-infrared $\mathcal{O}(\alpha^2 L)$ terms). It was exploited in such a way that the analytical integration (which is our main goal) over the phase-space in the later stage is feasible and as simple as possible. The complete ansatz for the type (B) distribution reads as follows:

$$\begin{aligned}
&D^{(2)B}_{[2,0]}(\tilde{\alpha}_1,\tilde{\beta}_1,\phi_1,\tilde{\alpha}_2,\tilde{\beta}_2,\phi_2) = \frac{4\pi\alpha^2}{t_p t_q}\\
&\times\Bigg[\tilde{S}_p(\tilde{\alpha}_1,\tilde{\beta}_1)\tilde{S}_p(\tilde{\alpha}_2,\tilde{\beta}_2)\theta((\tilde{\alpha}_1-\tilde{\beta}_1)(\tilde{\alpha}_2-\tilde{\beta}_2))\\
&\times\Bigg\{\theta(v_1-v_2)\ \tfrac{1}{2}\ \Big[H(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)\chi(v_2^*)+H(\tilde{\alpha}_1^*,\tilde{\beta}_1^*,\psi_p)\chi(v_2)\Big]\\
&+\theta(v_2-v_1)\ \tfrac{1}{2}\ \Big[H(\tilde{\alpha}_2,\tilde{\beta}_2,\psi_p)\chi(v_1^*)+H(\tilde{\alpha}_2^*,\tilde{\beta}_2^*,\psi_p)\chi(v_1)\Big]\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+\tilde{S}_p(\tilde{\alpha}_1,\tilde{\beta}_1)\tilde{S}_p(\tilde{\alpha}_2,\tilde{\beta}_2)\theta(\tilde{\alpha}_1-\tilde{\beta}_1)\theta(-\tilde{\alpha}_2+\tilde{\beta}_2)\\
&\times\Big\{\theta(v_1-v_2)H(\tilde{\alpha}_1^*,\tilde{\beta}_1^*,\psi_p)\chi(v_2)+\theta(v_2-v_1)H(\tilde{\alpha}_2,\tilde{\beta}_2,\psi_p)\chi(v_1^*)\Big\}\\
&+\tilde{S}_p(\tilde{\alpha}_1,\tilde{\beta}_1)\tilde{S}_p(\tilde{\alpha}_2,\tilde{\beta}_2)\theta(-\tilde{\alpha}_1+\tilde{\beta}_1)\theta(\tilde{\alpha}_2-\tilde{\beta}_2)\\
&\times\Big\{\theta(v_1-v_2)H(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)\chi(v_2^*)+\theta(v_2-v_1)H(\tilde{\alpha}_2^*,\tilde{\beta}_2^*,\psi_p)\chi(v_1)\Big\}\Big]\,, \qquad (9)
\end{aligned}
$$

where $\chi(x)=\frac{1}{2}(1+(1-x)^2)$ and, because of the "cascade" character of the double emission, we use "starred" variables, defined by:

$$
\begin{aligned}
v_1^* &= \frac{v_1}{(1-v_2)}\,, & v_2^* &= \frac{v_2}{(1-v_1)}\,,\\
\tilde{\alpha}_1^* &= \frac{\tilde{\alpha}_1}{(1-v_2)}\,, & \tilde{\alpha}_2^* &= \frac{\tilde{\alpha}_1}{(1-v_1)}\,,\\
\tilde{\beta}_1^* &= \frac{\tilde{\beta}_1}{(1-v_2)}\,, & \tilde{\beta}_2^* &= \frac{\tilde{\beta}_1}{(1-v_1)}\,. \qquad (10)
\end{aligned}
$$

We can easily check that we reproduce the proper soft photon limit in the case (1) when both photons are soft, *i.e.* $v_1,v_2\to 0$ and (2) when one photon is hard and one is soft, for instance $v_1\to 0$ and $v_2=\text{const}$. The first requirement is quite natural and is fulfilled trivially because in the soft limit $v_i\to 0$ both $H(\tilde{\alpha}_i,\tilde{\beta}_i,\psi_p)\to 1$ and $\chi(v_i)\to 1$, $i=1,2$. The second requirement is fulfilled thanks to the ordering of the energies. Do we reproduce the proper collinear (leading-log) limit when both photons are hard and collinear? If the first photon is hard and collinear, v_1 is finite and we have either $\tilde{\alpha}_1\to 0$ or $\tilde{\beta}_1\to 0$ (not both). For two hard collinear photons, we encounter two situations: (a) both photons are collinear with the same fermion (initial electron or final electron), (b) each photon is collinear with a different fermion, *i.e.* one is associated with the initial-state (beam) electron and the other with the final-state electron. We can easily check that for our ansatz both in the case of the collinear limit (a):

$$
b_{2UU}\sim\tfrac{1}{2}[\chi(v_1)\chi(v_2^*)+\chi(v_2)\chi(v_1^*)], \qquad (11)
$$

and in the anticollinear case (b):

$$
b_{2UU}\sim\chi(v_1^*)\chi(v_2) \qquad (12)
$$

we recover expressions expected from the convolution of the LL kernels. Let us note that the following property $H(\tilde{\alpha}_i,\tilde{\beta}_i,\psi_p)\to\chi(v_i)$ was instrumental in obtaining the above proper collinear limits.

Finally, let us turn to the LL ansatz for the double photon emission distribution for the simultaneous emission of one photon from the upper line and one photon from the lower line. We require that the same LL and soft limits are fulfilled. The type (B) ansatz reads as:

$$D^{(2)B}_{[1,1]}(\tilde{\alpha}_1,\tilde{\beta}_1,\phi_1,\tilde{\alpha}'_1,\tilde{\beta}'_1,\phi_1) = \frac{4\pi\alpha^2}{t_p t_q}$$
$$\times\Big[\theta(v_1 - v'_1)\tilde{S}(\tilde{\alpha}_1,\tilde{\beta}_1)\tilde{S}(\tilde{\alpha}'_1,\tilde{\beta}'_1)H(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)\chi(v'_1)$$
$$+\theta(v'_1 - v_1)\tilde{S}(\tilde{\alpha}_1,\tilde{\beta}_1)\tilde{S}(\tilde{\alpha}'_1,\tilde{\beta}'_1)H(\tilde{\alpha}'_1,\tilde{\beta}'_1,\psi_p)\chi(v_1)\Big]. \quad (13)$$

It is simpler than the previous ansatz because the two photons are now attached to different fermion lines and we do not need to use "starred" variables.

The LL expression for the case of double emission from the lower line $D^{(2)B}_{[0,2]}$ is constructed in complete analogy to $D^{(2)B}_{[2,0]}$.

We shall now present choice (A) for the $\mathcal{O}(\alpha^r)$, $r = 0,1,2$, distributions $D^{(r)A}_{[n,n']}$. The $D^{(r)A}_{[0,0]}, r = 0,1,2$, in the $\mathcal{O}(\alpha^r)_{\rm prag}$ reads

$$\begin{aligned} D^{(r)A}_{[0,0]} &= \frac{4\pi\alpha^2}{t_p t_q}\, G(p_1,p_2,q_1,q_2)\ (1+v^{(r)}), \\ G(a,b,c,d) &\equiv \frac{(ab)^2+(cd)^2+(ad)^2+(bc)^2}{4(ab)^2}, \end{aligned} \quad (14)$$

where the virtual corrections are the same as in Eq. (2). We note that in the case of $n+n'=0$ under discussion the identity $b_0 = G(p_1,p_2,q_1,q_2)$ holds. The single photon distributions, $D^{(r)}_{[1,0]}(k_1), r = 1,2$, are defined as follows

$$\begin{aligned} D^{(r)A}_{[1,0]}(k_1) &= \frac{4\pi\alpha^2}{t_p t_q}\, \tilde{S}_p(k_1)\left(1+v^{(r)}_{[1,0]}(\tilde{\alpha}_1,\tilde{\beta}_1)\right) \\ &\quad\times(1+\delta^*_p(\tilde{\alpha}_1,\tilde{\beta}_1))\, G(p_1,p_2,q_1,q_2), \end{aligned} \quad (15)$$

where $v^{(r)}_{[1,0]}$ are the same as in Eq. (4) and, for the one photon case under discussion, the identity $H(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p) = G(p_1,p_2,q_1,q_2)$ holds. As we see, up to $\mathcal{O}(\alpha)$ the two choices (A) and (B) are identical (no exponentiation!). The double bremsstrahlung distribution for two photons on the upper line reads as follows (second-order LL ansatz with correct soft limits):

$$
\begin{aligned}
&D^{(2)A}_{[2,0]}(k_1,k_2)\\
&= \tilde{S}_p(k_1)\tilde{S}_p(k_2)\theta((\tilde{\alpha}_1-\tilde{\beta}_1)(\tilde{\alpha}_2-\tilde{\beta}_2))\\
&\qquad\times\Big\{\theta(v_1-v_2)\ \tfrac{1}{2}\ [G(p_1,p_2-k_2,q_1,q_2)\chi(v_2^*)\\
&\qquad\qquad +G(p_1-k_2,p_2,q_1,q_2)\chi(v_2)]\\
&\qquad +\theta(v_2-v_1)\ \tfrac{1}{2}\ [G(p_1,p_2-k_1,q_1,q_2)\chi(v_1^*)\\
&\qquad\qquad +G(p_1-k_1,p_2,q_1,q_2)\chi(v_1)]\Big\}\\
&\quad +\tilde{S}_p(k_1)\tilde{S}_p(k_2)\theta(\tilde{\alpha}_1-\tilde{\beta}_1)\theta(-\tilde{\alpha}_2+\tilde{\beta}_2)\\
&\qquad\times\Big\{\theta(v_1-v_2)G(p_1,p_2-k_2,q_1,q_2)\chi(v_2)\\
&\qquad +\theta(v_2-v_1)G(p_1-k_1,p_2,q_1,q_2)\chi(v_1^*)\Big\}\\
&\quad +\tilde{S}_p(k_1)\tilde{S}_p(k_2)\theta(-\tilde{\alpha}_1+\tilde{\beta}_1)\theta(\tilde{\alpha}_2-\tilde{\beta}_2)\\
&\qquad\times\Big\{\theta(v_1-v_2)G(p_1-k_2,p_2,q_1,q_2)\chi(v_2^*)\\
&\qquad +\theta(v_2-v_1)G(p_1,p_2-k_1,q_1,q_2)\chi(v_1)\Big\}, \qquad (16)
\end{aligned}
$$

where v_i^* are defined as previously. Finally we define

$$
D^{(2)A}_{[1,1]}(k_1,k_1') = \frac{4\pi\alpha^2}{t_p t_q}\ \tilde{S}_p(k_1)\tilde{S}(k_1')G(p_1,p_2,q_1,q_2). \qquad (17)
$$

The above function, although remarkably simple, has all soft limits and the $\mathcal{O}(\alpha^2L^2)$ limit correct!

2.2. Exponentiated second order

The complete *master formula* for the $\mathcal{O}(\alpha^r)$, $r=0,1,2$, exponentiated total cross-section for the process $e^-(p_1)+e^+(q_1)\to e^-(p_2)+e^+(q_2)+n\gamma(k_j)+n'\gamma(k_l)$, as implemented in the BHLUMI 4.xx Monte Carlo program, reads

$$
\begin{aligned}
\sigma^{(r)} = &\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\frac{1}{n!}\frac{1}{n'!}\quad\int\frac{d^3p_2}{p_2^0}\int\frac{d^3q_2}{q_2^0}\ e^{Y(\Omega_U;p_1,p_2)}\,e^{Y(\Omega_L;q_1,q_2)}\\
&\times\prod_{j=1}^{n}\int_{k_j\notin\Omega_U}\frac{d^3k_j}{k_j^0}\tilde{S}(p_1,p_2;k_j)\prod_{l=1}^{n'}\int_{k_j\notin\Omega_U}\frac{d^3k'_l}{k'^0_l}\tilde{S}(q_1,q_2;k'_l)
\end{aligned}
$$

$$\times\ \delta^{(4)}\Big(p_1 - p_2 + q_1 - q_2 - \sum_{j=1}^{n} k_j - \sum_{l=1}^{n'} k'_l\Big)\Big\{\bar\beta_0^{(r)}(Q, p_1, p_2, q_1, q_2)$$
$$+ \sum_{j=1}^{n} \frac{\bar\beta_{1U}^{(r)}(Q, p_1, p_2, q_1, q_2, k_j)}{\tilde S(p_1, p_2; k_j)}$$
$$+ \sum_{l=1}^{n'} \frac{\bar\beta_{1L}^{(r)}(Q, p_1, p_2, q_1, q_2, k'_l)}{\tilde S(q_1, q_2; k'_l)}$$
$$+ \sum_{n\ge j>m\ge 1} \frac{\bar\beta_{2UU}^{(r)}(Q, p_1, p_2, q_1, q_2, k_j, k_m)}{\tilde S(p_1, p_2; k_j)\tilde S(p_1, p_2; k_m)}$$
$$+ \sum_{n'\ge l>m\ge 1} \frac{\bar\beta_{2LL}^{(r)}(Q, p_1, p_2, q_1, q_2, k'_l, k'_m)}{\tilde S(q_1, q_2; k'_l)\tilde S(q_1, q_2; k'_m)}$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n'} \frac{\bar\beta_{2UL}^{(r)}(Q, p_1, p_2, q_1, q_2, k_j, k'_l)}{\tilde S(p_1, p_2; k_j)\tilde S(q_1, q_2; k'_l)}\Big\}. \tag{18}$$

Let us also write the above expression in equivalent but more compact notation

$$\sigma^{(r)} = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} \frac{1}{n!}\frac{1}{n'!}\int \frac{d^3p_2}{p_2^0}\int \frac{d^3q_2}{q_2^0}$$
$$\times \prod_{j=1}^{n}\int_{k_j\notin\Omega_U} \frac{d^3k_j}{k_j^0}\tilde S_p(k_j) \prod_{l=1}^{n'}\int_{k'_l\notin\Omega_L} \frac{d^3k'_l}{k'^0_l}\tilde S_q(k'_l)$$
$$\times\ \delta^{(4)}\Big(p_1 - p_2 + q_1 - q_2 - \sum_{j=1}^{n} k_j - \sum_{l=1}^{n'} k'_l\Big)\ e^{Y_p(\Omega_U)+Y_q(\Omega_L)}$$
$$\times \Big\{\bar\beta_0^{(r)} + \sum_{j=1}^{n}\frac{\bar\beta_{1U}^{(r)}(k_j)}{\tilde S_p(k_j)} + \sum_{l=1}^{n'}\frac{\bar\beta_{1L}^{(r)}(k'_l)}{\tilde S_q(k'_l)} + \sum_{n\ge j>m\ge 1}\frac{\bar\beta_{2UU}^{(r)}(k_j, k_m)}{\tilde S_p(k_j)\tilde S_p(k_m)}$$
$$+ \sum_{n'\ge l>m\ge 1}\frac{\bar\beta_{2LL}^{(r)}(k_l, k_m)}{\tilde S_q(k'_l)\tilde S_q(k'_m)} + \sum_{j=1}^{n}\sum_{l=1}^{n'}\frac{\bar\beta_{2UL}^{(r)}(k_j, k'_l)}{\tilde S_p(k_j)\tilde S_q(k'_l)}\Big\}. \tag{19}$$

Let us explain all ingredients in the above expression.

The YFS form factor for one fermion line is generally defined as follows:

$$\begin{aligned} Y(\Omega_U; p_1, p_2) &= 2\alpha\tilde B(\Omega, p_1, p_2) + 2\alpha\Re B(p_1, p_2) \\ &= -\frac{\alpha}{4\pi^2}\int_{k\in\Omega_U} \frac{d^3k}{k^0}\Big(\frac{p_1}{kp_1} - \frac{p_2}{kp_2}\Big)^2 \end{aligned}$$

$$+2\alpha\Re\int\frac{d^4k}{k^2}\frac{i}{(2\pi)^3}\left(\frac{2p_1-k}{2kp_1-k^2}-\frac{2p_2-k}{2kp_2-k^2}\right)^2. \quad (20)$$

It is completely determined by the definition of the infrared domains $\Omega_{U,L}$. For our "rectangular" definition of Ω_U, see Eq. (8), the upper-line YFS form factor reads as follows

$$\begin{aligned}
Y_p(\Omega_U) &= \gamma_p \ln \Delta + \delta_{\rm YFS},\\
\delta_{\rm YFS} &= -\gamma_p \ln\left(1-\sum_{i=1}^{n}\tilde\beta_i\right)+\frac{1}{4}\gamma_p+\frac{\alpha}{\pi}\left(-\frac{1}{2}\right),\\
\gamma_p &= 2\frac{\alpha}{\pi}\left(\ln\frac{1}{\delta_p}-1\right),
\end{aligned} \quad (21)$$

and the lower-line form factor is completely analogous. Let us note that in the latter analytical calculations we shall switch to a "triangular" definition $\tilde\alpha_i+\tilde\beta_i<\delta$ of the infrared domain Ω, for which

$$\delta_{\rm YFS} = -\gamma_p \ln\left(1-\sum_{i=1}^{n}\tilde\beta_i\right)+\frac{1}{4}\gamma_p+\frac{\alpha}{\pi}\left(-\frac{1}{2}-\frac{\pi^2}{6}\right); \quad (22)$$

see also discussion in Ref. [10].

Let us start defining various components of the differential distribution with the expressions for $\bar\beta_0^{(r)}$ functions, $r=0,1,2$, in $\mathcal{O}(\alpha^r)_{\rm prag}$, respectively. These functions in the YFS scheme are generally defined as

$$\begin{aligned}
\bar\beta_0^{(r)} &= \left\{D^{(r)}_{[0,0]}\exp(-Y_p(\Omega_U)-Y_q(\Omega_L)\right\}\Big|_{\mathcal{O}(\alpha^r)},\\
\bar\beta_{1U}^{(r)}(k_i) &= \left\{D^{(r)}_{[1,0]}(k_i)\exp(-Y_p(\Omega_U)-Y_q(\Omega_L)\right\}\Big|_{\mathcal{O}(\alpha^r)}-\tilde S_p(k_i)\,\bar\beta_0^{(r-1)},\\
\bar\beta_{2UU}^{(2)}(k_i,k_j) &= D^{(2)}_{[2,0]}(k_i,k_j)-\bar\beta_{1U}^{(1)}(k_i)\tilde S_p(k_j)\\
&\quad -\bar\beta_{1U}^{(1)}(k_j)\tilde S_p(k_i)-\bar\beta_0^{(0)}\tilde S_p(k_i)\tilde S_p(k_j),\\
\bar\beta_{2UL}^{(2)}(k_i,k'_j) &= D^{(2)}_{[1,1]}(k_i,k'_j)-\bar\beta_{1U}^{(1)}(k_i)\tilde S_q(k'_j)\\
&\quad -\bar\beta_{1L}^{(1)}(k'_j)\tilde S_p(k_i)-\bar\beta_0^{(0)}\tilde S_p(k_i)\tilde S_q(k'_j).
\end{aligned} \quad (23)$$

We have to stress a very important feature of the above definitions. The raw distributions $D^{(r)}_{[n,n']}$ are originally defined in the corresponding N-body phase-space, $N=2+n+n'=0,1,2$, while $\bar\beta^{(r)}$ have to be defined in the presence of any number $L=1,2,...\infty$ of additional real photons, *i.e.* in the

$(N+L)$-particle phase-space. This requires an interpolation of the original formulas for $D^{(r)}_{[n,n']}$ to higher-dimensional phase-space. This interpolation is of course to some extent arbitrary and it is a well-known feature of the YFS scheme already discussed in the original paper [14]. For instance $D^{(r)B}_{[0,0]}$ is defined originally in the 2-body phase-space. In Eqs. (23) this function is used beyond the 2-body phase-space. This case is simple because $D^{(r)B}_{[0,0]}$ depends only on t and s and its extension to the N-photon case is trivial. The case of $D^{(r)B}_{[1,0]}(k_i)$ is already less trivial. In this case the interpolation to multiple-photon phase-space is done with the simple substitution $k_1 \to k_i$, *i.e.* $(\tilde{\alpha}_1, \tilde{\beta}_1) \to (\tilde{\alpha}_i, \tilde{\beta}_i)$ in Eq. (4). The same method using the mapping of the Sudakov variables is employed for $D^{(r)B}_{[1,1]}(k_i, k_j)$ and the two other double bremsstrahlung distributions. Generally, for the type (A) expressions, the interpolation is done through Mandelstam variables and Sudakov variables, while for the type (B) the interpolation is done using almost exclusively Sudakov variables.

For the purpose of the analytical phase-space integrations over the type (B) differential distributions, we write in the following the explicit expressions for the $\bar{\beta}$'s. In this case we express all distributions and the phase-space integral in terms of the Sudakov variables. In particular the "soft bremsstrahlung integration element" is parametrized as follows

$$\begin{aligned}
\int_{k_j \notin \Omega_U} \frac{d^3 k_i}{k_i^0} \tilde{S}_p(k_i) &= \int d\phi_i \int d\tilde{\alpha}_i d\tilde{\beta}_i \; \theta(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i \tilde{\beta}_i - \Delta) \; S_p(\tilde{\alpha}_1, \tilde{\beta}_i) \\
&= \int d\omega_i \; \theta(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i \tilde{\beta}_i - \Delta)\,, \\
S_p(\tilde{\alpha}_1, \tilde{\beta}_i) &= \frac{\alpha}{2\pi^2} \frac{\tilde{\alpha}_i \tilde{\beta}_i}{(\tilde{\alpha}_i + \delta_p \tilde{\beta}_i)^2 (\tilde{\beta}_i + \delta_p \tilde{\alpha}_i)^2}\,, \quad \delta_p = \frac{m_e^2}{|t_p|}.
\end{aligned} \tag{24}$$

The new soft factor $S_p(\tilde{\alpha}_1, \tilde{\beta}_i)$ differs only by normalization from the standard YFS soft factor

$$\tilde{S}(p_1, p_2, k) = S_p(\tilde{\alpha}_1, \tilde{\beta}_i) \; K_p^2/(p_1 p_2), \tag{25}$$

see Eq. (7) for the definition of K_p.

Let us begin with the $\mathcal{O}(\alpha^r)_{\mathrm{prag}}$ expression for $\bar{\beta}_0^{(r)B}$, $r = 0, 1, 2$:

$$\begin{aligned}
\bar{\beta}_0^{(r)B} &= \frac{4\pi\alpha^2}{t_p t_q} b_0 \; (1 + \kappa^{(r)}), \\
\kappa^{(0)} &= 0, \quad \kappa^{(1)} = \gamma \quad \kappa^{(2)} = \gamma + \tfrac{1}{2}\gamma^2.
\end{aligned} \tag{26}$$

Note that this choice is different from (simpler than) the corresponding one in Ref. [10]. Next, we also write explicit expressions for the upper line emission function $\bar{\beta}_{1U}^{(r)B}, r = 1, 2$.

$$
\begin{aligned}
\bar{\beta}_{1U}^{(r)B}(\tilde{\alpha}_i, \tilde{\beta}_i) &= \frac{4\pi\alpha^2}{t_p t_q} \frac{K_p^2}{(p_1 p_2)} b_{1U}^{(r)}(\tilde{\alpha}_i, \tilde{\beta}_i, \phi_i)\,, \\
b_{1U}^{(1)}(\tilde{\alpha}_i, \tilde{\beta}_i, \phi_i) &= S_p(\tilde{\alpha}_i, \tilde{\beta}_i) H(\tilde{\alpha}_i, \tilde{\beta}_i, \psi_p) - b_0 S_p(\tilde{\alpha}_i, \tilde{\beta}_i)\,, \\
b_{1U}^{(2)}(\tilde{\alpha}_i, \tilde{\beta}_i, \phi_i) &= S_p(\tilde{\alpha}_i, \tilde{\beta}_i)\Big(1 + \gamma\pi(\tilde{\alpha}_i, \tilde{\beta}_i)\Big) H(\tilde{\alpha}_i, \tilde{\beta}_i, \psi_p) \\
&\quad - b_0 S_p(\tilde{\alpha}_i, \tilde{\beta}_i)\Big(1 + \gamma\pi(0,0)\Big)\,, \\
\pi(\tilde{\alpha}_i, \tilde{\beta}_i) &= 1 + \frac{1}{4} \ln \frac{(1-\tilde{\beta}_i)^2}{(1-v_i)}\,.
\end{aligned} \tag{27}
$$

The lower-line function $\bar{\beta}_{1L}^{(r)}$ is defined in a completely analogous way. The explicit expressions for the $\mathcal{O}(\alpha^2)$ double photon emission $\bar{\beta}_2$ distribution

$$
\bar{\beta}_{2UU}^{(2)B}(\tilde{\alpha}_i, \tilde{\beta}_i, \phi_i, \tilde{\alpha}_j, \tilde{\beta}_j, \phi_j) = \frac{4\pi\alpha^2}{t_p t_q} \frac{K_p^4}{(p_1 p_2)^2} b_{2UU}(\tilde{\alpha}_i, \tilde{\beta}_i, \phi_i, \tilde{\alpha}_j, \tilde{\beta}_j, \phi_j), \tag{28}
$$

$$
\bar{\beta}_{2UL}^{(2)B}(\tilde{\alpha}_i, \tilde{\beta}_i, \phi_i, \tilde{\alpha}_{j'}, \tilde{\beta}_{j'}, \phi_{j'}) = \frac{4\pi\alpha^2}{t_p t_q} \frac{K_p^2 K_q^2}{(p_1 p_2)} b_{2UL}(\tilde{\alpha}_i, \tilde{\beta}_i, \phi_i, \tilde{\alpha}_{j'}, \tilde{\beta}_{j'}, \phi_{j'}), \tag{29}
$$

are easily deduced from their definitions.

Let us finally comment on the exponentiation of the type (A) matrix element. As in case (B) we substitute the corresponding $D_{[n,n']}^{(r)A}$ into Eqs (23) and the only non-trivial matter to be discussed is the (off-shell) extrapolation of the distributions $D_{[n,n']}^{(r)A}$ into the multi-photon phase-space. The extrapolation is here even simpler than in case (B) because instead of the function $H(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)$ we employ the simpler function $G(p_1, p_2, q_1, q_2)$, which contains momenta only of fermions, and is therefore, by construction, "blind" to any individual spectator photon. The usual substitutions $k_1 \to k_i$ (1-photon case) and $(k_1, k_2) \to (k_i, k_j)$ or $(k_1, k_1') \to (k_i, k_j')$ (2-photon case) are obvious in the realization of the extrapolation.

3. Semi-analytical integration

Analytical integration over the multi-photon phase-space for the *true* experimental ES, *i.e.* set of cuts, is practically impossible[7]. What we may try

[7] See Refs. [20, 21] for an example of semi-analytical integration over the phase-space for another unrealistic ES.

to achieve is to perform an analytical calculation for the ES as close as possible to the true experimental ES. The primary aim of this work is to obtain analytically the total cross section for the matrix element of the BHLUMI multi photon MCEG with a precision of at least 0.03%. This exercise is, first of all, a zero-level test of the correctness of the implementation of the $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element in BHLUMI. The exercise has great importance, even if it is done for *unrealistic* ESs, provided that the high precision below 0.03% is really achieved. In the above spirit the type of ES was chosen in such a way that the analytical calculation is maximally simple. Of course, gaining experience from this step we are now in a much better position to repeat a similar high-precision semi-analytical (SAN) calculation for a more realistic ES in the future.

Having the above in mind, we have defined here an "academic event selection", called for short an "academic ES" or AES, for which the task of analytical integration over the phase-space is feasible and the result is not overwhelmingly complicated. We define the cuts of our AES as follows: $|t_{\rm min}| < |t| < |t_{\rm max}|$ and $V < V_{\rm max}$, where the variable V represents some kind of measure of the *total* energy carried away by *all* emitted real photons. The requirement of $0 < V < 1$ represents the condition of completeness of the phase-space and the particular case $0 \le V \le \varepsilon$ represents the condition that all photons be soft. The V-variable we actually use is defined as

$$V = 1 - \mathcal{Z}_p \mathcal{Z}_q\,, \tag{30}$$

$$\mathcal{Z}_p = \frac{(p_1 p_2)\,|t|}{\Big[(p_1 p_2) + \sum_j (p_1 k_j)\Big]^2}\,, \qquad \mathcal{Z}_q = \frac{(q_1 q_2)\,|t|}{\Big[(q_1 q_2) + \sum_l (q_1 k'_l)\Big]^2}\,. \tag{31}$$

With the above definition of the phase-space window, it is quite straightforward to integrate the $\mathcal{O}(\alpha^2)_{\rm prag}$ matrix element, keeping all terms within the $\mathcal{O}(\alpha^2)_{\rm prag}$ approximation. This we found insufficient to establish a technical precision at the 0.03% level because some terms beyond $\mathcal{O}(\alpha^2)_{\rm prag}$ — especially for partially incomplete results — are of that size. See Sec. 1.3 for the definition of the $\mathcal{O}(\alpha^r)_{\rm prag}$ approximations and for a discussion of the numerical importance of the various perturbative corrections.

We have therefore decided to integrate analytically up to terms[8] of the $\mathcal{O}(\alpha^3)_{\rm prag}$. In the $\mathcal{O}(\alpha^3)_{\rm prag}$ approximation we include by definition all terms from $\mathcal{O}(\alpha^2)_{\rm prag}$ plus terms of $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L)$. In other words, terms of $\mathcal{O}(\alpha^2 L)$, due to our LL ansatz, and terms of $\mathcal{O}(\alpha^3 L^3)$, due to exponentiation,

[8] This is to our knowledge the only example of analytical integration over the full phase-space up to three photons.

are integrated analytically over the entire phase-space (within our AES) exactly! The first results of the analytical integration (without any details of the calculation) were presented in Ref. [5].

In the following we shall integrate analytically contributions from $\bar{\beta}_0$, $\bar{\beta}_1$ and $\bar{\beta}_2$. These three calculations differ substantially. For $\bar{\beta}_0$ the main difficulty will be in the very precise integration over "spectator photons", because $\bar{\beta}_0$ contributes most of the total cross sections. At the other extreme, for the $\bar{\beta}_2$ contributions, the integration over "spectator photons" is either absent or can be done easily in the LL approximation. However, the integration over the "active photon" variables, which sits directly in the $\bar{\beta}_2$, is very complicated and is the main source of difficulty. The case of $\bar{\beta}_1$ is intermediate and the most complicated, because both integrations, over "spectator photons" and "active photon", are difficult and interrelated. We start with the $\bar{\beta}_0$ case because results of integrations over "spectator photons" will be useful for the rest of the calculation. It is also well suited for an introduction of the notation and basic calculation methods.

3.1. Preliminaries

Here we calculate the contribution to the total cross section from the $\bar{\beta}_0$ part. For fixed $Q^2 = t$ let us consider the corresponding distribution

$$
\begin{aligned}
\frac{d\sigma^{(r)}_{\bar{\beta}_0}}{d|t|\,dV} &= \frac{4\pi\alpha^2}{|t|^2}\rho^{(r)}_{\bar{\beta}_0}(t,V) = \frac{4\pi\alpha^2}{|t|^2}\int dv dv' \\
&\times \int d\psi_p \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int d\omega_i \theta(\tilde{\alpha}_i+\tilde{\beta}_i-\tilde{\alpha}_i\tilde{\beta}_i-\Delta)\, \mathrm{e}^{\gamma_p \ln\Delta+\delta_{\mathrm{YFS}}} \\
&\times \Theta(1-\mathcal{Z}_p)\delta\Big(v-\sum_{i=1}^{n}(\tilde{\alpha}_i+\tilde{\beta}_i-\tilde{\alpha}_i\tilde{\beta}_i)\Big)\delta(\phi_{K_p}) \\
&\times \int d\psi_q \sum_{n'=1}^{\infty}\frac{1}{n'!}\prod_{j=1}^{n'}\int d\omega'_j \theta(\tilde{\alpha}'_i+\tilde{\beta}'_i-\tilde{\alpha}'_i\tilde{\beta}'_i-\Delta)\, \mathrm{e}^{\gamma_q \ln\Delta+\delta'_{\mathrm{YFS}}} \\
&\times \Theta(1-\mathcal{Z}_q)\delta\Big(v'-\sum_{i=1}^{n'}(\tilde{\alpha}'_i+\tilde{\beta}'_i-\tilde{\alpha}'_i\tilde{\beta}'_i)\Big)\delta(\phi_{K_q}) \\
&\times \bar{\beta}^{(r)}_0\ \delta(V-v-v'+vv')) \\
&= \frac{4\pi\alpha^2}{|t|^2} b_0(1+\kappa^{(r)})\int dv dv' \delta(V-v-v'+vv'))B_0(v)B_0(v')\,, \quad (32)
\end{aligned}
$$

where

$$B_0(v) = \int d\psi_p \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int d\omega_i \, \theta(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i\tilde{\beta}_i - \Delta) \; e^{\gamma_p \ln \Delta + \delta_{\rm YFS}}$$
$$\times \, \Theta(1 - \mathcal{Z}_p) \sum_{i=1}^{n} \delta\Big(v - \sum_{i=1}^{n} (\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i\tilde{\beta}_i)\Big) \; \delta(\phi_{K_p}) \qquad (33)$$

corresponds to the emission of photons from a single line and will be calculated in this Section.

Let us note that the constraint $\delta(\phi_{K_p})$ reflects the requirement that in the QRS_p frame where $p_1 = (E_p, 0, 0, -|p|)$ and $p_2 = (E_p, 0, 0, |p|)$, the total three-momentum of all photons $\vec{K} = \sum_{i=1}^{n} \vec{k}_i$ is in the x–z plane, *i.e.* $K_y = 0$. (In the actual Monte Carlo algorithm, this is realized easily with the help of the rotation around the z-axis, which makes $K_y = 0$.) Note that in the single photon case $\vec{k}$ is in the x–z plane while ψ_p is simply its azimuthal angle around the t-channel momentum transfer Q. This kind of parametrization of the single photon was employed in the early work reported in Ref. [22], which later led to Monte Carlo of Ref. [23]; it was also quite essential in the analytical $\mathcal{O}(\alpha)$ calculation of the luminosity cross section in Ref. [24]. The multi-photon generalization was given for the first time in BHLUMI 1.x [9].

Furthermore, for any semi-analytical calculation it is crucial that we also *know exactly and explicitly* the (upper) phase-space limits. Here they are given with the following condition[9]

$$\mathcal{Z}_p = \Big(1 - \sum_{i=1}^{n} \tilde{\alpha}_i\Big)\Big(1 - \sum_{i=1}^{n} \tilde{\beta}_i\Big) - \frac{1}{4}\tilde{K}^2 > 0, \qquad (34)$$

where $\tilde{K} = \sum_{i=1}^{n} \tilde{k}_i$ and the dimension-less $\tilde{k}_i$ are defined as in Ref. [9]. The above expression is totally equivalent (no approximations) to that of Eq. (31). Note also that the following identity holds

$$\delta_p = \frac{m_e^2}{|t_p|} = \frac{\mathcal{Z}_p}{\Big(1 - \sum_{i=1}^{n} \tilde{\beta}_i\Big)^2} \frac{m_e^2}{|t|}. \qquad (35)$$

We shall use the completely analogous parametrization in terms of variables $\tilde{\alpha}'_i$ and $\tilde{\beta}'_i$ for the lower line.

---

[9] The same condition has already been implemented in the BHLUMI Monte Carlo [9].

3.2. Collinearization

In the first step in our analytical calculation we introduce a series of $\mathcal{O}(\alpha^3)_{\rm prag}$ approximations, which leads to *collinearization* of the integral (3.1), *i.e.* in the resulting integrals we shall be able to factorize initial- and final-state photons and sum up infinite sums over photon multiplicity. A very similar *collinearization* procedure will also be applied in the calculation of the $\bar{\beta}_1$ contribution in the next Section.

We start with approximating the function $\mathcal{Z}_p$, which monitors the upper limit of the phase-space, as follows

$$\mathcal{Z}_p \to \bar{\mathcal{Z}}_p = 1 - \sum_{i=1}^{n}(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i\tilde{\beta}_i) = 1 - \sum_{i=1}^{n} v_i \,. \tag{36}$$

The above approximation leads of course to

$$V = 1 - \Big(1 - \sum_j(\tilde{\alpha}_j + \tilde{\beta}_j - \tilde{\alpha}_j\tilde{\beta}_j)\Big)\Big(1 - \sum_l(\tilde{\alpha}'_l + \tilde{\beta}'_l - \tilde{\alpha}'_l\tilde{\beta}'_l)\Big) \,. \tag{37}$$

The above two approximations are valid not only in $\mathcal{O}(\alpha^2)_{\rm prag}$ but also within the LL and next-to-LL approximations to any order!

The above ansatz is crucial for all further $\mathcal{O}(\alpha^3)_{\rm prag}$ approximations. It is valid for one non-collinear photon and an arbitrary number of collinear photons (as can be checked with explicit kinematic considerations) – it is therefore valid not only in $\mathcal{O}(\alpha^3)_{\rm prag}$ but in LL+NLL to infinite order. Let us now reorganize the integral as follows (no further approximations). We separate photons in the sum into two categories: *(i)* photons with $\tilde{\alpha}_i > \tilde{\beta}_i$, which will be referred to as initial-state photons and *(ii)* photons with $\tilde{\alpha}_i < \tilde{\beta}_i$ called final-state photons. Using the identity

$$d\omega_i = \theta(\tilde{\alpha}_i - \tilde{\beta}_i)d\omega_i + \theta(\tilde{\beta}_i - \tilde{\alpha}_i)d\omega_i = d\omega_i^I + d\omega_i^F \tag{38}$$

we obtain the following expression

$$\begin{aligned} B_0(t,v\;) = &\int dv_I dv_F dv_1 \delta(v - v_I - v_F) \int d\psi_p \theta(1 - v_I - v_F) \\ \times &\left[e^{\frac{1}{2}\gamma_p \ln \Delta} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{v_i > \Delta} d\omega_i^I \delta(v_I - \sum_{i=0}^{n} v_i) \right] \\ \times &\left[e^{\frac{1}{2}\gamma_p \ln \Delta} \sum_{n'=0}^{\infty} \frac{1}{n'!} \prod_{j=1}^{n'} \int_{v_j > \Delta} d\omega_j^F \delta(v_F - \sum_{i=0}^{n'} v_j) \right] e^{\delta_{\rm YFS}}\, \delta(\phi_{K_p}) . \end{aligned} \tag{39}$$

Our aim is to integrate and sum contributions from the initial- and final-state photons in square brackets, in the first place. This is non-trivial because all parts of the integral are interconnected through the variable

$$\delta_p = \frac{1-v}{\left(1-\tilde{\beta}_I-\tilde{\beta}_F\right)^2}\,\frac{m_e^2}{|t|}, \qquad \tilde{\beta}_I = \sum_{i=1}^{n} \tilde{\beta}_i^I, \qquad \tilde{\beta}_F = \sum_{j=1}^{n'} \tilde{\beta}_j^F, \tag{40}$$

which is present in all parts of integrand. We achieve the separation of the multi photon integration/summation by means of the following crucial *approximation*

$$\delta_p \to \bar{\delta}_p = \frac{1-v}{(1-v_F)^2}\,\frac{m_e^2}{|t|}. \tag{41}$$

As is shown in the dedicated Appendix A (Sect. 10) we are allowed, within $\mathcal{O}(\alpha^3)_{\mathbf{prag}}$, to do the above approximation in all bremsstrahlung distributions $d\omega_i \to d\bar{\omega}_i$ and in part of the form factor $e^{\frac{1}{2}\gamma_p \ln \Delta} \to e^{\frac{1}{2}\bar{\gamma}_p \ln \Delta}$. The rest of the form factor requires, however, more discussion. Much as in Appendix A, we may prove that for

$$\delta_{\rm YFS}(\gamma_p, \tilde{\beta}_I+\tilde{\beta}_F) = -\gamma_p \ln\left(1-\tilde{\beta}_I-\tilde{\beta}_F\right) + \frac{1}{4}\gamma_p + \frac{\alpha}{\pi}\left(-\frac{1}{2}-\frac{\pi^2}{6}\right) \tag{42}$$

we are allowed, within $\mathcal{O}(\alpha^3)_{\mathbf{prag}}$, to approximate[10]

$$\delta_{\rm YFS}(\gamma_p, \tilde{\beta}_I+\tilde{\beta}_F) \longrightarrow \delta_{\rm YFS}(\bar{\gamma}_p, \tilde{\beta}_I+\tilde{\beta}_F). \tag{43}$$

With all the discussed approximations we obtain the following expression:

$$\begin{aligned} B_0(t,v) =& \int\limits_0^1 dv_I dv_F \delta(v-v_I-v_F) \int d\psi_p \\ &\times\left[e^{\frac{1}{2}\bar{\gamma}_p \ln \Delta} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int\limits_{v_i>\Delta} d\bar{\omega}_i^I \delta(v_I - \sum_{i=0}^{n} v_i)\right] \\ &\times\left[e^{\frac{1}{2}\bar{\gamma}_p \ln \Delta} \sum_{n'=0}^{\infty} \frac{1}{n'!} \prod_{j=1}^{n'} \int\limits_{v_j>\Delta} d\bar{\omega}_j^F \delta(v_F - \sum_{i=0}^{n'} v_j)\right] \\ &\times \delta(\phi_{K_p})\ e^{\bar{\delta}_{\rm YFS}(\bar{\gamma}_p, \tilde{\beta}_I+\tilde{\beta}_F)}\,. \end{aligned} \tag{44}$$

[10] We have checked with a special dedicated MC run that the relative error introduced by the above approximations is below 10^{-5}!

In the above we have introduced the self-explanatory notation

$$\bar{\gamma}_p = 2\frac{\alpha}{\pi}\left(\ln\frac{(1-v_F)^2}{1-v}\frac{|t|}{m_e^2}-1\right) = \gamma + 2\frac{\alpha}{\pi}\ln\frac{(1-v_F)^2}{1-v}. \tag{45}$$

At this stage the only contribution that prevents us from separate integration and summations over multiple initial and final photons is the $\delta_{\rm YFS}$ form factor. Let us discuss this point in a bit more detail. We would like to replace $\tilde{\beta}_I \to 0$ and $\tilde{\beta}_F \to v_F$ as we did previously in δ_p. We cannot do it, however, because the difference $\exp(\delta_{\rm YFS}(\bar{\gamma}_p, \tilde{\beta}_I+\tilde{\beta}_F)) - \exp(\delta_{\rm YFS}(\bar{\gamma}_p, v_F))$ may give a non-zero $\mathcal{O}(\alpha\gamma)$ contribution. In fact only the first term in the expansion

$$\begin{aligned} &e^{\delta_{\rm YFS}(\bar{\gamma}_p,\tilde{\beta}_I+\tilde{\beta}_F)} - e^{\delta_{\rm YFS}(\bar{\gamma}_p,v_F)}\\ &= \bar{\gamma}_p\left(-\ln(1-\tilde{\beta}_I+\tilde{\beta}_F)+\ln(1-v_F)\right)+\mathcal{O}(\bar{\gamma}_p^2) \end{aligned} \tag{46}$$

matters. Furthermore, this term may yield a non-zero $\mathcal{O}(\alpha\gamma)$ contribution only in the *single-photon* case. This single-photon contribution to B_0 reads

$$\begin{aligned} B_0^{\rm sing}(t,v) = &\int_0^1 dv_I dv_F \delta(v-v_I-v_F)\int d\psi_p\\ \times&\left[\delta(v_F)\int d\bar{\omega}_1^I\, \delta(v_I-v_1)\,\gamma(-\ln(1-\tilde{\beta}_1))\right.\\ &\left.+\delta(v_I)\int d\bar{\omega}_1^F\, \delta(v_F-v_1)\,\gamma(\ln(1-v_1)-\ln(1-\tilde{\beta}_1))\right]\delta(\phi_{K_p}). \end{aligned} \tag{47}$$

Luckily, the two terms in the above formula (from initial- and final-state emissions) cancel after integration over $d\bar{\omega}_1$, so in principle we may drop out this contribution. (It has to be kept, however, for the future applications in which one will distinguish between the initial and final bremsstrahlung, so we shall calculate it later in this Section.) In the resulting expression we may finally pull the form factor completely out of the multi photon integrals

$$\begin{aligned} B_0(t,v) = &\int_0^1 dv_I dv_F \delta(v-v_I-v_F)\int d\psi_p\\ &\times e^{\delta_{\rm YFS}(\bar{\gamma}_p,v_F)}\\ &\times\left[e^{\frac{1}{2}\bar{\gamma}_p\ln\Delta}\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int_{v_i>\Delta} d\bar{\omega}_i^I\delta(v_I-\sum_{i=0}^{n}v_i)\right]\\ &\times\left[e^{\frac{1}{2}\bar{\gamma}_p\ln\Delta}\sum_{n'=0}^{\infty}\frac{1}{n'!}\prod_{j=1}^{n'}\int_{v_j>\Delta} d\bar{\omega}_j^F\delta(v_F-\sum_{i=0}^{n'}v_j)\right]\delta(\phi_{K_p}). \end{aligned} \tag{48}$$

The integration $\int d\psi_p$ can be done at the expense of $\delta(\phi_{K_p})$, which is not completely straightforward; for more details see the next Section on $\bar{\beta}_1$ calculation. The integral

$$B_0(t,v) = \int_0^1 dv_I dv_F \delta(v - v_I - v_F) \times e^{\bar{\delta}_{\rm YFS}(\bar{\gamma}_p, v_F)}\, f_1\left(\tfrac{1}{2}\bar{\gamma}_p, v_I\right) f_1\left(\tfrac{1}{2}\bar{\gamma}_p, v_F\right) \tag{49}$$

clearly factorizes into a convolution of the two functionally identical expressions

$$f_1\left(\frac{1}{2}\bar{\gamma}_p, x\right) = e^{\frac{1}{2}\bar{\gamma}_p \ln \Delta} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int_{v_i > \Delta} d\bar{\omega}_i^I \, \delta(x - \sum_{i=0}^{n} v_i). \tag{50}$$

Note that the convolution is not completely trivial because γ_p still depends on v_F.

For the sake of completeness let us also give an explicit expression for the initial-state contribution in $B_0^{\rm sing}$

$$\begin{aligned} B_{0I}^{\rm sing}(v) &= -\int d\bar{\omega}_1^I \, \delta(v - \tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\alpha}_1\tilde{\beta}_1)\, \gamma \ln(1 - \tilde{\beta}_1) \\ &= \gamma \frac{\alpha}{\pi}\frac{1}{v}\left(\frac{1}{8}\ln^2(1-v) + {\rm Li}_2(1-v) \right. \\ &\qquad \left. + {\rm Li}_2(1) - 2{\rm Li}_2\left(\sqrt{1-v}\right)\right). \end{aligned} \tag{51}$$

3.3. Single hemisphere multi photon integral

In the following we shall calculate the multi photon integral f_1. We immediately find that the integration over the photon angles leads to

$$\int_{v_i > \Delta} d\bar{\omega}_i^I = \tfrac{1}{2} \int_{v_i > \Delta} d\bar{\omega}_i = \tfrac{1}{2} \int_{v_i > \Delta} \frac{dv_i}{v_i}[\bar{\gamma}_p - \frac{\alpha}{\pi}\ln(1 - v_i)], \tag{52}$$

and hence

$$f_1\left(\frac{1}{2}\bar{\gamma}_p, x\right) = e^{\frac{1}{2}\bar{\gamma}_p \ln \Delta} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \times \int_{v_i > \Delta} \frac{dv_i}{v_i}[\bar{\gamma}_p - \frac{\alpha}{\pi} \ln(1 - v_i)]\, \delta\left(x - \sum_{i=0}^{n} v_i\right). \tag{53}$$

The function f_1 at the $x \to 0$ soft limit coincides with the well-known soft photon integral

$$f_0(g,x) = \mathrm{e}^{g\ln\Delta} \sum_{n=0}^{\infty} \frac{g^n}{n!} \prod_{i=1}^{n} \int_{v_i>\Delta} \frac{dv_i}{v_i} \delta(x - \sum_{i=0}^{n} v_i) = F(g) g x^{g-1}, \qquad (54)$$

where

$$F(g) = \frac{\mathrm{e}^{-Cg}}{\Gamma(1+g)} \qquad (55)$$

and C is the Euler constant. We did not attempt to obtain a close expression for f_1 distribution because it is rather easily expandable in powers of α and in fact, in the $\mathcal{O}(\alpha^3)_{\rm prag}$, we need only the first term beyond Eq. (55). More precisely we replace

$$\prod_{i=1}^{n} [\bar{\gamma}_p - \frac{\alpha}{\pi} \ln(1-v_i)] \to \bar{\gamma}_p^n - \bar{\gamma}_p^{n-1} \frac{\alpha}{\pi} \sum_{i=1}^{n} \ln(1-v_i), \qquad (56)$$

and we evaluate two multi photon integrals using Eq. (55):

$$f_1\left(\tfrac{1}{2}\bar{\gamma}_p, x\right) = f\left(\tfrac{1}{2}\bar{\gamma}_p, x\right) - \frac{1}{2}\frac{\alpha}{\pi} F\left(\tfrac{1}{2}\bar{\gamma}_p\right) \int_0^x dy \, \frac{\ln(1-y)}{y} \, \frac{1}{2}\bar{\gamma}_p (x-y)^{\frac{1}{2}\bar{\gamma}_p - 1}. \qquad (57)$$

Expanding the second integral in powers of $\bar{\gamma}_p$ we find

$$f_1\left(\frac{1}{2}\bar{\gamma}_p, x\right) = F\left(\frac{1}{2}\bar{\gamma}_p\right) x^{\frac{1}{2}\bar{\gamma}_p - 1} \times \left[\frac{1}{2}\bar{\gamma}_p - \frac{1}{2}\frac{\alpha}{\pi} \ln(1-x) - \frac{1}{8}\frac{\alpha}{\pi} \bar{\gamma}_p \ln^2(1-x)\right]. \qquad (58)$$

The above formula was checked with the help of the dedicated one-dimensional Monte Carlo program. We have checked numerically the transition from Eq. (53) to (58) and we have found agreement better than 3×10^{-5}. It should be noted that we have also calculated analytically the explicit $\mathcal{O}(\alpha^4)_{\rm prag}$ expression for the f_1 function (we do not include the relevant results here).

3.4. Convolution integral

Our integral

$$B_0(t,v) = \int_0^v dv_F\, e^{\bar{\delta}_{\rm YFS}(\bar{\gamma}_p(v_F),v_F)}\; f_1\left(\tfrac{1}{2}\bar{\gamma}_p(v_F), v - v_F\right) f_1\left(\tfrac{1}{2}\bar{\gamma}_p(v_F), v_F\right) \tag{59}$$

is singular (but integrable) at the end points $v_F = 1$ and $v_F = v$. For example its integrand behaves like $\sim v_F^{(1/2)\bar{\gamma}_p(0)-1}$ at $v_F \to 0$ and the integration $\int dv_F$ contributes $1/\bar{\gamma}_p(0)$. Because of that we cannot expand it simply up to $\mathcal{O}(\alpha^3)_{\rm prag}$ terms and integrate term by term. The proper way of proceeding is to isolate two singular contributions and integrate them separately. The non-singular remnant is truncated to $\mathcal{O}(\alpha^3)_{\rm prag}$ and integrated term by term[11]. Of course, there is a freedom in the choice of the two singular components and we shall choose them to be maximally simple. We define the first singular contribution as follows

$$B_{0A}(t,v) = \int_0^v dv_F\, e^{\bar{\delta}_{\rm YFS}(\bar{\gamma}_p(0),0)} \times f_1\left(\tfrac{1}{2}\bar{\gamma}_p(0), v\right) f_0\left(\tfrac{1}{2}\bar{\gamma}_p(0), v_F\right) \left(\frac{v - v_F}{v}\right)^{\frac{1}{2}\bar{\gamma}_p(0)}, \tag{60}$$

where the additional factor $((v - v_F)/v)^p$ at the end of the formula is chosen in such a way that it does not affect the residue, *i.e.* $((v - v_F)/v)^p = 1$ at $v_F \to 0$. It is introduced to facilitate the transition from $F^2(\bar{\gamma}_p(0)/2)$ to $F(\bar{\gamma}_p(0))$ and later to the standard normalization factor $F(\gamma)$. The second singular contribution we choose in the analogous way

$$B_{0B}(t,v) = \int_0^v dv_F\, e^{\bar{\delta}_{\rm YFS}(\bar{\gamma}_p(v),v_F)} \times f_0\left(\tfrac{1}{2}\bar{\gamma}_p(v), v - v_F\right) f_1\left(\tfrac{1}{2}\bar{\gamma}_p(v), v\right) \left(\frac{v_F}{v}\right)^{\frac{1}{2}\bar{\gamma}_p(v)} \tag{61}$$

The non-singular integral (expandable to $\mathcal{O}(\alpha^3)_{\rm prag}$ prior to integration) reads, of course, as follows

$$B_{0R}(t,v) = B_0(t,v) - B_{0A}(t,v) - B_{0B}(t,v). \tag{62}$$

[11] The actual calculation was done by hand and also using the program `form1` for algebraic manipulations [25].

Let us introduce a short-hand notation, which will be useful also in the following Sections:

$$\Delta_{\rm YFS}(\gamma) = \frac{1}{4}\gamma + \frac{\alpha}{\pi}\Big(- \frac{1}{2} - \frac{\pi^2}{6}\Big), \tag{63}$$

$$\gamma' = \gamma - 2\frac{\alpha}{\pi}\ln(1-v) = \bar{\gamma}_p(0), \tag{64}$$

$$\gamma''(v) = \gamma + 2\frac{\alpha}{\pi}\ln(1-v) = \bar{\gamma}_p(v). \tag{65}$$

The two singular integrals read

$$\begin{aligned} B_{0A}(t,v) &= e^{\Delta_{\rm YFS}(\gamma')}\, f_1\Big(\tfrac{1}{2}\gamma', v\Big) \\ &\times \int_0^v dv_F\, f_0\Big(\tfrac{1}{2}\gamma', v_F\Big) \Big(\frac{v-v_F}{v}\Big)^{\frac{1}{2}\gamma'} \\ &= e^{\Delta_{\rm YFS}(\gamma')}\; f_1\Big(\tfrac{1}{2}\gamma', v\Big)\, F\Big(\tfrac{1}{2}\gamma'\Big) \frac{\Gamma^2\left(1+\frac{1}{2}\gamma'\right)}{\Gamma(1+\gamma')} v^{\frac{1}{2}\gamma'}, \\ B_{0B}(t,v) &= e^{\Delta_{\rm YFS}(\gamma'')-\gamma''\ln(1-v)}\, f_1\Big(\tfrac{1}{2}\gamma'', v\Big) \\ &\times \int_0^v dv_F\; f_0\Big(\tfrac{1}{2}\gamma'', v-v_F\Big) \Big(\frac{v_F}{v}\Big)^{\frac{1}{2}\gamma''} \\ &= e^{\Delta_{\rm YFS}(\gamma'')-\gamma''\ln(1-v)}\, f_1\Big(\tfrac{1}{2}\gamma'', v\Big) F\Big(\tfrac{1}{2}\gamma''\Big) \frac{\Gamma^2\left(1+\frac{1}{2}\gamma''\right)}{\Gamma(1+\gamma'')} v^{\frac{1}{2}\gamma''}, \end{aligned} \tag{66}$$

and the non-singular is found, as usual up to $\mathcal{O}(\alpha^3)_{\rm prag}$, to be:

$$B_{0R}(t,v) = e^{\Delta_{\rm YFS}(\gamma)}\; F(\gamma) v^{\gamma-1}\; \Big(-\frac{1}{8}\gamma^3 + \frac{3}{4}\frac{\alpha}{\pi}\gamma\Big)\ln^2(1-v), \tag{67}$$

so that the total result is

$$\begin{aligned} B_0(t,v) = \quad & F(\gamma)\, v^{\gamma-1}\, e^{\Delta_{\rm YFS}(\gamma)} \Big\{ \gamma - \frac{\alpha}{\pi}\ln(1-v) \\ & -\frac{1}{2}\gamma^2\ln(1-v) + \frac{1}{8}\gamma^3\ln^2(1-v) - \frac{\alpha}{\pi}\gamma\ln^2(1-v)\Big\}. \end{aligned} \tag{68}$$

3.5. Combining upper and lower line

Combining upper and lower line is done using exactly the same methods as combining initial- and final-state contribution for the upper fermion line described in the previous Section.

The two-line contribution from $\bar{\beta}_0^{(r)}$ reads, in three consecutive orders $r = 1, 2, 3$, as follows

$$\rho_{\bar{\beta}_0}^{(r)}(t,V) = b_0\,(1+\kappa^{(r)}) \int dv dv' \delta(V - v - v' + vv'))B_0(v)\; B_0(v')$$
$$= b_0\; F\,(2\gamma)\, V^{2\gamma-1}\, e^{2\Delta_{\rm YFS}(\gamma)}(1+\kappa^{(r)}) \Big\{ 2\gamma - 2\frac{\alpha}{\pi}\ln(1-V)$$
$$-2\gamma^2 \ln(1-V) + \frac{3}{4}\gamma^3 \ln^2(1-V) - \gamma\frac{\alpha}{\pi}\ln^2(1-V)\Big\}. \quad (69)$$

3.6. Numerical results on $\bar{\beta}_0$

In Fig. 1 we compare our semi-analytical result of Eq. (69) with the numerical Monte Carlo result of BHLUMI. We plot the quantity

$$R(\bar{\beta}_0^{(0)}; t, V_{\rm max}) = \frac{\int_0^{V_{\rm max}} \frac{d\sigma^{(0)}_{\bar{\beta}_0}}{d|t|\, dV} dV}{\frac{d\sigma_{\rm Born}}{d|t|}} = \int_0^{V_{\rm max}} \rho_{\bar{\beta}_0}^{(r)}(t,V)\, dV \quad (70)$$

as a function of the cut on the total photon energy $V_{\rm max}$, for the fixed transfer

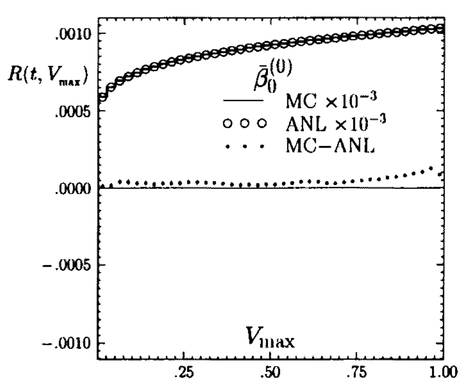

Fig. 1. The comparison of the Monte Carlo and semi-analytical results for the lowest-order $\bar{\beta}_0$.

$t = -4.612982$ GeV^2. We plot and examine the *difference* of the MC and the SAN results. The actual MC and SAN result are also plotted – they have to be multiplied by a factor 10^{-3} in order to be seen in the plot. In the semi-analytical result the integration over V is performed numerically (using the standard Gauss integration method). As we clearly see in the plot, the MC and SAN results agree to better then 1.5×10^{-4}! The same level of agreement was reached for $\bar{\beta}_0^{(1)}$ and $\bar{\beta}_0^{(2)}$. In spite of the simplicity of Eq. (69), the above result is very important – it is a cornerstone in establishing the overall normalization of the BHLUMI Monte Carlo at the level of 1×10^{-4}, simply because $\bar{\beta}_0$ represents 95% of the total cross section.

4. Contribution from $\bar{\beta}_1$

In the following we consider the emission from the upper line only, *i.e.* we concentrate on terms proportional to $\bar{\beta}_{1U}^{(r)}$, $r = 1, 2$. The total contribution from the lower line to total cross section is the same. For fixed $Q^2 = t$, let us consider the corresponding distribution

$$
\begin{aligned}
\frac{d\sigma_{1U}^{(r)}}{d|t|\, dV} &= \frac{4\pi\alpha^2}{|t|^2}\, \rho_{1U}^{(r)}(t,V) = \frac{4\pi\alpha^2}{|t|^2} \int dv\, dv' \\
&\times \int d\psi_p \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int d\omega_i\, \theta(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i\tilde{\beta}_i - \Delta)\ \mathrm{e}^{\gamma_p \ln \Delta + \delta_{\mathrm{YFS}}} \\
&\times \Theta(1 - \mathcal{Z}_p) \sum_{i=1}^{n} \frac{b_{1U}^{(r)}(\tilde{\alpha}_i, \tilde{\beta}_i)}{S_p(\tilde{\alpha}_1, \tilde{\beta}_i)}\, \delta\Big(v - \sum_{i=1}^{n}(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i\tilde{\beta}_i)\Big)\ \delta(\phi_{K_p}) \\
&\times \int d\psi_q \sum_{n'=1}^{\infty} \frac{1}{n'!} \prod_{j=1}^{n'} \int d\omega_j'\, \theta(\tilde{\alpha}_i' + \tilde{\beta}_i' - \tilde{\alpha}_i'\tilde{\beta}_i' - \Delta)\ \mathrm{e}^{\gamma_q \ln \Delta + \delta_{\mathrm{YFS}}'} \\
&\times \Theta(1 - \mathcal{Z}_q)\delta\Big(v' - \sum_{i=1}^{n'}(\tilde{\alpha}_i' + \tilde{\beta}_i' - \tilde{\alpha}_i'\tilde{\beta}_i')\Big)\ \delta(\phi_{K_q}) \\
&\times \delta(V - v - v' + vv') \\
&= \frac{4\pi\alpha^2}{|t|^2} \int dv dv' \delta(V - v - v' + vv') B_1^{(r)}(v)\ B_0(v')\,, \qquad (71)
\end{aligned}
$$

where we know the function $B_0(v')$ from the $\bar{\beta}_0$ calculation and the new function

$$
B_1^{(r)}(v) = \int d\psi_p \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \int d\omega_i \theta(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i\tilde{\beta}_i - \Delta)\, \mathrm{e}^{\gamma_p \ln \Delta + \delta_{\mathrm{YFS}}}
$$

$$\times\, \Theta(1-\mathcal{Z}_p)\sum_{i=1}^{n}\frac{b_{1U}^{(r)}(\tilde\alpha_i,\tilde\beta_i)}{S_p(\tilde\alpha_i,\tilde\beta_i)}\,\delta\Big(v-\sum_{i=1}^{n}(\tilde\alpha_i+\tilde\beta_i-\tilde\alpha_i\tilde\beta_i)\Big)\,\delta(\phi_{K_p}) \tag{72}$$

is to be calculate in this Section.

4.1. Collinearization

In the first step in our analytical calculation we introduce a series of $\mathcal{O}(\alpha^3)_{\rm prag}$ approximations, which lead to *collinearization* of the integral (4), *i.e.* summation/integration over an infinite series of photons. We start, as in the $\tilde\beta_0$ case, by approximating the function $\mathcal{Z}_p$, which monitors the upper limit of the phase-space, as follows

$$\mathcal{Z}_p \to \bar{\mathcal{Z}}_p = 1-\sum_{i=1}^{n}(\tilde\alpha_i+\tilde\beta_i-\tilde\alpha_i\tilde\beta_i) = 1-\sum_{i=1}^{n}v_i. \tag{73}$$

Let us now reorganize the integral as follows (no further approximations). First we separate the photons in the sum into two categories: these that enter into $b_{1U}^{(r)}$ and those that do not. Each of the above categories is split into two categories, which will be tagged with the index $K=I,F$: *(i)* photons with $\tilde\alpha_i>\tilde\beta_i$, which will be referred to as initial-state photons ($K=I$) and *(ii)* photons with $\tilde\alpha_i<\tilde\beta_i$, called final-state photons ($K=F$). Using the identity

$$d\omega_i=\theta(\tilde\alpha_i-\tilde\beta_i)d\omega_i+\theta(\tilde\beta_i-\tilde\alpha_i)d\omega_i=d\omega_i^I+d\omega_i^F \tag{74}$$

we come to the following expression

$$\begin{aligned}
B_1^{(r)}(t,v) &= B_{1I}^{(r)}(t,v)+B_{1F}^{(r)}(t,v)\\
B_{1K}^{(r)}(t,v) &= \int dv_I dv_F dv_1\;\delta(v-v_I-v_F-v_1)\\
&\quad\times\int d\psi_p\;\theta(1-v_I-v_F-v_1)\\
&\quad\times\int d\omega_1^K\delta(v_1-\tilde\alpha_1-\tilde\beta_1+\tilde\alpha_1\tilde\beta_1)\;b_{1U}^{(r)}(\tilde\alpha_1,\tilde\beta_1)/S_p(\tilde\alpha_1,\tilde\beta_1)\\
&\quad\times\Big[\mathrm{e}^{\frac12\gamma_p\ln\Delta}\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int_{v_i>\Delta}d\omega_i^I\delta(v_I-\sum_{i=0}^{n}v_i)\Big]\\
&\quad\times\Big[\mathrm{e}^{\frac12\gamma_p\ln\Delta}\sum_{n'=0}^{\infty}\frac{1}{n'!}\prod_{j=1}^{n'}\int_{v_j>\Delta}d\omega_j^F\delta(v_F-\sum_{i=0}^{n'}v_j)\Big]\,\mathrm{e}^{\delta_{\rm YFS}}\;\delta(\phi_{K_p})\,.
\end{aligned} \tag{75}$$

The next step in the collinearization is similar to what is done in the case of $\bar{\beta}_0$ (using the theorem proved in Appendix A) by means of the crucial $\mathcal{O}(\alpha^3)_{\rm prag}$ approximation:

$$\delta_p = \frac{1-v}{\left(1-\tilde{\beta}_1-\tilde{\beta}_I-\tilde{\beta}_F\right)^2}\frac{m_e^2}{|t|} \longrightarrow \bar{\delta}_p^K = \frac{1-v}{\left(1-v_{1K}-v_F\right)^2}\frac{m_e^2}{|t|}, \quad (76)$$

where $v_{1I} \equiv 0$ and $v_{1F} = v_1$, in all bremsstrahlung distributions $d\bar{\omega}_1^K$ and $d\bar{\omega}_i^K$ and also in part of the form factor $e^{\frac{1}{2}\gamma_p \ln \Delta}$. In the remaining part of the form factor within $\mathcal{O}(\alpha^3)_{\rm prag}$ we are able to substitute

$$\delta_{\rm YFS}(\gamma_p, \tilde{\beta}_I+\tilde{\beta}_F+\tilde{\beta}_1) \longrightarrow \delta_{\rm YFS}(\bar{\gamma}_p^K, \tilde{\beta}_I+\tilde{\beta}_F+\tilde{\beta}_1), \qquad K=I,F. \quad (77)$$

Here we have implicitly introduced the following self-explanatory definition:

$$\bar{\gamma}_p^K = 2\frac{\alpha}{\pi}\left(\ln\frac{(1-v_F-v_{1K})^2}{1-v}\frac{|t|}{m_e^2}-1\right) = \gamma + 2\frac{\alpha}{\pi}\ln\frac{(1-v_F-v_{1K})^2}{1-v}. \quad (78)$$

With the above approximations we arrive at the following expression:

$$\begin{aligned} B_{1K}^{(r)}(t,v) = &\int_0^1 dv_I dv_F dv_1 \delta(v-v_I-v_F-v_1) \int d\psi_p \\ &\times \int d\bar{\omega}_1^K \delta(v_1-\tilde{\alpha}_1-\tilde{\beta}_1+\tilde{\alpha}_1\tilde{\beta}_1)\frac{b_{1U}^{(r)}(\tilde{\alpha}_1,\tilde{\beta}_1)}{S_p(\tilde{\alpha}_1,\tilde{\beta}_1)} \\ &\times \left[e^{\frac{1}{2}\bar{\gamma}_p^K \ln\Delta}\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int_{v_i>\Delta} d\bar{\omega}_i^I \delta(v_I-\sum_{i=0}^{n} v_i)\right] \\ &\times \left[e^{\frac{1}{2}\bar{\gamma}_p^K \ln\Delta}\sum_{n'=0}^{\infty}\frac{1}{n'!}\prod_{j=1}^{n'}\int_{v_j>\Delta} d\bar{\omega}_j^F \delta(v_F-\sum_{i=0}^{n'} v_j)\right] \\ &\times e^{\delta_{\rm YFS}(\bar{\gamma}_p,\tilde{\beta}_I+\tilde{\beta}_F+\tilde{\beta}_1)}\,\delta(\phi_{K_p}). \end{aligned} \quad (79)$$

As we see, the only thing that now prevents us from integrating and summing over the infinite series of photon contributions is the form factor dependence on $\tilde{\beta}_I+\tilde{\beta}_F+\tilde{\beta}_1$. Similarly to the $\bar{\beta}_0$ case, we expand

$$\begin{aligned} e^{\delta_{\rm YFS}(\bar{\gamma}_p^K,\tilde{\beta}_I+\tilde{\beta}_F+\tilde{\beta}_1)} = & e^{\delta_{\rm YFS}(\bar{\gamma}_p^K, v_{1,K}+v_F)}\Big\{1+\bar{\gamma}_p^K[\ln(1-v_F-v_{1K}) \\ & -\ln(1-\tilde{\beta}_I-\tilde{\beta}_F-\tilde{\beta}_1)]+\mathcal{O}((\bar{\gamma}_p^K)^2)\Big\} \end{aligned} \quad (80)$$

and we are able to perform the integration and summation over the infinite series of photon contributions for the first term in the above expansion, while for the second term the summation is *irrelevant* because the only $\mathcal{O}(\alpha\gamma)$ contribution of interest comes from the *single-photon* configuration. Consequently, after performing the multi-photon integrations/summations we obtain

$$\begin{aligned} B_{1K}^{(r)}(t,v) &= \int dv_I dv_F dv_1 \delta(v-v_I-v_F-v_1) \\ &\times \int \frac{d\psi_p}{2\pi} \int d\bar{\omega}_{1K} \delta(v_1-\tilde{\alpha}_1-\tilde{\beta}_1+\tilde{\alpha}_1\tilde{\beta}_1) \\ &\times \frac{b_{1U}^{(r)}(\tilde{\alpha}_1,\tilde{\beta}_1)}{S_p(\tilde{\alpha}_1,\tilde{\beta}_1)} e^{\bar{\delta}_{\rm YFS}(\bar{\gamma}_p^K, v_F+v_1)} f_1\left(\tfrac{1}{2}\bar{\gamma}_p^K, v_I\right) f_1\left(\tfrac{1}{2}\bar{\gamma}_p^K, v_F\right) \\ &+ \int d\bar{\omega}_{1K} \int \frac{d\psi_p}{2\pi}\, \delta(v-\tilde{\alpha}_1-\tilde{\beta}_1+\tilde{\alpha}_1\tilde{\beta}_1) \frac{b_{1U}^{(r-1)}(\tilde{\alpha}_1,\tilde{\beta}_1)}{S_p(\tilde{\alpha}_1,\tilde{\beta}_1)} \\ &\times\gamma[\ln(1-v_{1K})-\ln(1-\tilde{\beta}_1)]\,. \end{aligned} \tag{81}$$

Note that the elimination of $\delta(\phi_{K_p})$ gives rise to the $1/2\pi$ factor[12]. Concentrating on the more general second-order case the corresponding integral is brought to the following nice form, which is the starting point for further work

$$\begin{aligned} B_{1K}^{(2)}(t,v) &= \int dv_I dv_F dv_1 \delta(v-v_I-v_F-v_1) \\ &\times \int \frac{d\psi_p}{2\pi} \int d\bar{\omega}_{1K}\; \delta(v_1-\tilde{\alpha}_1-\tilde{\beta}_1+\tilde{\alpha}_1\tilde{\beta}_1) \Big[b_0 \nu(v_1)\,(1+\gamma\pi(0)) \\ &+ b_0\chi(v_1)\gamma\,(\pi(v_{1K})-\pi(0)) + (1+\gamma\pi(v_{1K}))\, h(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)\Big] \\ &\times e^{\bar{\delta}_{\rm YFS}(\bar{\gamma}_p^K, v_F+v_{1K})}\; f_1\left(\tfrac{1}{2}\bar{\gamma}_p^K, v_I\right)\; f_1\left(\tfrac{1}{2}\bar{\gamma}_p^K, v_F\right) \\ &+ \int d\bar{\omega}_{1K} \int \frac{d\psi_p}{2\pi} \delta(v-\tilde{\alpha}_1-\tilde{\beta}_1+\tilde{\alpha}_1\tilde{\beta}_1)[b_0\nu(v_1) \\ &+ h(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)]\gamma[\ln(1-v_{1K})-\ln(1-\tilde{\beta}_1)]\,. \end{aligned} \tag{82}$$

In the above formula we have isolated the leading-logarithmic contribution, using the following decomposition into leading-log and sub-leading parts (prior to integration over the phase-space):

$$H(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p) = b_0\chi(v_1)+h(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p) = b_0+b_0\;\nu(v_1)+h(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)\,, \tag{83}$$

[12] The formal proof goes as follows: one may add integration $(2\pi)^{-1}\int d\phi \equiv 1$ over dummy ϕ angle, then rotate all photons with ϕ. All of the integrand is invariant under such rotation except $\delta(\phi_{K_p})$, which transforms into $\delta(\phi_{K_p}-\phi)$. This now can be removed at the expense of $\int d\phi$.

where

$$\chi(x) \equiv \tfrac{1}{2}(1+(1-x)^2), \qquad \nu(x) \equiv \chi(x) - 1 = x\left(-1+\tfrac{x}{2}\right). \tag{84}$$

Let us still reorder our integral, perform some relatively trivial azimuthal angle integrations and do the integration over ψ_p and $d\phi_1$ in $d\omega_1$. Neglecting unimportant terms of $\mathcal{O}(\alpha\xi)$ and $\mathcal{O}(m_e^2/t)$ we find[13]

$$\begin{aligned}&\int \frac{d\psi_p}{2\pi} \int d\bar{\omega}_1\, \delta(v_1 - \tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\alpha}_1\tilde{\beta}_1)\; h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)\\ &= \frac{\alpha}{\pi} \int d\tilde{\alpha}_1 d\tilde{\beta}_1 \delta(v_1 - \tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\alpha}_1\tilde{\beta}_1) g(\tilde{\alpha}_1, \tilde{\beta}_1)\,,\end{aligned} \tag{85}$$

where

$$\begin{aligned}&g(\tilde{\alpha}_1, \tilde{\beta}_1) = \frac{v_1^2}{2}\left(\frac{\bar{\delta}_p}{y_1^2} + \frac{\bar{\delta}_p}{z_1^2}\right) + 2(v_1 - 1) + \tfrac{1}{2}\tilde{\alpha}_1\tilde{\beta}_1 - \xi\, \frac{\chi(v_1)}{1-v_1}\,\frac{1}{z_1}\,,\\ &y_i = \tilde{\alpha}_i + \tilde{\beta}_i \bar{\delta}_p^K, \qquad z_i = \tilde{\beta}_i + \tilde{\alpha}_i \bar{\delta}_p^K\,.\end{aligned} \tag{86}$$

With the help of the above identity we obtain

$$\begin{aligned}&B^{(2)}_{1K}(t,v) = B^{(2)}_{1Km}(t,v) + B^{(2)}_{1K\mathrm{singl}}(t,v_1),\\ &B^{(2)}_{1Km}(t,v) = \int dv_I dv_F dv_1 \delta(v - v_I - v_F - v_1)\\ &\times \frac{\alpha}{\pi} \int_{1K} \left[\frac{\tilde{\alpha}_1\tilde{\beta}_1}{y_1^2 z_1^2}\left(b_0 \nu(v_1)\Big(1 + \gamma\pi(0)\Big) + b_0\chi(v_1)\gamma\Big(\pi(v_{1K}) - \pi(0)\Big)\right)\right.\\ &\left. + \Big(1+\gamma\pi(v_{1K})\Big) g(\tilde{\alpha}_1, \tilde{\beta}_1)\right] e^{\delta_{\mathrm{YFS}}(\bar{\gamma}_p^K, v_F + v_{1K})}\; f_1\left(\tfrac{1}{2}\bar{\gamma}_p^K, v_I\right)\; f_1\left(\tfrac{1}{2}\bar{\gamma}_p^K, v_F\right),\\ &B^{(2)}_{1K\mathrm{singl}}(t,v_1) = \frac{\alpha}{\pi}\gamma \int_{1K} \left[\frac{\tilde{\alpha}_1\tilde{\beta}_1}{y_1^2 z_1^2} b_0 \nu(v_1) + g(\tilde{\alpha}_1, \tilde{\beta}_1)\right] \ln\frac{1 - v_{1K}}{1 - \tilde{\beta}_1}\,,\end{aligned} \tag{87}$$

where we use the following short-hand notation

$$\begin{aligned}&\int_{1I} \equiv \int_{\tilde{\alpha}_1 > \tilde{\beta}_1} d\tilde{\alpha}_1 d\tilde{\beta}_1 \delta(v_1 - \tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\alpha}_1\tilde{\beta}_1)\,,\\ &\int_{1F} \equiv \int_{\tilde{\alpha}_1 < \tilde{\beta}_1} d\tilde{\alpha}_1 d\tilde{\beta}_1 \delta(v_1 - \tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\alpha}_1\tilde{\beta}_1)\,.\end{aligned} \tag{88}$$

[13] We have checked numerically with the help of a dedicated MC run that the approximations in the above equation introduce only 10^{-6} of relative error.

We are now ready to integrate over the direction of the first photon. Let us start with the single-photon sub-leading integral in $B^{(2)}_{1K\mathrm{sing}}(t,v_1)$. As in the $\bar{\beta}_0$ case we find that it does not contribute to the total result because contributions from the initial $K=I$ and final $K=F$ state do cancel due to $B^{(2)}_{1I\mathrm{sing}}(t,v) = -B^{(2)}_{1F\mathrm{sing}}(t,v)$. This is generally true for any integral of the form

$$A_K(v_1) = \int_{1K} f(\tilde{\alpha}_1,\tilde{\beta}_1)\ \ln\frac{1-v_{1K}}{1-\tilde{\beta}_1}\,, \tag{89}$$

where $f(\tilde{\alpha}_1,\tilde{\beta}_1)$ is symmetric. Note that here the non-symmetric $\mathcal{O}(\xi)$ part in $g(\tilde{\alpha}_1,\tilde{\beta}_1)$ can be neglected. Nevertheless we calculate this contribution explicitly, for the purpose of some important future tests, see Section 8:

$$\begin{aligned} B^{(2)}_{1K\mathrm{singl}}(t,v) \;=\; & \gamma\frac{\alpha}{\pi}\Big\{-1+\frac{v}{2}+\sqrt{1-v} \\ & +\ln^2(1-v)\left[\frac{3}{8}-\frac{5}{16}v+\frac{1}{8v}\right] \\ & +(1/v)[\mathrm{Li}_2(1)+\mathrm{Li}_2(1-v)-2\mathrm{Li}_2(\sqrt{(1-v)})]\Big\}. \end{aligned} \tag{90}$$

In the main leading logarithmic integral

$$\begin{aligned} B^{(2)}_{1Km}(t,v) \;=\; & \int dv_I dv_F dv_1 \delta(v-v_I-v_F-v_1) \\ & \times\frac{\alpha}{\pi}\int_{1K}\left[\frac{\tilde{\alpha}_1\tilde{\beta}_1}{y_1^2 z_1^2}\left\{b_0\nu(v_1)(1+\gamma)+b_0\chi(v_1)\gamma\frac{1}{4}\ln\frac{(1-v_{1K})^2}{1-v_1}\right\}\right. \\ & \left.+g(\tilde{\alpha}_1,\tilde{\beta}_1)\left\{1+\gamma\left(1+\frac{1}{4}\ln\frac{(1-v_{1K})^2}{1-v_1}\right)\right\}\right] \\ & \times e^{\delta_{\mathrm{YFS}}(\bar{\gamma}_p^K, v_F+v_{1K})}\, f_1\left(\tfrac{1}{2}\bar{\gamma}_p^K, v_I\right) f_1\left(\tfrac{1}{2}\bar{\gamma}_p^K, v_F\right) \end{aligned} \tag{91}$$

we need the following elementary integrals

$$\begin{aligned} \frac{\alpha}{\pi}G_a^K(v_1) \;&=\; \frac{\alpha}{\pi}\int_{1K}\frac{\tilde{\alpha}_1\tilde{\beta}_1}{y_1^2 z_1^2} = \frac{1}{2}\frac{1}{v_1}[\bar{\gamma}_p^K - \frac{\alpha}{\pi}\ \ln(1-v_1)], \quad K=I,F, \\ \frac{\alpha}{\pi}g_a^I(v_1) \;&=\; \frac{\alpha}{\pi}\int_{1I} g(\tilde{\alpha}_1,\tilde{\beta}_1) \\ &=\; \frac{\alpha}{\pi}\frac{1}{2}\left(-3+\frac{5}{2}v\right)\ln(1-v_1)+\xi\,\gamma\,\frac{-3-2v^2+4v}{8(1-v_1)}, \\ \frac{\alpha}{\pi}g_a^F(v_1) \;&=\; \frac{\alpha}{\pi}\int_{1F} g(\tilde{\alpha}_1,\tilde{\beta}_1) \\ &=\; \frac{\alpha}{\pi}\frac{1}{2}\left(-3+\frac{5}{2}v\right)\ln(1-v_1),+\xi\,\gamma\,\frac{-6+5v}{8(1-v_1)}\,. \end{aligned} \tag{92}$$

The integration over the photon direction with the help of the above formulas leads to the following results with a 2-fold convolution

$$
\begin{aligned}
&B_1^{(2)}(t,v) = B_{1Im}^{(2)}(t,v) + B_{1Fm}^{(2)}(t,v)\,,\\
&B_{1Im}^{(2)}(t,v) = \int dv_I dv_F dv_1 \delta(v - v_I - v_F - v_1)\; e^{\bar\delta_{\rm YFS}(\bar\gamma_p^I, v_F)}\\
&\times \left[\frac{1}{2}\frac{1}{v_1}\;\left\{\bar\gamma_p^I - \tfrac{\alpha}{\pi}\;\ln(1-v_1)\right\}\left(b_0\nu(v_1)(1+\gamma) - b_0\chi(v_1)\gamma\tfrac{1}{4}\ln(1-v_1)\right)\right.\\
&\left.+\frac{\alpha}{\pi}g_a^I(v_1)\left\{1+\gamma\left(1-\tfrac{1}{4}\ln(1-v_1)\right)\right\}\right] f_1\left(\tfrac{1}{2}\bar\gamma_p^I, v_I\right)\; f_1\left(\tfrac{1}{2}\bar\gamma_p^I, v_F\right),\\
&B_{1Fm}^{(2)}(t,v) = \int dv_I dv_F dv_1 \delta(v - v_I - v_F - v_1)\; e^{\bar\delta_{\rm YFS}(\bar\gamma_p^F, v_F+v_1)}\\
&\times \left[\frac{1}{2}\frac{1}{v_1}\;\left\{\bar\gamma_p^F - \tfrac{\alpha}{\pi}\;\ln(1-v_1)\right\}\left(b_0\nu(v_1)(1+\gamma) + b_0\chi(v_1)\gamma\tfrac{1}{4}\ln(1-v_1)\right)\right.\\
&\left.+\frac{\alpha}{\pi}g_a^F(v_1)\left\{1+\gamma\left(1+\tfrac{1}{4}\ln(1-v_1)\right)\right\}\right] f_1\left(\tfrac{1}{2}\bar\gamma_p^F, v_I\right) f_1\left(\tfrac{1}{2}\bar\gamma_p^F, v_F\right)\,.
\end{aligned}
\tag{93}
$$

4.2. Last integrations

In the following we shall evaluate two double-convolution integrals starting with $B_{1Fm}^{(2)}$, which is a little bit easier. Generally, we shall use the same "pole decomposition" procedure as in the $\bar\beta_0$ calculation and it will be used twice in the process of integration. We have some freedom in the order of integration, which we will exploit to facilitate the integration. In the case of $B_{1Fm}^{(2)}$ it is easier to perform first the sub-convolution in variables v_F and v_1 keeping $u = v_F + v_1$ constant because u enters in a natural way into

$$\bar\gamma_p^F(v_F+v_1) = \gamma + 2\frac{\alpha}{\pi}\ln\frac{(1-v_F-v_1)^2}{1-v} = \gamma''(v) + 4\frac{\alpha}{\pi}\ln\frac{1-v_F-v_1}{1-v}. \tag{94}$$

Let us recall also the definition of the form factor

$$
\begin{aligned}
\delta_{\rm YFS}(\bar\gamma_p^F(v_F+v_1), v_F+v_1) = &-\; \bar\gamma_p^F(v_F+v_1)\ln\left(1-v_F-v_1\right)\\
&+\;\frac{1}{4}\bar\gamma_p^F(v_F+v_1) + \frac{\alpha}{\pi}\left(-\frac{1}{2}-\frac{\pi^2}{6}\right),
\end{aligned}
\tag{95}
$$

see the notation in Eqs. (63)–(65). Summarizing, the non-trivial dependence of f_1 on $v_F+v_1 = v - v_I$ through $\bar\gamma_p^F$ dictates the following economical order of integration

$$B^{(2)}_{1Fm}(t,v) = \int_0^v dv_I du \delta(v - v_I - u) \times e^{\bar{\delta}_{YFS}(\bar{\gamma}_p^F(u),u)} f_1\left(\tfrac{1}{2}\bar{\gamma}_p^F(u), v_I\right) R_F^{(2)}(\bar{\gamma}_p^F(u), u), \quad (96)$$

where

$$R_F^{(2)}(\bar{\gamma}_p^F, u) = \int dv_F dv_1 \delta(u - v_F - v_1)\, f_1\left(\tfrac{1}{2}\bar{\gamma}_p^F, v_F\right) \times \left[\frac{1}{2}\frac{1}{v_1}\left\{\bar{\gamma}_p^F - \frac{\alpha}{\pi}\ln(1-v_1)\right\}\left(b_0\nu(v_1)(1+\gamma) + b_0\chi(v_1)\gamma\tfrac{1}{4}\ln(1-v_1)\right) + \frac{\alpha}{\pi}g_a^F(v_1)\left\{1+\gamma\left(1+\tfrac{1}{4}\ln(1-v_1)\right)\right\}\right]. \quad (97)$$

The inner convolution is done with the usual techniques, see the case of $\tilde{\beta}_0$, and the result reads as follows:

$$\begin{aligned} R_F^{(2)}(\gamma_F, u) &= F\left(\tfrac{1}{2}\bar{\gamma}_p^F\right) u^{\frac{1}{2}\bar{\gamma}_p^F} \\ &\times\Bigg\{\bar{\gamma}_p^F\left(-\frac{1}{2}+\frac{1}{4}u\right) + \frac{\alpha}{\pi}\ln(1-u)(-1+u) + (\bar{\gamma}_p^F)^2\left(-\frac{1}{8}u\right) \\ &+\gamma\bar{\gamma}_p^F\left(-\frac{1}{2}+\frac{1}{4}u\right) \\ &+\gamma\bar{\gamma}_p^F\ln(1-u)\left(-\frac{1}{4}+\frac{1}{8}u+\frac{1}{4u}\right) \\ &+\gamma^3\ln(1-u)\left(\frac{1}{16}-\frac{1}{16}u\right) + \gamma^3\mathrm{Li}_2\,u\left(-\frac{1}{8}+\frac{1}{16}u\right) \\ &+\gamma^3\ln(1-u)^2\left(-\frac{1}{16}+\frac{1}{32}u+\frac{1}{16}u\right) \\ &+\gamma\frac{\alpha}{\pi}\frac{3}{8}u + \gamma\frac{\alpha}{\pi}\ln(1-u)\left(-\frac{5}{8}+\frac{5}{8}u\right) \\ &+\gamma\frac{\alpha}{\pi}\ln(1-u)^2\left(-\frac{3}{4}+\frac{3}{4}u-\frac{1}{4u}\right) \\ &+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(u)\left(-\frac{3}{4}+\frac{5}{8}u\right)\Bigg\}, \end{aligned} \quad (98)$$

where we have replaced $\bar{\gamma}_p^F \to \gamma$ wherever possible. The second integration yields us the total second-order result

$$B^{(2)}_{1Fm}(t,v) = \tfrac{1}{2}b_0\, F(\gamma'')\, v^{\gamma''}\exp(\Delta_{YFS}(\gamma'') - \gamma''\ln(1-v)) \times\left\{\gamma''\left(-1+\frac{v}{2}\right) + \frac{\alpha}{\pi}\ln(1-v)(-2+2v) + \gamma''^2\left(-\frac{v}{2}\right)\right.$$

$$
+\gamma\gamma''\left(-1+\frac{v}{2}\right)+\gamma\gamma''\ln(1-v)\left(-\frac{1}{2}+\frac{v}{4}+\frac{1}{2v}\right)
+\gamma^3\ln(1-v)\left(-\frac{v}{4}\right)+\gamma^3\ln^2(1-v)\left(\frac{1}{4v}\right)
+\gamma\frac{\alpha}{\pi}\left(\frac{7v}{2}\right)+\gamma\frac{\alpha}{\pi}\ln(1-v)\left(\frac{3}{2}+\frac{v}{2}\right)
+\gamma\frac{\alpha}{\pi}\ln^2(1-v)\left(-4+3v-\frac{1}{2v}\right)+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(v)\left(-7+\frac{9v}{2}\right)\Big\}. \tag{99}
$$

In the latter discussion we shall also use the corresponding results for the first-order matrix element. It reads as follows

$$
B^{(1)}_{1Fm}(t,v)=\tfrac{1}{2}b_0\,F(\gamma'')\,v^{\gamma''}\exp(\Delta_{\rm YFS}(\gamma'')-\gamma''\ln(1-v))
\times\Big\{\gamma''\left(-1+\frac{v}{2}\right)+\frac{\alpha}{\pi}\ln(1-v)(-2+2v)+\gamma''^2\left(-\frac{v}{2}\right)
+\gamma^3\left(\frac{v}{4}\right)+\gamma^3\ln(1-v)\left(-\frac{1}{4}\right)+\gamma^3\ln^2(1-v)\left(\frac{1}{4}-\frac{v}{8}\right)
+\gamma^3\mathrm{Li}_2(v)\left(\frac{1}{2}-\frac{v}{4}\right)+\gamma\frac{\alpha}{\pi}\frac{7v}{2}
+\gamma\frac{\alpha}{\pi}\ln(1-v)\left(\frac{7}{2}-\frac{3v}{2}\right)+\gamma\frac{\alpha}{\pi}\ln^2(1-v)(-3+2v)
+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(v)\left(-7+\frac{9v}{2}\right)\Big\}. \tag{100}
$$

In the remaining initial-state contribution, the order of convolutions is dictated again by the v_F dependence in

$$
\bar{\gamma}^I_p(v_F)=\gamma+2\frac{\alpha}{\pi}\ln\frac{(1-v_F)^2}{1-v}=\gamma'(v)+4\frac{\alpha}{\pi}\ln(1-v_F), \tag{101}
$$

which suggests that we convolute first v_1 with v_I and next $u=v_1+v_I$ with v_F as follows

$$
B^{(2)}_{1Im}(t,v)=\int_0^v dv_F du\,\delta(v-v_F-u)\;e^{\bar{\delta}_{\rm YFS}(\bar{\gamma}^I_p(v_F),v_F)}
\times f_1\left(\tfrac{1}{2}\bar{\gamma}^I_p(v_F),v_F\right)R^{(2)}_I(\bar{\gamma}^I_p(v_F),u), \tag{102}
$$

where

$$
R^{(2)}_I(\bar{\gamma}^I_p,u)=\int dv_I dv_1\delta(u-v_I-v_1)\;f_1\left(\tfrac{1}{2}\bar{\gamma}^I_p,v_I\right)
$$

$$\times \left[\frac{1}{2}\frac{1}{v_1}\left\{\bar{\gamma}_p^I - \frac{\alpha}{\pi}\ln(1-v_1)\right\}\left(b_0\nu(v_1)(1+\gamma) - b_0\chi(v_1)\gamma\tfrac{1}{4}\ln(1-v_1)\right)\right.$$
$$\left. +\frac{\alpha}{\pi}g_a^I(v_1)\left\{1+\gamma\left(1-\tfrac{1}{4}\ln(1-v_1)\right)\right\}\right]. \tag{103}$$

The result of the inner convolution reads

$$R_I^{(2)}(\bar{\gamma}_p^I,u) = F\left(\tfrac{1}{2}\bar{\gamma}_p^I\right)u^{(1/2)\bar{\gamma}_p^I}$$
$$\times\left\{\bar{\gamma}_p^I\left(-\frac{1}{2}+\frac{1}{4}u\right)+\frac{\alpha}{\pi}\ln(1-u)(-1+u)+(\bar{\gamma}_p^I)^2\left(-\frac{1}{8}u\right)\right.$$
$$+\gamma\bar{\gamma}_p^I\left(-\frac{1}{2}+\frac{1}{4}u\right)+\gamma\bar{\gamma}_p^I\ln(1-u)\left(\frac{1}{4}-\frac{1}{8}u-\frac{1}{4u}\right)$$
$$+\gamma^3\left(-\frac{1}{8}u\right)+\gamma^3\ln(1-u)\left(-\frac{1}{16}+\frac{1}{16}u\right)$$
$$+\gamma^3\mathrm{Li}_2(u)\left(\frac{1}{8}-\frac{1}{16}u\right)$$
$$+\gamma^3\ln(1-u)^2\left(\frac{1}{16}-\frac{1}{32}u-\frac{1}{16u}\right)$$
$$+\gamma\frac{\alpha}{\pi}\frac{3}{8}u+\gamma\frac{\alpha}{\pi}\ln(1-u)\left(-\frac{5}{8}+\frac{5}{8}u\right)$$
$$+\gamma\frac{\alpha}{\pi}\ln(1-u)^2\left(\frac{1}{4}-\frac{1}{4}u+\frac{1}{4u}\right)$$
$$\left.+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(u)\left(-\frac{3}{4}+\frac{5}{8}u\right)-\gamma\xi\frac{\chi(u)}{2(1-u)}\right\}, \tag{104}$$

where we have replaced $\bar{\gamma}_p^I \to \gamma$ wherever it was possible. The second integration/convolution leads to the following results for the corresponding $\mathcal{O}(\alpha^{2,1})_{\rm prag}$ matrix elements

$$B_{1Im}^{(2)}(t,v) = \tfrac{1}{2}b_0\,F(\gamma')\,v^{\gamma'}\,e^{\Delta_{\rm YFS}(\gamma')}$$
$$\times\left\{\gamma'\left(-1+\frac{v}{2}\right)+\frac{\alpha}{\pi}\ln(1-v)(-2+2v)+\gamma'^2\left(-\frac{1}{2}v\right)\right.$$
$$+\gamma\gamma'\left(-1+\frac{v}{2}\right)+\gamma\gamma'\ln(1-v)\left(\frac{1}{2}-\frac{v}{4}-\frac{1}{2v}\right)$$
$$+\gamma^3\left(-\frac{v}{2}\right)+\gamma^3\ln(1-v)\left(-\frac{1}{2}+\frac{v}{2}\right)$$
$$+\gamma^3\ln^2(1-v)\left(\frac{1}{4}-\frac{v}{8}-\frac{1}{4v}\right)+\gamma\frac{\alpha}{\pi}\left(\frac{7v}{2}\right)$$
$$+\gamma\frac{\alpha}{\pi}\ln(1-v)\left(\frac{3}{2}-\frac{3v}{2}\right)+\gamma\frac{\alpha}{\pi}\ln^2(1-v)\frac{1}{2v}$$

$$+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(v)\left(1+\frac{v}{2}\right)-\xi\gamma\theta(1-v-\xi)\frac{\chi(v)}{1-v}\Bigg\},\qquad(105)$$

$$\begin{aligned}B^{(1)}_{1Im}(t,v)&=\tfrac{1}{2}b_0\,F(\gamma')v^{\gamma'}\,\mathrm{e}^{\Delta_{\mathrm{YFS}}(\gamma')}\\&\times\Bigg\{\gamma'\left(-1+\frac{v}{2}\right)+\frac{\alpha}{\pi}\ln(1-v)(-2+2v)+\gamma'^2\left(-\frac{v}{2}\right)\\&+\gamma^3\left(\frac{v}{4}\right)+\gamma^3\ln(1-v)\left(-\frac{1}{4}+\frac{v}{4}\right)\\&+\gamma^3\mathrm{Li}_2(v)\left(-\frac{1}{2}+\frac{v}{4}\right)\\&+\gamma\frac{\alpha}{\pi}\left(\frac{7v}{2}\right)+\gamma\frac{\alpha}{\pi}\ln(1-v)\left(\frac{7}{2}-\frac{7v}{2}\right)\\&+\gamma\frac{\alpha}{\pi}\ln^2(1-v)(-1+v)\\&+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(v)\left(1+\frac{v}{2}\right)-\xi\gamma\theta(1-v-\xi)\frac{\chi(v)}{1-v}\Bigg\}.\end{aligned}\qquad(106)$$

In the above expressions we still kept the γ' and γ'' resulting directly from the integrations. In the final expression for the B_1-function we expand them

$$\begin{aligned}B^{(2)}_1(v)&=b_0\,F(\gamma)\,v^{\gamma}\,\mathrm{e}^{\Delta_{\mathrm{YFS}}(\gamma)}\\&\times\Bigg\{\gamma\left(-1+\frac{v}{2}\right)+\frac{\alpha}{\pi}\ln(1-v)\,(-2+2v)+\gamma^2(-1)\\&+\gamma^2\ln(1-v)\left(\frac{1}{2}-\frac{v}{4}\right)+\gamma^3\left(-\frac{v}{4}\right)+\gamma^3\ln(1-v)\left(\frac{1}{4}+\frac{v}{8}\right)\\&+\gamma^3\ln^2(1-v)\left(\frac{1}{8}-\frac{v}{16}-\frac{1}{4v}\right)+\gamma\frac{\alpha}{\pi}\frac{7v}{2}\\&+\gamma\frac{\alpha}{\pi}\ln(1-v)\left(\frac{3}{2}-\frac{v}{2}\right)+\gamma\frac{\alpha}{\pi}\ln^2\frac{(1-v)}{v}\\&+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(v)\left(-3+\frac{5v}{2}\right)+\xi\gamma\theta(1-v-\xi)\frac{\chi(v)}{1-v}\left(-\frac{1}{2}\right)\Bigg\},\end{aligned}$$

(107)

$$
\begin{aligned}
B_1^{(1)}(v) &= b_0\, F(\gamma)\, v^{\gamma}\ \mathrm{e}^{\Delta_{\rm YFS}(\gamma)} \\
&\times\Bigg\{\gamma\Big(-1+\frac{v}{2}\Big)+\frac{\alpha}{\pi}\ln(1-v)(-2+2v)+\gamma^2\left(-\frac{v}{2}\right) \\
&+\gamma^2\ln(1-v)\left(\frac{1}{2}-\frac{v}{4}\right)+\gamma^3\left(\frac{v}{4}\right) \\
&+\gamma^3\ln(1-v)\left(-\frac{1}{4}+\frac{3v}{8}\right)+\gamma^3\ln^2(1-v)\left(-\frac{1}{8}+\frac{v}{16}\right) \\
&+\gamma\frac{\alpha}{\pi}\left(\frac{7v}{2}\right)+\gamma\frac{\alpha}{\pi}\ln(1-v)\left(\frac{7}{2}-\frac{5v}{2}\right) \\
&+\gamma\frac{\alpha}{\pi}\ln^2(1-v)\left(1-\frac{v}{2}\right)+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(v)\left(-3+\frac{5v}{2}\right) \\
&+\xi\gamma\theta(1-v-\xi)\frac{\chi(v)}{(1-v)}\left(-\frac{1}{2}\right)\Bigg\}. \qquad (108)
\end{aligned}
$$

The above result represents the $\bar{\beta}_1$ contribution in the total absence of the photon emission from the lower line. In the presence of the photon emission from the lower line (at the $\bar{\beta}_0$ level) we have to perform the convolution of the above results with the function B_0 from the lower line

$$
\rho_{1U}^{(r)}(t,V)=\int dv dv'\delta(V-v-v'+vv')B_1^{(r)}(v)\ B_0(v'), \qquad (109)
$$

see also Eq. (4). The corresponding second- and first-order results read (remember that virtual corrections are here as for emission from two fermion lines!)

$$
\begin{aligned}
\rho_{1U}^{(2)}(t,V) &= b_0\ F^2(\gamma)V^{2\gamma}\ \mathrm{e}^{2\Delta_{\rm YFS}(\gamma)} \\
&\times\Bigg\{\gamma\Big(-1+\frac{V}{2}\Big)+\frac{\alpha}{\pi}\ln(1-V)(-2+2V) \\
&+\gamma^2\left(-1-\frac{V}{2}\right)+\gamma^2\ln(1-V)\left(\frac{3}{2}-\frac{3V}{4}\right) \\
&+\gamma^3\left(\frac{V}{4}\right)+\gamma^3\ln(1-V)\left(\frac{3}{4}+\frac{7}{8}V\right) \\
&+\gamma^3\ln^2(1-V)\left(-\frac{3}{8}+\frac{3V}{16}-\frac{1}{4V}\right)+\gamma^3\mathrm{Li}_2(V)\left(1-\frac{V}{2}\right) \\
&+\frac{\alpha}{\pi}\gamma(6V)+\frac{\alpha}{\pi}\gamma\ln(1-V)\left(4-\frac{5V}{2}\right)
\end{aligned}
$$

$$+ \frac{\alpha}{\pi}\gamma \ln(1-V)^2 \left(\frac{1}{2} - \frac{3V}{4} + \frac{1}{V}\right) + \frac{\alpha}{\pi}\gamma \mathrm{Li}_2(V)(-6+5V)$$
$$- \frac{1}{2}\gamma\xi\theta(1-v-\xi)\frac{\chi(V)}{(1-V)}\Bigg\}, \tag{110}$$

$$\rho^{(1)}_{1U}(t,V) = b_0\, F^2(\gamma)V^{2\gamma}\, \mathrm{e}^{2\Delta_{\mathrm{YFS}}(\gamma)}$$
$$\times\Bigg\{\gamma\left(-1+\frac{V}{2}\right) + \frac{\alpha}{\pi}\ln(1-V)(-2+2V) + \gamma^2(-V)$$
$$+\gamma^2\ln(1-V)\left(\frac{3}{2} - \frac{3V}{4}\right) + \gamma^3\left(\frac{5V}{4}\right)$$
$$+\gamma^3\ln(1-V)\left(-\frac{3}{4} + \frac{13V}{8}\right) + \gamma^3\ln^2(1-V)\left(-\frac{5}{8} + \frac{5V}{16}\right)$$
$$+\gamma^3\mathrm{Li}_2(V)\left(1-\frac{V}{2}\right) + \frac{\alpha}{\pi}\gamma(6V)$$
$$+\frac{\alpha}{\pi}\gamma\ln(1-V)\left(6-\frac{9V}{2}\right) + \frac{\alpha}{\pi}\gamma\ln^2(1-V)\left(\frac{3}{2}-\frac{5V}{4}\right)$$
$$+\frac{\alpha}{\pi}\gamma\mathrm{Li}_2(V)(-6+5V) - \frac{1}{2}\gamma\xi\theta(1-v-\xi)\frac{\chi(V)}{(1-V)}\Bigg\}. \tag{111}$$

The total $\bar{\beta}_1$ contribution is the sum of the above with an analogous contribution from the lower line. It is simply twice the upper line result.

4.3. Numerical results on $\bar{\beta}_1$

In Fig. 2 we compare our semi-analytical result of Eq. (110) with the numerical result of BHLUMI. We plot the quantity $R(\bar{\beta}^{(2)}_1; t, V_{\max})$ defined in a way analogous to the definition of $\bar{\beta}_0$ in Eq. (70), as a function of the cut of the total photon energy $V_{\max}$, for the same fixed value of the transfer t. As before, we plot the difference between the MC and semi-analytical results, showing in addition $\bar{\beta}_1$ itself, multiplied by a factor 10^{-2}. Again, the MC and SAN results agree to better than 1×10^{-4} (for the contribution from one line) in units of the Born cross section. This result marks an important step towards a similar agreement for the second order total cross section, because $\bar{\beta}_1$ is relatively complicated and, at the same time, numerically sizeable.

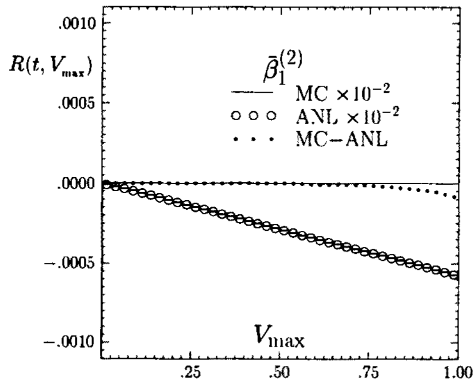

Fig. 2. The comparison of the Monte Carlo and semi-analytical results for the second-order $\bar{\beta}_1$.

5. Contribution from $\bar{\beta}_2$

5.1. Upper line emission $\bar{\beta}_{2UU}$

In the following we shall calculate the contributions to the differential distribution from the simultaneous emission of two real photons from one electron (upper) line as defined by Eq. (29). Again we shall split photons into those that enter directly into $\bar{\beta}_{2UU}$ and the other "spectator" photons that do not, as in the case of $\bar{\beta}_{0,1}$. Here, summation over spectator multiple photons is generally done more easily because $\bar{\beta}_{2UU}$ is already $\mathcal{O}(\alpha^2)$ from the start, so that additional smearing due to spectator photons will be sufficient to discuss in the leading-log approximation (keeping however the correct soft limit as usual!). Introducing $d\omega_i = d\omega_i^I + d\omega_i^F$ we obtain for "non-spectator" photons

$$
\begin{aligned}
&d\omega_1 d\omega_2\, b_{2UU}(\tilde{\alpha}_1, \tilde{\beta}_1, \phi_1, \tilde{\alpha}_2, \tilde{\beta}_2, \phi_2)\\
&\quad = (d\omega_1^I d\omega_2^I + d\omega_1^F d\omega_2^F)\\
&\qquad \times \Big[\theta(v_1 - v_2)\ \tfrac{1}{2}\{H(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)\chi(v_2^*) + H(\tilde{\alpha}_1^*, \tilde{\beta}_1^*, \psi_p)\chi(v_2)\}\\
&\qquad +\theta(v_2 - v_1)\ \tfrac{1}{2}\{H(\tilde{\alpha}_2, \tilde{\beta}_2, \psi_p)\chi(v_1^*) + H(\tilde{\alpha}_2^*, \tilde{\beta}_2^*, \psi_p)\chi(v_1)\}\\
&\qquad\qquad -(H(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p) - b_0) - (H(\tilde{\alpha}_2, \tilde{\beta}_2, \psi_q) - b_0) - b_0\Big]\\
&\quad +d\omega_1^I d\omega_2^F \Big[\theta(v_1 - v_2)\ H(\tilde{\alpha}_1^*, \tilde{\beta}_1^*, \psi_p)\chi(v_2)\\
&\qquad\qquad +\theta(v_2 - v_1)\ H(\tilde{\alpha}_2, \tilde{\beta}_2, \psi_p)\chi(v_1^*)
\end{aligned}
$$

$$-(H(\tilde\alpha_1,\tilde\beta_1,\psi_p)-b_0)-(H(\tilde\alpha_2,\tilde\beta_2,\psi_q)-b_0)-b_0\Big]$$

$$+d\omega_1^F d\omega_2^I\Big[\theta(v_2-v_1)\ H(\tilde\alpha_1^*,\tilde\beta_1^*,\psi_p)\chi(v_2)$$

$$+\theta(v_1-v_2)\ H(\tilde\alpha_2,\tilde\beta_2,\psi_p)\chi(v_1^*)$$

$$-(H(\tilde\alpha_1,\tilde\beta_1,\psi_p)-b_0)-(H(\tilde\alpha_2,\tilde\beta_2,\psi_q)-b_0)-b_0\Big]. \quad (112)$$

When both photons are in the initial-state or both in the final-state, using the usual decomposition $H(\tilde\alpha,\tilde\beta,\psi)=b_0\chi(v)+h(\tilde\alpha,\tilde\beta,\psi)$ into leading-log and sub-leading parts, the above expression can be split into leading and non-leading parts:

$$d\omega_1^K d\omega_2^K\ b_{2UU}(\tilde\alpha_1,\tilde\beta_1,\phi_1,\tilde\alpha_2,\tilde\beta_2,\phi_2)=d\omega_1^K d\omega_2^K\ \Big\{b_0\mathcal{K}^*_{KK}(v_1,v_2)$$

$$+\theta(v_1-v_2)\Big[\frac{h(\tilde\alpha_1,\tilde\beta_1,\psi_p)\chi(v_2^*)}{2}+\frac{h(\tilde\alpha_1^*,\tilde\beta_1^*,\psi_p)\chi(v_2)}{2}$$

$$-h(\tilde\alpha_1,\tilde\beta_1,\psi_p)-h(\tilde\alpha_2,\tilde\beta_2,\psi_q)\Big]$$

$$+\theta(v_2-v_1)\Big[\frac{h(\tilde\alpha_2,\tilde\beta_2,\psi_p)\chi(v_1^*)}{2}+\frac{h(\tilde\alpha_2^*,\tilde\beta_2^*,\psi_p)\chi(v_1)}{2}$$

$$-h(\tilde\alpha_2,\tilde\beta_2,\psi_p)-h(\tilde\alpha_1,\tilde\beta_1,\psi_q)\Big]\Big\}, \quad (113)$$

where $K=I,F$. In the remaining case of one photon in the initial and one in the final state, we obtain, for instance:

$$d\omega_1^I d\omega_2^F b_{2UU}(\tilde\alpha_1,\tilde\beta_1,\phi_1,\tilde\alpha_2,\tilde\beta_2,\phi_2)=d\omega_1^I d\omega_2^F\ \Big\{b_0\mathcal{K}^*_{IF}(v_1,v_2)$$

$$+\theta(v_1-v_2)\Big[h(\tilde\alpha_1^*,\tilde\beta_1^*,\psi_p)\chi(v_2)-h(\tilde\alpha_1,\tilde\beta_1,\psi_p)-h(\tilde\alpha_2,\tilde\beta_2,\psi_q)\Big]$$

$$+\theta(v_2-v_1)\Big[\chi(v_1^*)h(\tilde\alpha_2,\tilde\beta_2,\psi_p)-h(\tilde\alpha_2,\tilde\beta_2,\psi_p)-h(\tilde\alpha_1,\tilde\beta_1,\psi_q)\Big]\Big\}, \quad (114)$$

and the expression for $d\omega_1^F d\omega_2^I\ b_{2UU}$ is quite similar. In the above formulas we have introduced the following short-hand notation for the leading logarithmic part

$$\mathcal{K}^*_{KL}(v_1,v_2)\equiv v_1,v_2\ k^*_{KL}(v_1,v_2),\quad K,L=I,F,$$

$$
\begin{aligned}
k^*_{II}(v_1,v_2) &= k^*_{FF}(v_1,v_2) = \tfrac{1}{2}[k^*_{IF}(v_1,v_2)+k^*_{IF}(v_2,v_1)],\\
k^*_{IF}(v_1,v_2) &= k^*_{FI}(v_2,v_1) = \chi\Big(\frac{v_1}{1-v_2}\Big)\,\chi(v_2)-\chi(v_1)-\chi(v_2)+1\\
&= \frac{1}{2}-\frac{1}{2}\,\frac{1}{1-v_2}+\frac{1}{4}\,\frac{v_1}{1-v_2}+\frac{1}{4}\,\frac{v_1}{(1-v_2)^2}\,. \qquad (115)
\end{aligned}
$$

The integrated contribution including the spectator multiple photons now reads:

$$
\begin{aligned}
&\frac{d\sigma_{2UU}}{d|t|\,dV} = \frac{4\pi\alpha^2}{|t|^2}\,\rho_{\bar\beta_{2UU}}(t,V) = \frac{4\pi\alpha^2}{|t|^2}\int dv\,dv'\\
&\times\int d\psi_p \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{i=2}^{n}\int d\omega_i\;\theta(\tilde\alpha_i+\tilde\beta_i-\tilde\alpha_i\tilde\beta_i-\Delta)\exp(\gamma_p\ln\Delta+\delta_{\rm YFS})\\
&\times\sum_{n\ge j>k\ge 1}\frac{b_{2UU}(\tilde\alpha_j,\tilde\beta_j,\tilde\alpha_k,\tilde\beta_k)}{\tilde S_p(\tilde\alpha_j,\tilde\beta_j)\tilde S_p(\tilde\alpha_k,\tilde\beta_k)}\\
&\times\Theta(1-\mathcal{Z}_p)\delta\Big(v-\sum_{i=1}^{n}(\tilde\alpha_i+\tilde\beta_i-\tilde\alpha_i\tilde\beta_i)\Big)\delta(\phi_{K_p})\\
&\times\int d\psi_q \sum_{n'=1}^{\infty}\frac{1}{n'!}\prod_{j=1}^{n'}\int d\omega'_j\;\theta(\tilde\alpha'_i+\tilde\beta'_i-\tilde\alpha'_i\tilde\beta'_i-\Delta)\exp(\gamma_q\ln\Delta+\delta'_{\rm YFS})\\
&\times\Theta(1-\mathcal{Z}_q)\delta\Big(v'-\sum_{i=1}^{n'}(\tilde\alpha'_i+\tilde\beta'_i-\tilde\alpha'_i\tilde\beta'_i)\Big)\delta(\phi_{K_q})\\
&\times\delta(V-v-v'-vv')\\
&= \frac{4\pi\alpha^2}{|t|^2}\int dv dv'\delta(V-v-v'-vv'))B_0(t,v')\;B_2(t,v)\,; \qquad (116)
\end{aligned}
$$

we know $B_0(t,v')$ from the $\bar\beta_0$ calculation and the new function $B_2(t,v)$, after substitution $d\omega_i = d\omega_i^I + d\omega_i^F$ for all photons, can be expressed as follows

$$
\begin{aligned}
&B_2(t,v) = B_2^{II}(t,v)+B_2^{FF}(t,v)+B_2^{IF}(t,v)+B_2^{FI}(t,v)\,,\\
&B_2^{KL}(t,v) = \frac{1}{2!}\int dv_1\,dv_2\,dv_I\,dv_F\;\delta(v-v_1-v_2-v_I-v_F)\\
&\times\int\frac{d\psi_p}{2\pi}\int d\omega_1^K\delta(v_1-\tilde\alpha_1-\tilde\beta_1+\tilde\alpha_1\tilde\beta_1)\int d\omega_2^L\delta(v_2-\tilde\alpha_2-\tilde\beta_2+\tilde\alpha_2\tilde\beta_2)\\
&\times b_{2UU}(\tilde\alpha_1,\tilde\beta_1,\phi_1,\tilde\alpha_2,\tilde\beta_2,\phi_2)\;f_1\Big(\frac{\bar\gamma_p}{2},v_I\Big)f_1\Big(\frac{\bar\gamma_p}{2},v_F\Big)\\
&\times\exp\big(\bar\delta_{\rm YFS}(\bar\gamma_p,v_F+\tilde\beta_1+\tilde\beta_2)\big)\,. \qquad (117)
\end{aligned}
$$

Note that in the current case the form factor and γ_p depend on $\tilde{\beta}_1+\tilde{\beta}_2$ in the following way:

$$\begin{aligned}\gamma_p(1-v_F-\tilde{\beta}_1-\tilde{\beta}_2) &= \gamma+2\frac{\alpha}{\pi}\ln\frac{(1-v_F-\tilde{\beta}_1-\tilde{\beta}_2)^2}{1-v},\\ \delta_{\rm YFS}(\gamma_p,v_F+\tilde{\beta}_1+\tilde{\beta}_2) &= -\gamma_p\ln(1-v_F-\tilde{\beta}_1-\tilde{\beta}_2)\\ &\quad+\frac{1}{4}\gamma_p+\frac{\alpha}{\pi}\left(-\frac{1}{2}-\frac{\pi^2}{6}\right). \end{aligned}\tag{118}$$

Our immediate aim is now to integrate over photon directions. The first step is again a "collinearization" procedure (as in $\bar{\beta}_{0,1}$ cases) which allows us to replace

$$\begin{aligned}\gamma_p(v_F+\tilde{\beta}_1+\tilde{\beta}_2) &\longrightarrow \bar{\gamma}_p(v_F+v_{1K}+v_{2L}),\\ \delta_{\rm YFS}(\gamma_p,v_F+\tilde{\beta}_1+\tilde{\beta}_2) &\longrightarrow \delta_{\rm YFS}(\bar{\gamma}_p,v_F+\tilde{\beta}_1+\tilde{\beta}_2),\end{aligned}\tag{119}$$

where $v_{1I}=v_{2I}=0$, $v_{1F}=v_1, v_{2F}=v_2$. Since we are dealing with the genuine $\mathcal{O}(\alpha^2)$ contribution we are allowed to simplify even further

$$\begin{aligned}\gamma_p(v_F+\tilde{\beta}_1+\tilde{\beta}_2) &\longrightarrow \gamma,\\ \delta_{\rm YFS}(\gamma_p,v_F+\tilde{\beta}_1+\tilde{\beta}_2) &\longrightarrow \delta_{\rm YFS}(\gamma,v_F+v_{1K}+v_{2L}),\end{aligned}\tag{120}$$

thus obtaining a nicer expression

$$\begin{aligned}B_2^{KL}(t,v) &= \frac{1}{2}\int dv_1\,dv_2\,dv_I\,dv_F\,\delta(v-v_1-v_2-v_I-v_F)\\ &\times\int\frac{d\psi_p}{2\pi}\int d\omega_1^K\delta(v_1-\tilde{\alpha}_1-\tilde{\beta}_1+\tilde{\alpha}_1\tilde{\beta}_1)\int d\omega_2^L\delta(v_2-\tilde{\alpha}_2-\tilde{\beta}_2+\tilde{\alpha}_2\tilde{\beta}_2)\\ &\times b_{2UU}(\tilde{\alpha}_1,\tilde{\beta}_1,\phi_1,\tilde{\alpha}_2,\tilde{\beta}_2,\phi_2)\,f_1\left(\frac{\gamma}{2},v_I\right)f_1\left(\frac{\gamma}{2},v_F\right)\\ &\times\exp\left(\delta_{\rm YFS}(\gamma,v_F+v_{1K}+v_{2L})\right).\end{aligned}\tag{121}$$

The integrations over photon directions will be done differently for the leading and sub-leading contributions. We split B_2^{KL} accordingly into a leading part

$$\begin{aligned}B_{2m}^{KL}(t,v) &= \frac{1}{2}\int dv_1\,dv_2\,dv_I\,dv_F\,\delta(v-v_1-v_2-v_I-v_F)\\ &\times\int\frac{d\psi_p}{2\pi}\int d\omega_1^K\delta(v_1-\tilde{\alpha}_1-\tilde{\beta}_1+\tilde{\alpha}_1\tilde{\beta}_1)\int d\omega_2^L\delta(v_2-\tilde{\alpha}_2-\tilde{\beta}_2+\tilde{\alpha}_2\tilde{\beta}_2)\\ &\times b_0\,\mathcal{K}^*_{KL}(v_1,v_2)f_1\left(\frac{\gamma}{2},v_I\right)f_1\left(\frac{\gamma}{2},v_F\right)\\ &\times\exp\left(\delta_{\rm YFS}(\gamma,v_F+v_{1K}+v_{2L})\right)\end{aligned}\tag{122}$$

and a sub-leading part

$$B_{2s}^{KK}(t,v) = \frac{1}{2}\int dv_1\, dv_2\, \delta(v-v_1-v_2)\int\frac{d\psi_p}{2\pi}$$
$$\times\int d\omega_1^K \delta(v_1-\tilde\alpha_1-\tilde\beta_1+\tilde\alpha_1\tilde\beta_1)\int d\omega_2^K\delta(v_2-\tilde\alpha_2-\tilde\beta_2+\tilde\alpha_2\tilde\beta_2)$$
$$\times 2\theta(v_1-v_2)\Big[\tfrac{1}{2}h(\tilde\alpha_1,\tilde\beta_1,\psi_p)\chi(v_2^*)+\tfrac{1}{2}h(\tilde\alpha_1^*,\tilde\beta_1^*,\psi_p)\chi(v_2)$$
$$-h(\tilde\alpha_1,\tilde\beta_1,\psi_p)-h(\tilde\alpha_2,\tilde\beta_2,\psi_q)\Big], \tag{123}$$

$$B_{2s}^{IF}(t,v) = \frac{1}{2}\int dv_1\, dv_2\, \delta(v-v_1-v_2)\int\frac{d\psi_p}{2\pi}$$
$$\times\int d\omega_1^I \delta(v_1-\tilde\alpha_1-\tilde\beta_1+\tilde\alpha_1\tilde\beta_1)\int d\omega_2^F\delta(v_2-\tilde\alpha_2-\tilde\beta_2+\tilde\alpha_2\tilde\beta_2)$$
$$\times\Big\{\theta(v_1-v_2)\Big[h(\tilde\alpha_1^*,\tilde\beta_1^*,\psi_p)\chi(v_2)-h(\tilde\alpha_1,\tilde\beta_1,\psi_p)-h(\tilde\alpha_2,\tilde\beta_2,\psi_q)\Big]$$
$$+\theta(v_2-v_1)\Big[\chi(v_1^*)h(\tilde\alpha_2,\tilde\beta_2,\psi_p)-h(\tilde\alpha_2,\tilde\beta_2,\psi_p)-h(\tilde\alpha_1,\tilde\beta_1,\psi_q)\Big]\Big\}, \tag{124}$$

where, due to the fact that it is of pure $\mathcal{O}(\gamma\alpha)$, we have neglected the convolution with the "spectator photons" completely. In B_{2s}^{KK} we have also folded the two cases $v_1 > v_2$ and $v_1 < v_2$ into one.

5.1.1. Leading part of $\bar\beta_{2UU}$

Let us now concentrate on the photon angular integrations for the leading part B_{2m}^{KL}. With the help of the integrals in Eq. (92) over photon momenta, we find

$$B_{2m}^{KL}(t,v) = b_0\frac{1}{2}\int dv_1\, dv_2\, dv_I\, dv_F\, \delta(v-v_1-v_2-v_I-v_F)$$
$$\times\frac{1}{2}\frac{1}{v_1}\left(\gamma-\frac{\alpha}{\pi}\ln(1-v_1)\right)\frac{1}{2}\frac{1}{v_2}\left(\gamma-\frac{\alpha}{\pi}\ln(1-v_2)\right),$$
$$\mathcal{K}_{KL}^*(v_1,v_2)\,\exp\left(\delta_{\rm YFS}(\gamma,v_F+v_{1K}+v_{2L})\right)f_1\left(\frac{\gamma}{2},v_I\right)f_1\left(\frac{\gamma}{2},v_F\right). \tag{125}$$

The above leading-logarithmic contribution can be rewritten as

$$B_{2m}^{KL}(t,v) = b_0\frac{1}{8}\int dv_1\, dv_2\, dv_I\, dv_F\, \delta(v-v_1-v_2-v_I-v_F)$$
$$\times\left(\gamma^2-\gamma\frac{\alpha}{\pi}\ln(1-v_1)-\gamma\frac{\alpha}{\pi}\ln(1-v_2)\right)$$

$$\times\, k^*_{KL}(v_1,v_2)\exp\left(\delta_{\rm YFS}(\gamma,v_F+v_{1K}+v_{2L})\right) f_0\left(\frac{\gamma}{2},v_I\right) f_0\left(\frac{\gamma}{2},v_F\right). \quad (126)$$

We have to consider all cases $K,L=I,F$ separately, starting with the case of both "active photons" in the initial state:

$$\begin{aligned} B^{II}_{2m}(t,v) &= b_0\,\frac{1}{8}\int du\; dv_F\;\delta(v-x_I-v_F)\\ &\qquad\times\exp\left(\delta_{\rm YFS}(\gamma,v_F)\right)U(\gamma,x_I)f_0\left(\frac{\gamma}{2},v_F\right), \end{aligned} \quad (127)$$

where

$$\begin{aligned} U(\gamma,x_I) &= \int dv_1\; dv_2\; dv_I\;\delta(x_I-v_1-v_2-v_I)f_0\left(\frac{\gamma}{2},v_I\right)\\ &\quad\times\left(\gamma^2-\gamma\frac{\alpha}{\pi}\,\ln(1-v_1)-\gamma\frac{\alpha}{\pi}\,\ln(1-v_2)\right)k^*_{II}(v_1,v_2)\\ &= \int dv_{12}\; dv_I\;\delta(x_I-v_{12}-v_I)f_0\left(\frac{\gamma}{2},v_I\right)\\ &\quad\times\left(\gamma^2 d^*_0(v_{12})-2\gamma\frac{\alpha}{\pi}\,d^*_2(v_{12})\right)\\ &= x_I^{\frac{1}{2}\gamma}\left(\gamma^2 d^*_0(x_I)+\tfrac{1}{2}\gamma^3\,d^*_1(x_I)-2\gamma\frac{\alpha}{\pi}\,d^*_2(x_I)\right). \end{aligned} \quad (128)$$

In the above calculation we have used the following identities and integrals:

$$\begin{aligned} d^*_0(v) &= \int dv_1\; dv_2\;\delta(v-v_1-v_2)k^*_{II}(v_1,v_2)\\ &= \frac{1}{2}v+\frac{1}{2}\left(1-\frac{v}{2}\right)\ln(1-v)\,, \end{aligned} \quad (129)$$

$$\int_0^v dx\;\gamma x^{\gamma-1}\;d^*_0(v-x) = v^\gamma d^*_0(v)+\gamma d^*_1(v)\,, \quad (130)$$

$$\begin{aligned} d^*_1(v) &= \int_0^v dx\;\frac{d^*_0(v-x)-d^*_0(v)}{x}\\ &= -\frac{3}{4}v-\frac{1}{4}(1-v)\ln(1-v)+\left(-\frac{1}{2}+\frac{1}{4}v\right){\rm Li}_2\left(\frac{-v}{1+v}\right), \end{aligned} \quad (131)$$

$$\begin{aligned} d^*_{2II}(v) = d^*_{2FF}(v) &= \int dv_1\; dv_2\;\delta(v-v_1-v_2)\;k^*_{II}(v_1,v_2)\ln(1-v_1)\\ &= -\frac{7}{8}v+\frac{2-v}{8}\left[{\rm Li}_2\left(\frac{1}{2-v}\right)-{\rm Li}_2\left(\frac{1-v}{2-v}\right)\right] \end{aligned}$$

$$+\Big(-1+\frac{5}{8}v+\frac{1}{4}\frac{1}{2-v}\Big)\ln(1-v)$$
$$+\frac{2-v}{16}\ln^2(1-v)+\frac{2-v}{8}\ln(1-v)\ln(2-v)\,. \qquad (132)$$

We omit indices $K,L=I,F$ wherever the relevant function is independent of them. In the second convolution, see Eq. (127), we may neglect the convolution altogether for "saturated" terms of $\mathcal{O}(\alpha\gamma)$ and $\mathcal{O}(\gamma^3)$ setting simply $x_I\to v$ and similarly $v_F\to 0$ in the YFS form factor. The only non-trivial integral involves the term d_0 in the U, and it can be evaluated using the identity of Eq. (130). The final result reads

$$\begin{aligned}B^{II}_{2m}(t,v) &= \frac{1}{8}\gamma^2 b_0\,\mathrm{e}^{\Delta_{\rm YFS}(\gamma)}\,v^\gamma d_0^*(v)\\ &+\frac{1}{8}\gamma^3 b_0\,\mathrm{e}^{\Delta_{\rm YFS}(\gamma)}\,d_1^*(v)-\frac{1}{4}\gamma\frac{\alpha}{\pi}\,b_0\,\mathrm{e}^{\Delta_{\rm YFS}(\gamma)}\,d^*_{2II}(v). \qquad (133)\end{aligned}$$

The case of both "active photons" in the final state looks almost identical

$$\begin{aligned}B^{FF}_{2m}(t,v) &= b_0\frac{1}{8}\int du\,dv_F\,\delta(v-v_I-x_F)\\ &\times\exp\left(\delta_{\rm YFS}(\gamma,x_F)\right)f_0\Big(\frac{\gamma}{2},v_I\Big)U(\gamma,x_F). \qquad (134)\end{aligned}$$

The only important difference is in the YFS form factor, where we set $v_F\to v$. This leads to an additional term of $\mathcal{O}(\gamma^3)$ in the final result

$$B^{FF}_{2m}(t,v)=B^{II}_{2m}(t,v)-\frac{1}{4}\gamma\frac{\alpha}{\pi}\,b_0\,\mathrm{e}^{\Delta_{\rm YFS}(\gamma)}\ln(1-v)d_0^*(v). \qquad (135)$$

The case when one of the "active photons" is in the final state and the other is in the initial state is the most complicated, because we have to convolve $k^*_{IF}(v_1,v_2)$ with f_0; using the following identities

$$\begin{aligned}\int dv_2\,dv_I\;\;\delta(x_I &- v_I-v_2)k^*_{IF}(v_1,v_2)\;f_0\Big(\gamma,v_I\Big)\\ &\equiv x_I^\gamma\;k^*_{IF}(x_I,v_2)+\gamma w'^*_{IF}(x_I,v_2),\\ \int dv_1\,dv_F\;\delta(x_F &- v_1-v_F)k^*_{IF}(v_1,v_2)\;f_0\Big(\gamma,v_F\Big)\\ &\equiv x_F^\gamma\;k^*_{IF}(v_1,x_F)+\gamma w''^*_{IF}(v_1,x_F),\\ w'^*_{IF}(v_1,v_2) &= \int_0^{v_1}dx\;\frac{k^*_{IF}(v_1-x,v_2)-k^*_{IF}(v_1,v_2)}{x},\\ w''^*_{IF}(v_1,v_2) &= \int_0^{v_2}dx\;\frac{k^*_{IF}(v_1,v_2-x)-k^*_{IF}(v_1,v_2)}{x},\end{aligned}$$

$$\begin{aligned} w^*_{IF}(v_1,v_2) &= w'^*_{IF}(v_1,v_2) + w''^*_{IF}(v_1,v_2) \\ -\frac{1}{2}\,\frac{v_1}{1-v_2} + \ln(1-v_1) &\left[-\frac{1}{2}\,\frac{1}{1-v_2} + \frac{1}{4}\,\frac{v_1}{1-v_2} + \frac{1}{4}\,\frac{v_1}{(1-v_2)^2}\right], \end{aligned} \quad (136)$$

we obtain the following results:

$$\begin{aligned} B^{IF}_{2m}(t,v) &= b_0 \frac{1}{8} \int dx_I\, dx_F\, \delta(v - x_I - x_F) \\ &\quad \times \exp\left(\delta_{\rm YFS}(\gamma, x_F)\right) V(\gamma, x_I, x_F), \\ V(\gamma, x_I, x_F) &= \gamma^2\, x_I^{\frac{1}{2}\gamma} x_F^{\frac{1}{2}\gamma}\, k^*_{IF}(x_I,x_F) + 2\tfrac{1}{2}\gamma^3 w^*_{IF}(x_I,x_F) \\ -\gamma\frac{\alpha}{\pi}\, \ln(1 &- x_I) k^*_{IF}(x_I,x_F) - \gamma\frac{\alpha}{\pi}\, \ln(1-x_F) k^*_{IF}(x_I,x_F). \end{aligned} \quad (137)$$

Again, we do not really need to integrate directly all of the integral in Eq. (137). It is enough to observe that Eq. (125) can be rewritten as

$$\begin{aligned} B^{KL}_{2m}(t,v) &= \frac{1}{8}\, b_0\, e^{\Delta_{\rm YFS}(\gamma)} \int dv_1\, dv_2\, dv_I\, dv_F\, \delta(v - v_1 - v_2 - v_I - v_F) \\ &\quad \times f_1\left(\frac{\gamma}{2}, v_I\right) f_1\left(\frac{\gamma}{2}, v_F\right) k^*_{KL}(v_1,v_2) \Big[\gamma^2 - \gamma\frac{\alpha}{\pi}\, \ln(1-v_1) \\ &\quad -\gamma\frac{\alpha}{\pi}\, \ln(1-v_2) - \gamma^3 \ln(1 - v_F - v_{1K} - v_{2L})\Big]. \end{aligned} \quad (138)$$

If we neglect the last term of $\mathcal{O}(\gamma^3)$ from the YFS form factor then the integrand can be symmetrized in variables v_1 and v_2, leading to the replacement $k^*_{KL}(v_1,v_2) \to k^*_{II}(v_1,v_2)$; consequently, the result is the same for all $K,L = I,F$ and we have to calculate it only for one case, for example $(K,L) = (I,I)$. (In fact for $(K,L) = (I,I)$ the contribution from the YFS form factor is zero.) For cases other than $(K,L) = (I,I)$ we use

$$\begin{aligned} B^{KL}_{2m}(t,v) &= B^{II}_{2m}(t,v) - \frac{1}{8}\gamma^3\, b_0\, e^{\Delta_{\rm YFS}(\gamma)} \int dv_1\, dv_2\, \delta(v - v_1 - v_2) \\ &\quad \times k^*_{KL}(v_1,v_2) \ln(1 - v_{1K} - v_{2L}), \end{aligned} \quad (139)$$

where we see explicitly the $\mathcal{O}(\gamma^3)$ contribution from the YFS form factor, which has to be recalculated for each $(K,L) \neq (I,I)$. In the present case the result of the second convolution integration reads

$$B^{IF}_{2m}(t,v) = B^{II}_{2m}(t,v) - \tfrac{1}{8}\gamma^3 b_0 \exp(\Delta_{\rm YFS}(\gamma))\, d^*_{2IF}(v), \quad (140)$$

where

$$d^*_{2IF}(v) = \int dv_1\, dv_2\, \delta(v - v_1 - v_2)\, k^*_{IF}(v_1,v_2) \ln(1-v_1)$$

$$= -\frac{3}{4}v + \Big(-1 + \frac{1}{2}v + \frac{1}{2}\frac{1}{2-v}\Big)\ln(1-v)$$
$$+\frac{2-v}{4}\Big[\mathrm{Li}_2\Big(\frac{1}{2-v}\Big) - \mathrm{Li}_2\Big(\frac{1-v}{2-v}\Big)$$
$$+\ln(1-v)\ln(2-v)\Big]. \qquad (141)$$

The total sum is

$$\begin{aligned} B_{2m}(t,v) &= B^{II}_{2m}(t,v) + B^{FF}_{2m}(t,v) + 2B^{IF}_{2m}(t,v) \\ &= \frac{1}{2}\gamma^2 b_0 \exp(\Delta_{\rm YFS}(\gamma)) v^\gamma d_0^*(v) \\ &\quad +\frac{1}{2}\gamma^3 b_0 \exp(\Delta_{\rm YFS}(\gamma)) d_1^*(v) \\ &\quad -\gamma\frac{\alpha}{\pi}\, b_0 \exp(\Delta_{\rm YFS}(\gamma)) d_{2II}^*(v) \\ &\quad -\frac{1}{8}\gamma^3 b_0 \exp(\Delta_{\rm YFS}(\gamma))\ln(1-v) d_0^*(v) \\ &\quad -\frac{1}{4}\gamma^3 b_0 \exp(\Delta_{\rm YFS}(\gamma)) d_{2IF}^*(v). \end{aligned} \qquad (142)$$

5.1.2. Sub-leading part of $\bar\beta_{2UU}$

Let us now calculate the sub-leading part B_{2s}, starting with the function $B^{II}_{2s}(t,v) = B^{FF}_{2s}(t,v)$. In order to integrate over the photon angles, we need to introduce, in addition to integrals in Eq. (92), the new integral

$$\begin{aligned} &\frac{\alpha}{\pi}\, g_a^{*I}(v_1,v_2) \\ &= \int\limits_{\tilde\alpha_1<\tilde\beta_1} d\tilde\omega_1\; \delta(v_1 - \tilde\alpha_1 - \tilde\beta_1 + \tilde\alpha_1\tilde\beta_1)\; h(\tilde\alpha_1^*, \tilde\beta_1^*, \psi_p) = \frac{\alpha}{\pi}\int_{1I} g(\tilde\alpha_1^*, \tilde\beta_1^*) \\ &= -\frac{1}{2}\frac{6v_1v_2}{(1-v_2)^4} \\ &\quad +\frac{1}{2}\ln(1-v_1)\Big[\frac{-3-v_2-v_2^2-v_2^3}{(1-v_2)^4} + v_1\frac{5/2+v_2-v_2^2/2}{(1-v_2)^4}\Big]. \end{aligned} \qquad (143)$$

With the help of the above we may now write

$$B^{II}_{2s}(t,v) = B^{FF}_{2s}(t,v) = \frac{1}{2}\frac{\alpha}{\pi}\,\gamma\, b_0\; e^{\Delta_{\rm YFS}(\gamma)}\, d_3^*(v), \qquad (144)$$

$$d_3^*(v) = \int dv_1 dv_2\; \delta(v - v_1 - v_2)\; \theta(v_1 - v_2)\; s(v_1,v_2) \qquad (145)$$

$$s(v_1,v_2) = \tfrac{1}{2}g_a^1(v_1)\frac{1}{v_2}\chi\left(\frac{v_2}{1-v_1}\right) + \tfrac{1}{2}g_a^{*I}(v_1,v_2)\frac{1}{v_2}\chi(v_2) - \frac{1}{v_2}g_a^1(v_1) - \frac{1}{v_1}g_a^1(v_2). \qquad (146)$$

Let us note that the integral defining $d_3^*(v)$, although it may not be immediately obvious, is finite. It is so because in the limit $v_2 \to 0$ we have $\chi(v_2^*) \to \chi(v_2)$ and $h_a^*(v_1,v_2) \to h_a(v_1)$. The apparent singularity $1/v_1$ is out of the integration domain due to ordering. This integral is more complicated than others due to presence of the "shifted" variable v_2^* and the ordering $v_1 > v_2$. The function $d_3^*(v)$ reads as follows

$$\begin{aligned}
d_3^*(v) =& \left(-3+\frac{5}{2}v\right)\left[\mathrm{Li}_2\left(-\frac{v}{1-v}\right) - \mathrm{Li}_2\left(-\frac{v}{2(1-v)}\right)\right]\\
&+\left(\frac{3}{4}-\frac{11}{8}v\right)\left[\mathrm{Li}_2\left(\frac{1}{2-v}\right) - \mathrm{Li}_2\left(\frac{1}{2}\right)\right]\\
&+\ln^2\left(1-\tfrac{1}{2}v\right)\left(\frac{3}{8}-\frac{5}{16}v\right)\\
&+\ln\left(1-\frac{v}{2}\right)\left(-\frac{9}{4}+\frac{3}{8}v-\frac{2}{2-v}+\frac{1}{(2-v)^2}-\frac{2}{(2-v)^3}\right)\\
&+\ln^2(1-v)\left(-\frac{3}{8}+\frac{5}{16}v\right)+\ln(1-v)\ln(v)\left(3-\frac{5}{2}v\right)\\
&+\ln\left(1-\frac{v}{2}\right)\ln(2-v)\left(\frac{3}{4}-\frac{11}{8}v\right)+\ln(1-v)\ln\left(\tfrac{1}{2}v\right)\left(-3+\frac{5}{2}v\right)\\
&+\ln(1-v)\left(\frac{7}{8}-\frac{1}{4}v-\frac{3}{4}\frac{1}{2-v}+\frac{1}{2}\frac{1}{(2-v)^2}-\frac{1}{4}\frac{1}{(2-v)^3}\right)\\
&+\frac{9}{8}-\frac{1}{8}v-\frac{13}{8}\frac{1}{2-v}+\frac{1}{4}\frac{1}{(2-v)^2}-3\frac{1}{(2-v)^3}.
\end{aligned} \qquad (147)$$

The "spectator photons" contribute beyond $\mathcal{O}(\alpha^3)_{\rm prag}$ and therefore the convolution with them may be kept or neglected. Below we show a variant of the B_{2s}^{KK} with explicit "spectator photons" for the latter purpose

$$\begin{aligned}
B_{2s}^{II}(t,v) =& \frac{1}{2}\frac{\alpha}{\pi}\,\gamma\, b_0 \int dx_I\, dx_F\, \delta(v-x_I-x_F)\\
&\times \exp(\delta_{\rm YFS}(\gamma,x_F))d_3^*(x_I)f_0\left(\frac{\gamma}{2},x_F\right),\\
B_{2s}^{FF}(t,v) =& \frac{1}{2}\frac{\alpha}{\pi}\,\gamma\, b_0 \int dx_I\, dx_F\, \delta(v-x_I-x_F)\\
&\times \exp(\delta_{\rm YFS}(\gamma,x_F))f_0\left(\frac{\gamma}{2},x_I\right)d_3^*(x_F).
\end{aligned} \qquad (148)$$

We now come to the two-hemisphere case, see Eq. (124), which reads as

$$\begin{aligned}
B_{2s}^{IF}(t,v) &= \tfrac{1}{2}\frac{\alpha}{\pi}\,\gamma\, b_0\, \exp(\varDelta_{\rm YFS}(\gamma)) \int dx_I\, dx_F\, \delta(v-x_I-x_F) S(x_I,x_F),\\
S(x_I,x_F) &= \theta(x_I-x_F)\, S_I(x_I,x_F) + \theta(x_F-x_I)\, S_F(x_I,x_F),\\
S_I(x_I,x_F) &\equiv g_a^{*I}(x_I,x_F)\frac{\chi(x_F)}{x_F} - g_a^1(x_I)\frac{1}{x_F} - g_a^1(x_F)\frac{1}{x_I},\\
S_F(x_I,x_F) &\equiv \frac{\chi(x_I^*)}{x_I} g_a^1(x_F) - g_a^1(x_F)\frac{1}{x_I} - g_a^1(x_I)\frac{1}{x_F}.
\end{aligned} \tag{149}$$

Folding together two cases ($x_I > x_F$ and $x_I < x_F$) and using the relation between S_K and $s(v_1,v_2)$ of Eq. (146)

$$S_I(v_1,v_2) + S_F(v_2,v_1) = 2s(v_1,v_2), \tag{150}$$

we find

$$B_{2s}^{IF}(t,v) = B_{2s}^{II}(t,v) = \frac{1}{2}\frac{\alpha}{\pi}\,\gamma\, b_0\, \exp(\varDelta_{\rm YFS}(\gamma)) d_3^*(v). \tag{151}$$

This completes the calculation of the sub-leading part B_{2s} of $\bar\beta_{2UU}$.

Combining the above partial results we obtain

$$\begin{aligned}
B_2(t,v) &= b_0\, \exp(\varDelta_{\rm YFS}(\gamma))\, v^\gamma \left(\frac{1}{2}\gamma^2\, d_0^*(v) + \frac{1}{2}\gamma^3\, d_1^*(v) - \gamma\frac{\alpha}{\pi}\, d_{2II}^*(v) \right.\\
&\left. -\frac{1}{8}\gamma^3\, \ln(1-v)\, d_0^*(v) - \frac{1}{4}\gamma^3\, d_{2IF}^*(v) + \frac{\alpha}{\pi}\,\gamma\, d_3^*(v) \right).
\end{aligned} \tag{152}$$

The final result reads

$$\begin{aligned}
&B_2(t,v) = b_0\, F(\gamma) v^\gamma\, \exp(\varDelta_{\rm YFS}(\gamma))\\
&\times\Bigg\{ \gamma^2\left(\frac{v}{4}\right) + \gamma^2 \ln(1-v)\left(\frac{1}{4}-\frac{1}{8}v\right)\\
&+\gamma^3\left(-\frac{3v}{16}\right) + \gamma^3\ln(1-v)\left(\frac{1}{8}-\frac{1}{16}v-\frac{1}{8}(2-v)^{-1}\right)\\
&+\gamma^3\ln^2(1-v)\left(-\frac{1}{16}+\frac{1v}{32}\right) + \gamma^3 {\rm Li}_2\left(-\frac{v}{1-v}\right)\left(-\frac{1}{4}+\frac{v}{8}\right)\\
&+\gamma^3\ln(2-v)\ln(1-v)\left(-\frac{1}{8}+\frac{v}{16}\right)\\
&+\gamma^3{\rm Li}_2\left(\frac{1-v}{2-v}\right)\left(\frac{1}{8}-\frac{v}{16}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\gamma^3 \mathrm{Li}_2\left(\frac{1}{2-v}\right)\left(-\frac{1}{8}+\frac{v}{16}\right)\\
&+\gamma\frac{\alpha}{\pi}\left(\frac{9}{8}+\frac{3v}{4}-\frac{13}{8}(2-v)^{-1}+\frac{1}{4}(2-v)^{-2}-3(2-v)^{-3}\right)\\
&+\gamma\frac{\alpha}{\pi}\ln(1-v)\left(\frac{15}{8}-\frac{7v}{8}\right.\\
&\qquad\qquad\left.-(2-v)^{-1}+\frac{1}{2}(2-v)^{-2}-\frac{1}{4}(2-v)^{-3}\right)\\
&+\gamma\frac{\alpha}{\pi}\ln\left(1-\frac{v}{2}\right)\left(-\frac{9}{4}+\frac{3v}{8}\right.\\
&\qquad\qquad\left.-2(2-v)^{-1}+(2-v)^{-2}-2(2-v)^{-3}\right)\\
&+\gamma\frac{\alpha}{\pi}\ln^2(1-v)\left(-\frac{1}{2}+\frac{3v}{8}\right)+\gamma\frac{\alpha}{\pi}\ln^2\left(1-\frac{v}{2}\right)\left(\frac{3}{8}-\frac{5v}{16}\right)\\
&+\gamma\frac{\alpha}{\pi}\ln(1-v)\ln(2-v)\left(-\frac{1}{4}+\frac{1}{8}v\right)\\
&+\gamma\frac{\alpha}{\pi}\ln(1-v)\ln v\left(3-\frac{5v}{2}\right)\\
&+\gamma\frac{\alpha}{\pi}\ln(1-v)\ln\frac{v}{2}\left(-3+\frac{5v}{2}\right)\\
&+\gamma\frac{\alpha}{\pi}\ln(2-v)\ln\left(1-\frac{v}{2}\right)\left(\frac{3}{4}-\frac{11v}{8}\right)\\
&+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(\frac{1}{2}\right)\left(-\frac{3}{4}+\frac{11v}{8}\right)+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(\frac{1-v}{2-v}\right)\left(\frac{1}{4}-\frac{v}{8}\right)\\
&+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(\frac{1}{2-v}\right)\left(\frac{1}{2}-\frac{5}{4}v\right)+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(-\frac{v}{2(1-v)}\right)\left(3-\frac{5v}{2}\right)\\
&\left.+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(\frac{-v}{1-v}\right)\left(-3+\frac{5v}{2}\right)\right\}. \qquad (153)
\end{aligned}
$$

5.1.3. Total result for $\bar{\beta}_{2UU}$

The total contribution to $\bar{\beta}_2$ due to double emission from the upper line alone reads:

$$
\begin{aligned}
&\rho_{\bar{\beta}_{2UU}}(t,V)=\tfrac{1}{2}b_0\,F^2(\gamma)V^{2\gamma}\,\exp(2\Delta_{\mathrm{YFS}}(\gamma))\\
&\times\left\{\gamma^2\left(\frac{V}{4}\right)+\gamma^2\ln(1-V)\left(\frac{1}{4}-\frac{V}{8}\right)\right.\\
&+\gamma^3\left(-\frac{9V}{16}\right)+\gamma^3\ln(1-V)\left(-\frac{3}{16}V-\frac{1}{8}(2-V)^{-1}\right)\\
&+\gamma^3\ln^2(1-V)\left(-\frac{1}{16}+\frac{1}{32}V\right)+\gamma^3\mathrm{Li}_2(V)\left(\frac{1}{2}-\frac{V}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma^3 \ln(1-V)\ln(2-V)\left(-\frac{1}{8}+\frac{V}{16}\right) \\
&+\gamma^3 \mathrm{Li}_2\left(\frac{1-V}{2-V}\right)\left(\frac{1}{8}-\frac{V}{16}\right) \\
&+\gamma^3 \mathrm{Li}_2\left(\frac{1}{2-V}\right)\left(-\frac{1}{8}+\frac{V}{16}\right) \\
&+\frac{\alpha}{\pi}\gamma\left(\frac{9}{8}+\frac{3}{4}V\right. \\
&\qquad\left.-\frac{13}{8}(2-v)^{-1}+\frac{1}{4}(2-v)^{-2}-3(2-v)^{-3}\right) \\
&+\frac{\alpha}{\pi}\gamma\ln(1-V)\left(\frac{15}{8}-\frac{7V}{8}\right. \\
&\qquad\left.-(2-v)^{-1}+\frac{1}{2}(2-v)^{-2}-\frac{1}{4}(2-v)^{-3}\right) \\
&+\frac{\alpha}{\pi}\gamma\ln\left(1-\frac{V}{2}\right)\left(-\frac{9}{4}+\frac{3V}{8}\right. \\
&\qquad\left.-2(2-v)^{-1}+(2-v)^{-2}-2(2-v)^{-3}\right) \\
&+\frac{\alpha}{\pi}\gamma\ln^2(1-V)\left(1-\frac{7V}{8}\right)+\frac{\alpha}{\pi}\gamma\ln^2\left(1-\frac{V}{2}\right)\left(\frac{3}{8}-\frac{5V}{16}\right) \\
&+\frac{\alpha}{\pi}\gamma\ln(1-V)\ln(2-V)\left(-\frac{1}{4}+\frac{V}{8}\right) \\
&+\frac{\alpha}{\pi}\gamma\ln(1-V)\ln(V)\left(3-\frac{5V}{2}\right) \\
&+\frac{\alpha}{\pi}\gamma\ln(1-V)\ln\frac{V}{2}\left(-3+\frac{5V}{2}\right) \\
&+\frac{\alpha}{\pi}\gamma\ln(2-V)\ln\left(1-\frac{V}{2}\right)\left(\frac{3}{4}-\frac{11V}{8}\right) \\
&+\frac{\alpha}{\pi}\gamma\mathrm{Li}_2\left(\frac{1}{2}\right)\left(-\frac{3}{4}+\frac{11V}{8}\right) \\
&+\frac{\alpha}{\pi}\gamma\mathrm{Li}_2\left(\frac{1-V}{2-V}\right)\left(\frac{1}{4}-\frac{V}{8}\right) \\
&+\frac{\alpha}{\pi}\gamma\mathrm{Li}_2\left(\frac{1}{2-V}\right)\left(\frac{1}{2}-\frac{5V}{4}\right) \\
&+\frac{\alpha}{\pi}\gamma\mathrm{Li}_2\left(-\frac{V}{2(1-V)}\right)\left(3-\frac{5V}{2}\right) \\
&+\frac{\alpha}{\pi}\gamma\mathrm{Li}_2(V)\left(3-\frac{5V}{2}\right)\Bigg\}. \qquad (154)
\end{aligned}
$$

5.2. Numerical results on $\bar\beta_{2UU}$

In Fig. 3 we compare our semi-analytical result of Eq. (154) with the numerical result of BHLUMI. We plot $R(\bar\beta^{(2)}_{2UU}; t, V_{\max})$, the quantity defined in a way analogous to that of Eq. (70) for $\bar\beta_0$, as a function of the cut on the total photon energy $V_{\max}$, for the same fixed value of the transfer t. We plot the MC and semi-analytical results and their difference. As we see, although $\bar\beta^{(2)}_{2UU}$ is analytically the most complicated of all beta's, it is numerically very small. It is at most 2.5×10^{-4} (contribution from one fermion line), and the difference between MC and semi-analytical results is completely negligible, much below 1×10^{-4}.

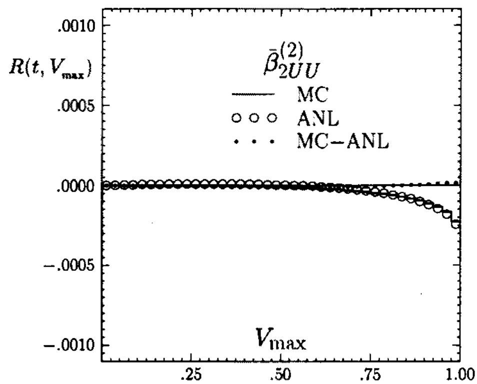

Fig. 3. The comparison of the Monte Carlo and semi-analytical results for the second-order $\bar\beta_2$, both photons on one fermion line.

5.3. Upper and lower line emission $\bar\beta_{2UL}$

The contribution from $\bar\beta_{2UL}$ is due to double real photon emission, one from the upper line and one from the lower one. Generally, the calculation of the $\bar\beta_{2UL}$ contribution is easier than that of the $\bar\beta_{2UU}$, because up to terms beyond $\mathcal{O}(\alpha^2)_{\rm prag}$, $\bar\beta_{2UL}$ is a product of two $\mathcal{O}(\alpha^2)_{\rm prag}$ contributions of the $\bar\beta_1$ type, *i.e.* $b_{2UL} \sim b^{(1)}_{1U} b^{(1)}_{1L}$, so that we may use the results of calculations that were already done for the $\bar\beta_1$ case. More precisely we have

$$\begin{aligned}
&b_{2UL}(\tilde\alpha_1, \tilde\beta_1, \phi_1, \tilde\alpha_{1'}, \tilde\beta_{1'}, \phi_{1'}) d\omega_1 d\omega'_1 \\
&\quad = \theta(v_1 - v'_1) d\omega_1 d\omega'_1 [H(\tilde\alpha_1, \tilde\beta_1, \psi_p)\chi(v'_1) \\
&\qquad - (H(\tilde\alpha_1, \tilde\beta_1, \psi_p) - b_0) - (H(\tilde\alpha'_1, \tilde\beta'_1, \psi_q) - b_0) - b_0] \\
&\quad +\theta(v'_1 - v_1) d\omega_1 d\omega'_1 [H(\tilde\alpha'_1, \tilde\beta'_1, \psi_p)\chi(v_1) \\
&\qquad - (H(\tilde\alpha_1, \tilde\beta_1, \psi_p) - b_0) - (H(\tilde\alpha'_1, \tilde\beta'_1, \psi_q) - b_0) - b_0]. \quad (155)
\end{aligned}$$

Employing the usual decomposition $H(\tilde{\alpha},\tilde{\beta},\psi) = b_0\chi(v) + h(\tilde{\alpha},\tilde{\beta},\psi)$ the above expression can be rewritten as

$$
\begin{aligned}
&b_{2UL}(\tilde{\alpha}_1,\tilde{\beta}_1,\phi_1,\tilde{\alpha}_{1'},\tilde{\beta}_{1'},\phi_{1'})d\omega_1 d\omega_1' = d\omega_1 d\omega_1' \\
&\times\Big\{\theta(v_1 - v_1')[b_0\nu(v_1)\nu(v_1') + h(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)\nu(v_1') - h(\tilde{\alpha}_1',\tilde{\beta}_1',\psi_q)] \\
&+\theta(v_1' - v_1)[b_0\nu(v_1)\nu(v_1') + h(\tilde{\alpha}_1',\tilde{\beta}_1',\psi_p)\nu(v_1) - h(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)]\Big\},
\end{aligned}
\tag{156}
$$

where $\nu(x) = \chi(x) - 1 = x(-1 + \frac{1}{2}x)$.

The integrated contribution reads as follows:

$$
\begin{aligned}
&\frac{d\sigma_{2UL}}{d|t|\,dV} = \frac{4\pi\alpha^2}{|t|^2}\,\rho_{\bar{\beta}_{2UL}}(t,V) = \frac{4\pi\alpha^2}{|t|^2}\int dv\,dv' \\
&\times\int d\psi_p \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int d\omega_i\;\theta(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i\tilde{\beta}_i - \Delta)\;\exp(\gamma_p\ln\Delta + \delta_{\rm YFS}) \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n'}\frac{\bar{\beta}^{(r)}_{2UL}(\tilde{\alpha}_j,\tilde{\beta}_j,\tilde{\alpha}_l',\tilde{\beta}_l')}{\tilde{S}_p(\tilde{\alpha}_j,\tilde{\beta}_j)\tilde{S}_q(\tilde{\alpha}_l',\tilde{\beta}_l')}\Big\}. \\
&\times\Theta(1 - \mathcal{Z}_p)\delta\Big(v - \sum_{i=1}^{n}(\tilde{\alpha}_i + \tilde{\beta}_i - \tilde{\alpha}_i\tilde{\beta}_i)\Big)\;\delta(\phi_{K_p}) \\
&\times\int d\psi_q \sum_{n'=1}^{\infty}\frac{1}{n'!}\prod_{j=1}^{n'}\int d\omega_j'\;\theta(\tilde{\alpha}_i' + \tilde{\beta}_i' - \tilde{\alpha}_i'\tilde{\beta}_i' - \Delta)\;\exp(\gamma_q\ln\Delta + \delta_{\rm YFS}') \\
&\times\Theta(1 - \mathcal{Z}_q)\delta\Big(v' - \sum_{i=1}^{n'}(\tilde{\alpha}_i' + \tilde{\beta}_i' - \tilde{\alpha}_i'\tilde{\beta}_i')\Big)\;\delta(\phi_{K_q}) \\
&\times\delta(V - v - v' + vv').
\end{aligned}
\tag{157}
$$

With the usual decomposition $d\omega_i = d\omega_i^I + d\omega_i^F$, the total contribution splits as follows:

$$
\begin{aligned}
\rho_{\bar{\beta}_{2UL}} &= \rho^{II}_{\bar{\beta}_{2UL}} + \rho^{FF}_{\bar{\beta}_{2UL}} + \rho^{IF}_{\bar{\beta}_{2UL}} + \rho^{FI}_{\bar{\beta}_{2UL}} \\
\rho^{KJ}_{\bar{\beta}_{2UL}} &= b_0\int dv dv'\delta(V - v - v' + vv')\int dv_I dv_F\int dv_I' dv_F' \\
&\times\int\frac{d\psi_p}{2\pi}d\omega_1^K\delta(v - v_1 - v_I - v_F)\int\frac{d\psi_q}{2\pi}d\omega'^J_1\delta(v' - v_1' - v_I' - v_F') \\
&\times\Big(\theta(v_1 - v_1')[\nu(v_1)\nu(v_1') + h(\tilde{\alpha}_1,\tilde{\beta}_1,\psi_p)\nu(v_1') - h(\tilde{\alpha}_1',\tilde{\beta}_1',\psi_q)]
\end{aligned}
$$

$$+\theta(v_1' - v_1)[\nu(v_1')\nu(v_1) + h(\tilde{\alpha}_1', \tilde{\beta}_1', \psi_q)\nu(v_1) - h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)]\Big)$$
$$\times f_1\left(\frac{\gamma}{2}, v_I\right) f_1\left(\frac{\gamma}{2}, v_F\right) \exp(\bar{\delta}_{\rm YFS}(\bar{\gamma}_p, v_F + v_{1K}))$$
$$\times f_1\left(\frac{\gamma}{2}, v_I'\right) f_1\left(\frac{\gamma}{2}, v_F'\right) \exp(\bar{\delta}_{\rm YFS}(\bar{\gamma}_q, v_F' + v_{1J'})), \quad (158)$$

where we have also done the maximum simplifications allowed in $\mathcal{O}(\alpha^3)_{\rm prag}$ similarly to the $\bar{\beta}_{2UU}$ case[14]. The above should be compared with the $\mathcal{O}(\alpha^2)_{\rm prag}$ expression, see Eq. (82), from the $\bar{\beta}_1$ calculation

$$B_{1K}^{(1)}(t, v) = \int dv_I dv_F dv_1 \delta(v - v_I - v_F - v_1)$$
$$\times \int \frac{d\psi_p}{2\pi} \int d\bar{\omega}_{1K}\ \delta(v_1 - \tilde{\alpha}_1 - \tilde{\beta}_1 + \tilde{\alpha}_1\tilde{\beta}_1)\ b_0\Big[\nu(v_1) + h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)\Big]$$
$$\times \exp(\bar{\delta}_{\rm YFS}(\gamma, v_F + v_{1K})) f_1\left(\tfrac{1}{2}\gamma, v_I\right)\ f_1\left(\tfrac{1}{2}\gamma, v_F\right). \quad (159)$$

Neglecting purely non-logarithmic $\mathcal{O}(\alpha^2)$ contributions we may use the relation

$$[\nu(v_1)\ +\ h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)][\nu(v_1') + h(\tilde{\alpha}_1', \tilde{\beta}_1', \psi_p)]$$
$$=\ \nu(v_1)\nu(v_1') + h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p) + h(\tilde{\alpha}_1', \tilde{\beta}_1', \psi_p) \quad (160)$$

in order to get a partial factorization:

$$\theta(v_1 - v_1')[\nu(v_1)\nu(v_1') + h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)\nu(v_1') - h(\tilde{\alpha}_1', \tilde{\beta}_1', \psi_q)]$$
$$+\theta(v_1' - v_1)[\nu(v_1')\nu(v_1) + h(\tilde{\alpha}_1', \tilde{\beta}_1', \psi_q)\nu(v_1) - h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)]$$
$$= [\nu(v_1) + h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_p)][\nu(v_1') + h(\tilde{\alpha}_1', \tilde{\beta}_1', \psi_p)]$$
$$-\theta(v_1 - v_1')\chi(v_1)h(\tilde{\alpha}_1', \tilde{\beta}_1', \psi_q) - \theta(v_1' - v_1)\chi(v_1')h(\tilde{\alpha}_1, \tilde{\beta}_1, \psi_q). \quad (161)$$

Regrouping terms as above and using the elementary integrals (92) for the photon angular integrations we find

$$\rho^{KL}_{\bar{\beta}_{2UL}} = \rho'^{KJ}_{\bar{\beta}_{2UL}} + \rho''^{KJ}_{\bar{\beta}_{2UL}}, \quad (162)$$

$$\rho'^{KJ}_{\bar{\beta}_{2UL}} = \frac{1}{4} b_0 \int dv dv' \delta(V - v - v' + vv') \int dv_I dv_F \int dv_I' dv_F'$$
$$\times \int dv_1 \delta(v - v_1 - v_I - v_F) \int dv_1' \delta(v' - v_1' - v_I' - v_F')$$

[14] The $b_0 = \chi(\xi)$ could be moved out in front because we keep only $\mathcal{O}(\xi\gamma)$ contributions.

$$\times \exp\left(\bar{\delta}_{\rm YFS}(\bar{\gamma}_p, v_F + v_{1K}) + \bar{\delta}_{\rm YFS}(\bar{\gamma}_q, v'_F + v'_{1L})\right)$$
$$\times f_1\left(\frac{\gamma}{2}, v_I\right) f_1\left(\frac{\gamma}{2}, v_F\right) f_1\left(\frac{\gamma}{2}, v'_I\right) f_1\left(\frac{\gamma}{2}, v'_F\right)$$
$$\times \left[\nu(v_1)\frac{1}{v_1}\left(\gamma - \frac{\alpha}{\pi}\ \ln(1-v_1)\right) + \frac{\alpha}{\pi} g_a^K(v_1)\right]$$
$$\times \left[\nu(v'_1)\frac{1}{v'_1}\left(\gamma - \frac{\alpha}{\pi}\ \ln(1-v'_1)\right) + \frac{\alpha}{\pi} g_a^L(v'_1)\right], \tag{163}$$

$$\rho''^{KJ}_{\bar{\beta}_{2UL}} = \frac{1}{4} b_0 \int dv dv' \delta(V - v - v' + vv') \int dv_I dv_F \int dv'_I dv'_F$$
$$\times \int dv_1 \delta(v - v_1 - v_I - v_F) \int dv'_1 \delta(v' - v'_1 - v'_I - v'_F)$$
$$\times \exp\left(\bar{\delta}_{\rm YFS}(\bar{\gamma}_p, v_F + v_{1K}) + \bar{\delta}_{\rm YFS}(\bar{\gamma}_q, v'_F + v'_{1L})\right)$$
$$\times f_1\left(\frac{\gamma}{2}, v_I\right) f_1\left(\frac{\gamma}{2}, v_F\right) f_1\left(\frac{\gamma}{2}, v'_I\right) f_1\left(\frac{\gamma}{2}, v'_F\right)\ \gamma \frac{\alpha}{\pi} W(v_1, v'_1), \tag{164}$$

where, in

$$W(v_1, v'_1) = -\theta(v_1 - v'_1)\frac{\chi(v_1)}{v_1} g_a^1(v'_1) - \theta(v'_1 - v_1)\frac{\chi(v'_1)}{v'_1} g_a^1(v_1) \tag{165}$$

we are able to neglect the ξ dependence of $g_a^K(v)$, getting a truncated version

$$g_a^1(v) = \frac{\alpha}{\pi}\frac{1}{2} \ln\left(-3 + \frac{5}{2} v\right). \tag{166}$$

In the leading-log part $\rho'^{KJ}_{\bar{\beta}_{2UL}}$ the only dependence on $K, L = I, F$ survives in the form factors $\bar{\delta}_{\rm YFS}(\gamma, v_F + v_{1K}$ and $\bar{\delta}_{\rm YFS}(\gamma, v'_F + v'_{1L})$. The sub-leading part $\rho''^{KJ}_{\bar{\beta}_{2UL}}$ is completely symmetric in $K, L = I, F$ and the integrations over the spectator photons can be neglected.

The leading-log integral can be rewritten as

$$\rho'^{KL}_{\bar{\beta}_{2UL}} = \frac{1}{4} b_0^{-1} \int dv dv' \delta(V - v - v' + vv') B^{(1)}_{1K}(t, v) B^{(1)}_{1L}(t, v'), \tag{167}$$

where the $B^{(1)}_{1K}$ functions are $\mathcal{O}(\alpha)$ versions of the similar $\mathcal{O}(\alpha^2)$ functions in the calculation of $\bar{\beta}_1$. Let us give them explicitly for the upper line in a form suitable for further exercises as described in Sect. 8:

$$B^{(1)}_{1Im}(t, v)$$
$$= \int_0^v dv_F du \delta(v - v_F - u)\ \exp(\bar{\delta}_{\rm YFS}(\gamma, v_F))\ f_1\left(\frac{1}{2}\gamma, v_F\right) R^{(1)}_I(\bar{\gamma}^I_p(v_F), u),$$

$$B^{(1)}_{1Fm}(t,v) = \int_0^v dv_I du \delta(v - v_I - u)\ \exp(\bar{\delta}_{\rm YFS}(\gamma,u)) f_1\left(\frac{1}{2}\gamma, v_I\right) R^{(1)}_F(\bar{\gamma}^F_p(u),u), \tag{168}$$

where

$$R^{(1)}_K(\bar{\gamma}^K_p,u) = \int dv_I dv_1 \delta(u - v_I - v_1)\ f_1\left(\frac{1}{2}\gamma, v_I\right) \times\left[\tfrac{1}{2}\frac{1}{v_1}\ \left\{\bar{\gamma}^K_p - \frac{\alpha}{\pi}\ \ln(1-v_1)\right\}\nu(v_1) + \frac{\alpha}{\pi}g^1_a(v_1)\right] = F\left(\tfrac{1}{2}\gamma\right) u^{\frac{1}{2}\gamma}\left\{\bar{\gamma}^K_p\left(-\frac{1}{2}+\frac{1}{4}u\right) + \frac{\alpha}{\pi}\ln(1-u)(-1+u) - \gamma^2\frac{1}{8}u\right\}. \tag{169}$$

The resulting functions are given in Eqs. (100) and (108). In fact we need here only their versions truncated to $\mathcal{O}(\alpha^2)_{\rm prag}$.

The non-leading part can be brought to the following form

$$\rho''^{KJ}_{\bar{\beta}_{2UL}}(t,V) = \frac{1}{4}\gamma\frac{\alpha}{\pi}b_0 \int dv dv' \delta(V - v - v' + vv')\ W(v_1,v_1') = \frac{1}{4}b_0\ F(\gamma)\, V^{2\gamma}\ \exp(2\Delta_{\rm YFS}(\gamma)) \times\left\{\gamma\frac{\alpha}{\pi}\ln(V)\ln(1-V)(6-5V) + \gamma\frac{\alpha}{\pi}\ln(1-V)^2\left(\frac{3}{8}-\frac{5}{16}V\right)\right. +\gamma\frac{\alpha}{\pi}\ln(1-V)\ln[1-(1-V)^{1/2}](-6+5V) +\gamma\frac{\alpha}{\pi}\ln(1-V)(1-V)^{1/2}\left(\frac{-3}{2}\right) +\gamma\frac{\alpha}{\pi}{\rm Li}_2(V)(6-5V) + \gamma\frac{\alpha}{\pi}{\rm Li}_2\left[1-(1-V)^{(-1/2)}\right](6-5V) \left. +\gamma\frac{\alpha}{\pi}\left[-2-\frac{1}{2}V+2(1-V)^{1/2}\right]\right\}. \tag{170}$$

The total result is given as a sum of the above two and over all initial/final state configurations:

$$\rho^{KJ}_{\bar{\beta}_{2UL}} = \sum_{K,J=I,F} \rho'^{KJ}_{\bar{\beta}_{2UL}} + \rho''^{KJ}_{\bar{\beta}_{2UL}}, \tag{171}$$

and it reads as

$$\rho_{\bar{\beta}_{2UL}}(t,V) = b_0\ F(\gamma)\, V^{2\gamma}\ \exp(2\Delta_{\rm YFS}(\gamma))$$

$$\times\Big\{\gamma^2\left(\frac{V}{2}\right)+\gamma^2\ln(1-V)\left(-\frac{1}{2}+\frac{V}{4}\right)$$
$$+\gamma\frac{\alpha}{\pi}\left[-2-\frac{5}{2}V+2(1-V)^{1/2}\right]$$
$$+\gamma\frac{\alpha}{\pi}\ln(1-V)\left[-2+2V-\frac{3}{2}(1-V)^{1/2}\right]$$
$$+\gamma\frac{\alpha}{\pi}\ln(1-V)^2\left(-\frac{5}{8}+\frac{11V}{16}\right)+\gamma\frac{\alpha}{\pi}\ln(V)\ln(1-V)(6-5V)$$
$$+\gamma\frac{\alpha}{\pi}\ln(1-V)\ln[1-(1-V)^{1/2}](-6+5V)$$
$$+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2(V)(6-5V)$$
$$+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2[1-(1-V)^{-1/2}](6-5V)+\gamma^3\left(-\frac{V}{2}\right)$$
$$+\gamma^3\ln(1-V)\left(\frac{1}{2}-\frac{5V}{4}\right)+\gamma^3\ln(1-V)^2\left(\frac{1}{4}-\frac{V}{8}\right)$$
$$+\gamma^3\mathrm{Li}_2(V)\left(-1+\frac{V}{2}\right)\Big\}. \qquad (172)$$

5.4. Numerical results on $\bar{\beta}_{2UL}$

In Fig. 4 we compare our semi-analytical result of Eq. (172) with the numerical result of BHLUMI. We plot the quantity $R(\bar{\beta}^{(2)}_{2UL}; t, V_{\max})$ defined in a way analogous to that done in Eq. (70) for $\bar{\beta}_0$, as a function of the cut

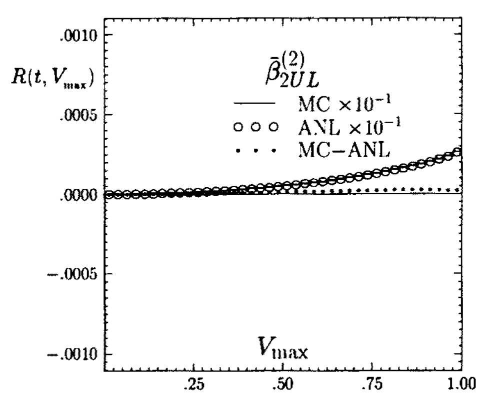

Fig. 4. The comparison of the Monte Carlo and semi-analytical results for the second-order $\bar{\beta}_2$, both photons on one fermion line.

of total photon energy $V_{\rm max}$, for the same fixed value of the transfer t. As before, we plot the difference between the MC and semi-analytical results, showing also $\bar\beta^{(2)}_{2UL}$ itself, multiplied by factor 10^{-1} (it is of order of half a per cent). As we see, the MC and semi-analytical results agree very well, *i.e.* within 1×10^{-4}.

6. Total result

Finally we add the contributions from all $\bar\beta_n$, $n=0,1,2,$

$$\rho^{(2)}_{\rm tot} = \rho^{(z)}_{\bar\beta_0}(t,V) + 2\rho^{(2)}_{1U}(t,V) + 2\rho_{\bar\beta_{2UU}}(t,V) + \rho_{\bar\beta_{2UL}}(t,V), \tag{173}$$

getting the total $\mathcal{O}(\alpha^2)_{\rm prag}$ result, which explicitly reads as follows:

$$\begin{aligned}
\rho^{(2)}_{\rm tot} &= b_0\, F(2\gamma)\, \exp(2\Delta_{\rm YFS}(\gamma)) 2\gamma V^{2\gamma-1}\left\{1+\gamma+\tfrac{1}{2}\gamma^2\right\}\\
&+b_0 F(2\gamma)\exp(2\Delta_{\rm YFS}(\gamma))V^{2\gamma}\\
&\times\Bigg\{\gamma(-2+V)+\frac{\alpha}{\pi}\ln(1-V)(-4+4V-2V^{-1})\\
&+\gamma^2(-2)+\gamma^2\ln(1-V)\left(3-\frac{3V}{2}-2V^{-1}\right)\\
&+\gamma^3\left(-\frac{9V}{8}\right)+\gamma^3\ln(1-V)\left[2+\frac{1}{8V}-2V^{-1}-\frac{1}{4}(2-V)^{-1}\right]\\
&+\gamma^3\ln(1-V)^2\left[-\frac{7}{8}+\frac{7V}{16}+\frac{1}{2}V^{-1}\right]+\gamma^3{\rm Li}_2(V)(2-V)\\
&+\gamma^3\ln(1-V)\ln(2-V)\left(-\frac{1}{4}+\frac{V}{8}\right)+\gamma^3{\rm Li}_2\left(\frac{1-V}{2-V}\right)\left(\frac{1}{4}-\frac{V}{8}\right)\\
&+\gamma^3{\rm Li}_2\left(\frac{1}{2-V}\right)\left(-\frac{1}{4}+\frac{V}{8}\right)\\
&+\gamma\frac{\alpha}{\pi}\Bigg[\frac{1}{4}+11V\\
&\qquad -\frac{13}{4}(2-V)^{-1}+\frac{1}{2}(2-V)^{-2}-6(2-V)^{-3}+2(1-V)^{1/2}\Bigg]\\
&+\gamma\frac{\alpha}{\pi}\ln(1-V)\Bigg[\frac{39}{4}-\frac{19V}{4}-2V^{-1}\\
&\qquad -2(2-V)^{-1}+(2-V)^{-2}-\frac{1}{2}(2-V)^{-3}-\frac{3}{2}(1-V)^{1/2}\Bigg]\\
&+\gamma\frac{\alpha}{\pi}\ln\left(1-\frac{V}{2}\right)\left[-\frac{9}{2}+\frac{3V}{4}-4(2-V)^{-1}+2(2-V)^{-2}-4(2-V)^{-3}\right]\\
&+\gamma\frac{\alpha}{\pi}\ln(1-V)^2\left[\frac{27}{8}-\frac{49V}{16}\right]+\gamma\frac{\alpha}{\pi}\ln(1-V)\ln(2-V)\left(-\frac{1}{2}+\frac{V}{4}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\gamma\frac{\alpha}{\pi}\ln(1-V)\ln(V)(12-10V)+\gamma\frac{\alpha}{\pi}\ln(1-V)\ln\frac{V}{2}(-6+5V)\\
&+\gamma\frac{\alpha}{\pi}\ln(1-V)\ln[1-(1-V)^{1/2}](-6+5V)\\
&+\gamma\frac{\alpha}{\pi}\ln(2-V)\ln\left(1-\frac{V}{2}\right)\left(\frac{3}{2}-\frac{11V}{4}\right)\\
&+\gamma\frac{\alpha}{\pi}\ln^2\left(1-\frac{V}{2}\right)\left(\frac{3}{4}-\frac{5V}{8}\right)+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(\frac{1}{2}\right)\left(-\frac{3}{2}+\frac{11V}{4}\right)\\
&+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(\frac{1-V}{2-V}\right)\left(\frac{1}{2}-\frac{V}{4}\right)+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(\frac{1}{2-V}\right)\left(1-\frac{5V}{2}\right)\\
&+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2\left(-\frac{V}{2(1-V)}\right)(6-5V)+\gamma\frac{\alpha}{\pi}\mathrm{Li}_2[1-(1-V)^{-1/2}](6-5V)\\
&\left.-\xi\gamma\frac{\chi(V)}{1-V}\right\}. \qquad (174)
\end{aligned}
$$

We have also derived the analogous analytical formulas for the total cross section for the matrix element without exponentiation and compared it with the corresponding BHLUMI result, also without exponentiation. Very good agreement between unexponentiated semi-analytical and Monte Carlo results has been obtained (a little bit worse, however, than for exponentiated ones). This variant of the calculation for the moment remains unpublished.

6.1. Numerical results for total cross section

In Fig. 5 we compare our semi-analytical result of Eq. (174) for the total cross section with the numerical result of BHLUMI. We plot the following quantity

$$
R^{(2)}(t,V_{\max})=\frac{\int_0^{V_{\max}}\frac{d\sigma^{(2)}}{d|t|\,dV}dV}{\frac{d\sigma_{\mathrm{Born}}}{d|t|}}=\int_0^{V_{\max}}\rho^{(2)}_{\mathrm{tot}}(t,V)\,dV \qquad (175)
$$

as a function of the cut on the total photon energy $V_{\max}$, for the fixed transfer $t = -4.612982$ GeV^2. We plot the difference between the MC and semi-analytical results, showing in addition $R^{(2)}$ itself, multiplied by factor 10^{-3}. As we see in the plot, the MC and semi-analytical results agree to better then 1.7×10^{-4}! As seen from the previous plots the dominant contribution to the difference comes from $\bar{\beta}_0$, see Fig. 1. The above is the main numerical result for the academic event selection (AES). Although AES is far from the typical experimental ES, this result together with the previous results for individual $\bar{\beta}_n$ is nevertheless quite precious and important because

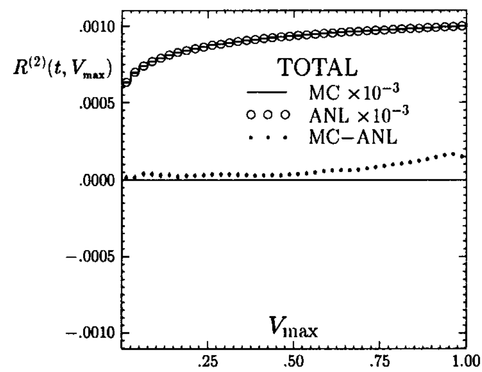

Fig. 5. The comparison of the Monte Carlo and semi-analytical results for the total cross section.

(a) it provides an important test of the correct implementation of the matrix element in BHLUMI[15], (b) for the "trivial" matrix element $\bar{\beta}_0^{(0}$ it tests very precisely the numerical correctness (technical precision) of the basic Monte Carlo algorithm of BHLUMI (independently of the matrix element). The main advantage of the above test was that any discrepancy between the MC and SAN that would have occurred at the early stage of its realization could be traced back to some mistake either in semi-analytical integration or in the matrix element in BHLUMI[16]. The main disadvantage is the lack of flexibility in the choice of ES in the semi-analytical part of the test.

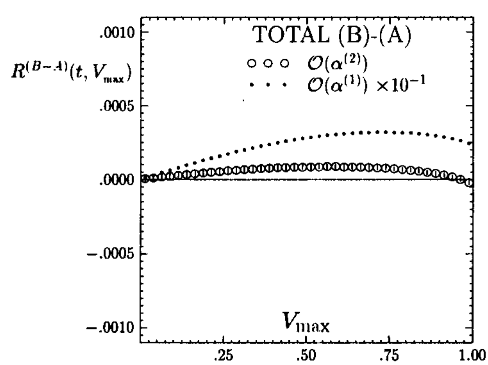

Fig. 6. Difference between cross section types (B) and (A) of the matrix element. Monte Carlo result only.

[15] Thanks to the above numerical test we could identify and correct a few bugs in the early implementations of the matrix element of BHLUMI 4.x.

[16] This is almost impossible to do in the comparison of two different MC programs.

Finally, in Fig. 6 we show the difference of the Monte Carlo total cross sections for two types of matrix element, type (B) for which an analytical integration is available and type (A) for which we have only the Monte Carlo result. As we see, the difference at $\mathcal{O}(\alpha^2)_{\rm exp}$ is negligible, below 0.01%, while at $\mathcal{O}(\alpha^1)_{\rm exp}$ it is quite sizeable, up to 0.3%. We have also checked (the relevant plot is not shown) that the difference of the total cross sections $\mathcal{O}(\alpha^2)_{\rm exp} - \mathcal{O}(\alpha^1)_{\rm exp}$ is much smaller for the matrix element type (A) than for (B).

7. Cross-check of the leading logs

Let us consider photon emission from one, for instance, the upper, electron line. The $\mathcal{O}(\gamma^2)_{\rm prag}$ leading-logarithmic formula for the distribution of the variable $v = v_I + v_F$ reads

$$\begin{aligned} B^{(2)}_{\rm LLog}(v) &= \int\limits_0^1 dz_1 \int\limits_0^1 dz_2\, \delta\Big(1 - v - z_1 z_2\Big) D^{(2)}_{\rm YFS}\left(\tfrac{1}{2}\gamma, z_1\right)\; D^{(2)}_{\rm YFS}\left(\tfrac{1}{2}\gamma, z_2\right) \\ &= D^{(2)}_{\rm YFS}(\gamma, 1 - v), \end{aligned} \tag{176}$$

where

$$\begin{aligned} D^{(2)}_{\rm YFS}(\gamma, z) = F(\gamma)\; e^{\frac{3}{4}\gamma}\, (1 - z)^{\gamma - 1} \Big\{ \gamma \left[1 - \frac{1}{2}(1 - z^2) \right] \\ + \gamma^2 \left[- \frac{1}{8}(1 + 3z^2) \ln(z) - \frac{1}{4}(1 - z)^2 \right] \Big\} \end{aligned} \tag{177}$$

is the $\mathcal{O}(\gamma^2)_{\rm prag}$ non-singlet (valence) "exponentiated" structure function for finding an electron carrying the energy fraction z, within an electron, see for example Ref. [19] and references therein. In the above, z_1 equals the fraction of energy of the initial-state electron after (collinear) emission of photons while z_2 describes a similar phenomenon in the final state. In order to see how to get the above formula, let us note that we start with $\delta\Big(v - \tilde{\alpha}_I(z_1, z_2) - \tilde{\beta}_F(z_1, z_2)\Big)$ and, with a simple kinematic exercise, we find that $\tilde{\alpha}_I(z_1, z_2) = z_2(1 - z_1)$ and $\tilde{\beta}_F(z_1, z_2) = 1 - z_2$. We have explicitly exploited the well-known self-reproduction property of the non-singlet structure in the convolution

$$D_{NS}(\gamma_1 + \gamma_2, z) = \int\limits_0^1 dz_1 \int\limits_0^1 dz_2\; \delta(z - z_1 z_2)\; D_{NS}(\gamma_1, z_1) D_{NS}(\gamma_2, z_2). \tag{178}$$

The upper and lower line contributions get combined in exactly the same way, thanks to our definition of the variable V

$$\begin{aligned}
\rho^{(2)}_{\rm LLog} &= \frac{4\pi\alpha^2}{|t|^2}\, b_0 \int dv dv' \delta(V - v - v' + vv')\; B^{(2)}_{\rm LLog}(v)\; B^{(2)}_{\rm LLog}(v') \\
&= \frac{4\pi\alpha^2}{|t|^2}\, b_0\, D'^{(2)}_{\rm YFS}(2\gamma, 1-V), \\
&= \frac{4\pi\alpha^2}{|t|^2}\, b_0\, F(2\gamma)\; e^{\frac{1}{2}\gamma}\; V^{2\gamma}\Big\{ 2\gamma V^{-1}\left(1+\gamma+\tfrac{1}{2}\gamma^2\right) \\
&+ \left[\gamma(-2+V) + \gamma^2(-2) + \gamma^2 \ln(1-V)\Big(3 - \frac{3V}{2} - \frac{2}{V}\Big)\right]\Big\}. \qquad (179)
\end{aligned}$$

As we see the leading logarithmic terms within $\mathcal{O}(\gamma^2)$ coincide with the analogous terms in Eq. (174), as expected. Note that in the above formula we used a variant of the structure function $D'^{(2)}_{\rm YFS}(\gamma, z)$ in which we factorize off the factor $\exp(\gamma/2)$ instead of $\exp(3\gamma/4)$.

8. Calorimetric event selection

In the academic event selection (AES), see Sect. 3, used throughout the present paper, the total energy of photons is restricted from above using the variable V, without any regard as to whether the photons are emitted closer to initial-state fermions or final-state fermions. In real LEP luminometers, photons close to final-state electrons (positrons) are effectively combined with the electron into a "cluster", and only the total energy of the cluster is restricted. This is called the calorimetric type of ES. In this case the energy of the photon close to a final electron is effectively unrestricted, even if two final clusters are required to carry most of the energy, while the energy of the photons close to beams can be in such a case limited quite strongly.

In our analytical calculation we have integrated first over the transverse momenta of photons, dividing the photon phase-space into initial-state and final-state parts, see Eq. (38), and later the energy of the initial-state photons v_I was combined with the energy of the final-state photons v_F, see for instance Eq. (49). In essence, this was a purely technical calculational trick and in our final analytical results for AES there is no real distinction whatsoever between initial state and final state.

Let us, however, stress that our calculation method, summarized briefly in the above paragraph, opens the way to the introduction of a certain type of "calorimetric academic event selection", CAES, in which an integration over energy of the final state v_F is performed and v_F does not enter into the overall photon energy cut. This is still not a very realistic ES, so we may ask: is the comparison of semi-analytical results with BHLUMI for CAES feasible, and is it interesting? The analytical integration over v_F and v'_F, keeping for instance $V_I = v_I + v'_I + v_I v'_I$ fixed, is probably feasible but it is not yet done and it is probably not worth being done. What can be done relatively easily is to implement in the collinear MC of the LUMLOG type the analytical distribution $d\sigma/dv_F dv_I dv'_F dv'_I$ (which is a by-product of the calculations of the previous Sections) and to integrate over v_F and v'_F numerically.

The above implementation in LUMLOG was partly realized. Only the LL version of $d\sigma/dv_F dv_I dv'_F dv'_I$ is now implemented in BHLUMI 4.04, see Refs. [11, 12]. Why is it that the LL version was realized first? It was done first because it was very important to check that the second order LL content of $d\sigma/dv_F dv_I dv'_F dv'_I$ is functionally identical to the product of the four structure functions. This test is even stronger than that of the previous Sect. 7. The other important and urgent application was the numerical evaluation of the so-called missing third-order LL correction in BHLUMI, presented in Ref. [12].

Would it be interesting to implement not only the LL version of the $d\sigma/dv_F dv_I dv'_F dv'_I$ in LUMLOG but its full form given below? Yes, because it would provide a unique example of comparison between the BHLUMI MC and SAN calculation, at the level of 10^{-4} for *any* kind of calorimetric ES. Generally, the calorimetric ES is substantially different from the non-calorimetric one and for certain errors in the matrix element numerical effects may cancel between the initial- and final-state emission, while being non-zero for calorimetric ES (remember that real luminometers are calorimetric!). The new test for CAES would provide another valuable test of the BHLUMI matrix element. In this Section we essentially provide the basis for such a test, hoping that it will be realized numerically in the future.

8.1. Master formula — sub-leading included

Looking into the simplest example of $\bar{\beta}_0^{(0)}$ in Eqs. (49) and (69), we see that this contribution was written at a certain stage of our calculation as an integral over the four-photon longitudinal variables v_I, v_F, v'_I, v'_F, each of them for photons from one of the initial/final state fermions

$$\sigma = \int_{|t|_{\min}}^{|t|_{\max}} d|t| \int_0^{V_{\max}} dV \int dv dv' \, \delta(V - v - v' + vv') \times \int dv_I dv_F \, \delta(v - v_I - v_F) \int dv'_I dv'_F \, \delta(v' - v'_I - v'_F) \, X, \quad (180)$$

where

$$X = \frac{4\pi\alpha^2}{|t|^2}\, b_0 \exp\left(\bar{\delta}_{\rm YFS}(\bar{\gamma}_p, v_F) + \bar{\delta}_{\rm YFS}(\bar{\gamma}_q, v'_F)\right) \times f_1(\bar{\gamma}_p/2, v_I)\, f_1(\bar{\gamma}_p/2, v_F)\, f_1(\bar{\gamma}_q/2, v'_I)\, f_1(\bar{\gamma}_q/2, v'_F). \qquad (181)$$

Simple kinematical considerations lead to relations between v_I, v_F, v'_I, v'_F and the standard LL variables z_1, z_2, z_3, z_4,

$$v_I = (1 - z_1)z_3, \quad v_F = 1 - z_3, \quad v'_I = (1 - z_2)z_4, \quad v'_F = 1 - z_4, \qquad (182)$$

see also Fig. 7. The phase-space integral transforms into

$$\sigma = \int_{|t|_{\rm min}}^{|t|_{\rm max}} d|t| \int_{1-V_{\rm max}}^{1} dz \int dz_1 dz_2 dz_3 dz_4\, \delta(z - z_1 z_2 z_3 z_4)\, X\, z_1 z_4. \qquad (183)$$

The above integral is ready for implementation in the LUMLOG MC, see Refs. [11, 12] for more details.

What is very important and non-trivial is that the contributions from the other $\bar{\beta}_n$ can also be written in the form of Eq. (180). Close examination of the calculations from the previous Sections lead to the following formula for the integrand in Eq. (180):

$$X = \exp\left(\bar{\delta}_{\rm YFS}(\bar{\gamma}_p, v_F) + \bar{\delta}_{\rm YFS}(\bar{\gamma}_q, v'_F)\right) \times f_1(\bar{\gamma}_p/2, v_I)\, f_1(\bar{\gamma}_p/2, v_F)\, f_1(\bar{\gamma}_q/2, v'_I)\, f_1(\bar{\gamma}_q/2, v'_F)$$

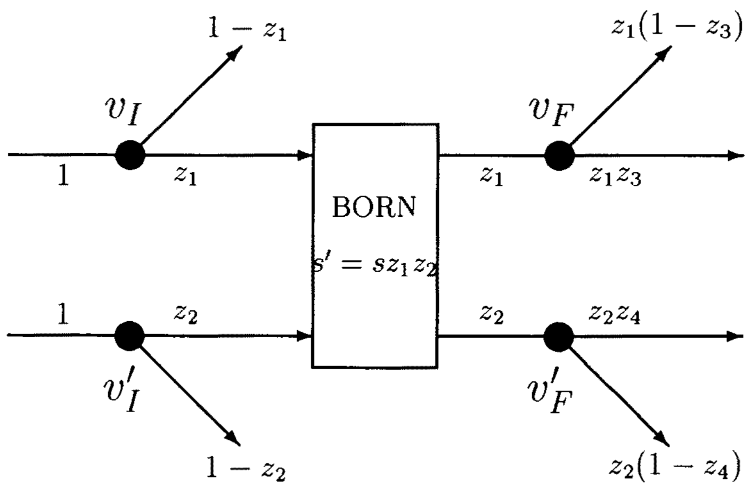

Fig. 7. Leading logarithmic kinematics.

$$\times\Bigg\{X[\bar\beta_0](\gamma)+\sum_{K=I,F}\frac{X[\bar\beta^{(2)}_{1K}](\bar\gamma_p,v_K)}{f_1(\bar\gamma_p/2,v_K)}+\sum_{L=I,F}\frac{X[\bar\beta^{(2)}_{1L}](\bar\gamma_q,v'_L)}{f_1(\bar\gamma_q/2,v'_L)}$$
$$+\sum_{K,L=I,F}\frac{X[\bar\beta^{(2)}_{2UL}](\gamma,K,L,v_K,v'_L)}{f_1(\bar\gamma_p/2,v_K)\;f_1(\bar\gamma_q/2,v'_L)}$$
$$+\sum_{K,L=I,F}\frac{X[\bar\beta^{(2)}_{2UU}](\gamma,K,L,v_K,v_L)}{f_1(\bar\gamma_p/2,v_K)\;f_1(\bar\gamma_p/2,v_L)}$$
$$+\sum_{K,L=I,F}\frac{X[\bar\beta^{(2)}_{2LL}](\gamma,K,L,u_K,v'_L)}{f_1(\bar\gamma_q/2,u_K)\;f_1(\bar\gamma_q/2,v'_L)}\Bigg\}, \tag{184}$$

where

$$X[\bar\beta_0](\gamma)=b_0\left(1+\gamma+\frac{1}{2}\gamma^2\right), \tag{185}$$
$$X[\bar\beta^{(2)}_{1K}](\gamma,v)=R^{(2)}_K(\gamma,v)+B^{(2)}_{1K\,\mathrm{singl}}(\gamma,v), \tag{186}$$
$$X[\bar\beta^{(2)}_{2UL}](\gamma,K,L,v_K,v'_L)=R^{(1)}_K(\gamma,v_K)\;R^{(1)}_L(\gamma,v'_L)+\frac{1}{4}\frac{\alpha}{\pi}\gamma\;W(v_K,v'_L), \tag{187}$$
$$X[\bar\beta^{(2)}_{2UU}](\gamma,I,I,v_I,v_F)=\left[\frac{1}{8}\;U(\gamma,v_I)+\tfrac{1}{2}\frac{\alpha}{\pi}\gamma d^*_3(v_I)\right]\;f_1(\bar\gamma_p/2,v_F), \tag{188}$$
$$X[\bar\beta^{(2)}_{2UU}](\gamma,F,F,v_I,v_F)=f_1(\bar\gamma_p/2,v_I)\;\left[\frac{1}{8}\;U(\gamma,v_F)+\frac{1}{2}\frac{\alpha}{\pi}\gamma d^*_3(v_F)\right], \tag{189}$$
$$X[\bar\beta^{(2)}_{2UU}](\gamma,I,F,v_I,v_F)=\frac{1}{8}V(\gamma,v_I,v_F)+\tfrac{1}{2}\frac{\alpha}{\pi}\gamma S(v_I,v_F). \tag{190}$$

In the above collection Eq. (185) is derived from Eqs. (26,59,69), Eq. (186) is derived from Eqs. (90,97,103), Eq. (187) is derived from Eqs. (164,167,168), Eqs. (188,188) are derived from Eqs. (127,148) and finally Eq. (190) is derived from Eqs. (137,149).

The version of Eq. (184) truncated to leading logarithmic approximation is shown explicitly in Refs. [11, 12]. It is already implemented in LUMLOG MCEG within BHLUMI 4.04 [11]. The complete Eq. (184) is not yet implemented in LUMLOG.

9. Summary

The aim of this paper was to summarize the third-order analytical calculations of the total cross section, which were (and will be) instrumental in the task of the high precision calculation of the small-angle Bhabha process. We presented in detail the calculation technique and the numerical comparisons with the Monte Carlo results; we also discussed future extensions of the calculations. The presented analytical calculations are relevant to the question of the *technical precision* of the calculation of the small-angle Bhabha (SABH) process, because it has allowed us to test the matrix element implemented in BHLUMI Monte Carlo term by term with the precision of 0.01%, and its basic MC integration algorithm (for $\bar{\beta}_0^{(0)}$) with the same precision. They are also helpful to partially solve the problem of the *physical precision* of the QED calculation of the SABH process because we were able

- to cross-check the correctness of the $\mathcal{O}(\alpha^2 L^2)$ matrix element and phase-space integration to within 1.7×10^{-4},
- to calculate the missing $\mathcal{O}(\alpha^3 L^3)$ in the BHLUMI cross section,
- to get analytical insight into the incomplete $\mathcal{O}(\alpha^2 L)$ component in the BHLUMI cross section,
- to gain direct analytical insight into the mechanism of "inclusive exponentiation" in the first order and beyond.

The above wealth of information and the calculation technology will be very useful in the next step, which consists in bringing the total theoretical precision of the SABH process below the level of 0.1%. In particular we have in hand all methods to calculate analytically the contribution of the missing $\mathcal{O}(\alpha^2 L)$ component in the BHLUMI cross section. Most of the presented results are restricted to the unrealistic academic event selection. We have indicated, however, the path to calculation for more realistic calorimetric event selection. The methods and results presented in this paper are major contribution to the future, more precise calculation of the SABH luminosity cross section.

The authors thank, for support and kind hospitality, the CERN Theory Division, where a substantial part of this work was performed, and, for their support the ALEPH and OPAL Collaborations. One of us (B.F.L. W.) thanks Prof. C. Prescott of SLAC for the kind hospitality of SLAC Group A while this work was completed.

10. Appendix A: Collinearization in $\mathcal{O}(\alpha^3)_{\rm prag}$

In the following we shall prove that within $\mathcal{O}(\alpha^3)_{\rm prag}$ we are allowed to do a replacement

$$\delta_p = \frac{1-v}{\left(1-\tilde{\beta}^I-\tilde{\beta}^F_j\right)^2}\,\frac{m_e^2}{|t|} \longrightarrow \bar{\delta}_p = \frac{1-v}{(1-v_F)^2}\,\frac{m_e^2}{|t|} \tag{191}$$

in the real bremsstrahlung distributions $d\omega_i(\delta_p)$. In our calculations this leads to a very useful "collinearization" of our integrals at the early stages of the calculations. As a preparatory step let us examine more closely the real bremsstrahlung distribution for one photon $d\omega_i$ (on the upper fermion line)

$$\int d\omega_i\ \delta(v_i - \tilde{\alpha}_i - \tilde{\beta}_i + \tilde{\alpha}_i\tilde{\beta}_i)$$
$$= \frac{\alpha}{2\pi^2}\int d\tilde{\alpha}_i d\tilde{\beta}_i d\phi_i\ \delta(v_i - \tilde{\alpha}_i - \tilde{\beta}_i + \tilde{\alpha}_i\tilde{\beta}_i)\ \frac{\tilde{\alpha}_i\tilde{\beta}_i}{(\tilde{\alpha}_i + \delta_p\tilde{\beta}_i)^2(\tilde{\beta}_i + \delta_p\tilde{\alpha}_i)^2}. \quad (192)$$

In order to see more clearly its singularities we may (in the presence of $\delta(v_i - \tilde{\alpha}_i - \tilde{\beta}_i + \tilde{\alpha}_i\tilde{\beta}_i)$) decompose it as follows

$$\frac{\tilde{\alpha}_i\tilde{\beta}_i}{(\tilde{\alpha}_i + \delta_p\tilde{\beta}_i)^2(\tilde{\beta}_i + \delta_p\tilde{\alpha}_i)^2}$$
$$= \frac{1}{v}\left(\frac{1}{\tilde{\alpha}_i + \delta_p\tilde{\beta}_i} + \frac{1}{\tilde{\beta}_i + \delta_p\tilde{\alpha}_i} - \frac{\delta_p v}{(\tilde{\alpha}_i + \delta_p\tilde{\beta}_i)^2} - \frac{\delta_p v}{(\tilde{\beta}_i + \delta_p\tilde{\alpha}_i)^2} - 1\right). \quad (193)$$

The first two terms directly lead to leading-logarithmic contributions, while the next two so-called "mass terms" provided finite non-log contributions coming from δ-narrow collinear regions (photon collinear with one of the fermions) — for instance

$$\int d\tilde{\alpha}_i \frac{\delta_p v}{(\tilde{\alpha}_i + \delta_p\tilde{\beta}_i)^2} = \int d\tilde{\alpha}_i \delta(\tilde{\alpha}_i). \quad (194)$$

In the above equation, the actual value of δ_p drops out, *i.e.* the only important thing is that $\delta_p \to 0$. Consequently in this kind of mass term we can do the substitution $\delta_p \to \bar{\delta}_p$ freely. On the other hand, for any of the leading-log poles in the photon angle, we have

$$\frac{1}{\tilde{\alpha}_i + \delta_p\tilde{\beta}_i} = \frac{1}{\tilde{\alpha}_i + \bar{\delta}_p\tilde{\beta}_i} - \frac{(\delta_p - \bar{\delta}_p)\tilde{\beta}_i}{(\tilde{\alpha}_i + \delta_p\tilde{\beta}_i)^2} + \mathcal{O}(\delta_p^2). \quad (195)$$

The correction term $\sim (\delta_p - \bar{\delta}_p)$ has two important features:

(i) it is non-logarithmic and strictly collinear for the i-th photon, similarly to the mass term in Eq. (194) and

(ii) it is proportional to $\bar{\Delta}\delta_p = \delta_p - \bar{\delta}_p$ which is equal to zero if all other photons $k \neq i$ are collinear (this is true by construction of $\bar{\delta}_p$).

Let us denote by $\bar{\Delta}$ the variation due to the operation $\delta_p \to \bar{\delta}_p$ and consider the general case of emission of n photons[17]

$$\bar{\Delta}\left(\prod_i^n d\omega_i\right) = \sum_{k=1}^n \bar{\Delta}(d\omega_k) \prod_{i\neq k} d\omega_i + \mathcal{O}(\alpha^2), \quad (196)$$

[17] In the n-photon case discussed below, we mean by $\mathcal{O}(\alpha^2)$ two powers of α not accompanied by big logs.

where we have used property (i) to eliminate higher powers of $\bar{\Delta}$, *i.e.* for instance $\bar{\Delta}(d\omega_k)\bar{\Delta}(d\omega_i) \sim \mathcal{O}(\alpha^2)$. The above remnant with a single power of $\bar{\Delta}$ is at most $\mathcal{O}(\gamma\alpha)$. In fact, it is even of $\mathcal{O}(\alpha^2)$, because according to (ii), $\bar{\Delta}d\omega_k \sim \bar{\Delta}\delta_p$, which is zero if all other photons $k \neq i$ are collinear; this means that we lose at least one big collinear log (*i.e.* we gain one pure non-log factor α) during integration over other photons directions.

To summarize our proof: we have shown that the effects of the $\delta_p \to \bar{\delta}_p$ substitution in the differential and integrated distributions $\prod_{i=1}^{n} d\omega_i$ are beyond $\mathcal{O}(\alpha^3)_{\rm prag}$. The same substitution can be and has to be done simultaneously in the related form factor $\exp\left(\frac{1}{2}\gamma_p(\delta_p)\ln\Delta\right)$ because here $\gamma_p(\delta_p)$ is directly related through an integration to $d\omega(\delta_p)$.

Although we have presented our proof for arbitrary numbers of photons it is really essential and sufficient to consider the cases with one and two photons (also in the version without exponentiation). We recommend dedicated readers to do this exercise.

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