BACKREACTION OF EXCITATIONS ON A DOMAIN WALL

R. Pełka

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Cracow, Poland

(Received March 3, 1997)

In this paper we investigate backreaction of excitations on a planar domain wall in a real scalar field model. The backreaction is investigated in the cases of homogeneous, plane wave and wave packet type excitations. We find that the excited domain wall radiates. The method of calculating backreaction for the general forms of excitations is also presented.

PACS numbers: 11.10. Lm

1. Introduction

The presence of topological defects in the field theoretical models with a degenerate vacuum is an important aspect of the structure of these models. Dynamics of topological defects is to be extracted from nonlinear equations describing evolution of fields they are composed of. The results, even though it is a formidable task to get them, are of great interest for particle physics (*e.g.* dynamics of a flux-tube in QCD [1]), for field theoretical cosmology (*e.g.* cosmic strings [2, 3]) and for condensed matter physics (*e.g.* domain walls in magnetics, vortices in superconductors or in superliquid helium, defects in liquid crystals [4–8]).

This paper is devoted to dynamics of domain walls governed by a Poincaré invariant wave equation. Domain walls appear in the models with the nontrivial zeroth homotopy group of the vacuum manifold. In the papers [9–11] two main approaches to the dynamics of domain walls have been presented. The first one is the polynomial approximation, the second one is the expansion in the width of the wall. Calculations made in the framework of these two methods indicate existence of an oscillating component in the width of the domain wall. This suggests that the domain wall can radiate. The radiation was also observed in computer simulations [12].

The problem we address ourselves to is connected with this special aspect of dynamics of domain walls, namely the radiation. We calculate the

radiation with the help taken from [13, 14] where analogous problem was considered in the abelian Higgs model. The method consists of two steps. The first one is to find the excitations of a static domain wall. They are investigated in the linear approximation and treated as small corrections to the basic domain wall field which is localized on the wall. The second step consists of looking for the effects of this excitations on the evolution of the domain wall, *i.e.* a backreaction. This procedure can be treated as the expansion of the domain wall field in the amplitude of the excitation A ($A \ll 1$). The zeroth order term is then the static, exact planar domain wall solution ϕ_0, the first order term is the excitation ϕ_1 and the second order term is the backreaction

$$\phi = \phi_0 + A\phi_1 + A^2\phi_2 + O(A^3). \tag{1}$$

The terms of the higher order can be interpreted as more complicated effects, *e.g.* the third order term as the selfinteraction of the excitation. We shall discuss three main cases of the excitations, namely the most simple of them — the homogeneous excitation, the second case — a plane wave type excitation and the last one — a localized excitation. In this paper we shall consider the dynamics of the domain wall in the simplest scalar field model which has a potential with two degenerate minima.

The plan of our paper is the following. In the next Section we present the calculation of the backreaction of the homogeneously excited domain wall. In Section 3 we describe the method of calculating backreaction for the general forms of excitations. Section 4 is devoted to the detailed analysis of the backreaction in the cases of plane wave and wave packet type excitations. In Section 5 we summarize the main points of our work.

2. Radiation from a homogeneously excited domain wall

We consider the planar domain wall in the model defined by the Lagrangian density,

$$\mathcal{L} = -\frac{1}{2}\eta_{\mu\nu}\partial^\mu\Phi\partial^\nu\Phi - \frac{\lambda}{2}(\Phi^2 - v^2)^2, \tag{2}$$

where $\eta_{\mu\nu}$ =diag$(-1,1,1,1)$ and λ, v are positive constants. The corresponding evolution equation for the scalar field Φ has the form:

$$\partial^\mu\partial_\mu\Phi - 2\lambda(\Phi^2 - v^2)\Phi = 0. \tag{3}$$

Exact, static solution representing the domain wall localized around the (x^1, x^2)-plane is given by the formula:

$$\phi_0 = v \tanh(\alpha x^3), \tag{4}$$

where $\alpha = \sqrt{\lambda v^2}$.

Let us rescale the scalar field Φ and the space-time coordinates x^μ,

$$\begin{aligned} \Phi &= v\phi\,, \\ \tilde{x}^\mu &= \alpha x^\mu\,, \end{aligned} \tag{5}$$

where $\phi(\tilde{x}^\mu)$ and $\tilde{x}^\mu$ are dimensionless. Moreover, for $\tilde{x}^0$ and $\tilde{x}^3$ we shall use the following notation,

$$\begin{aligned} \tilde{x}^0 &= \tau\,, \\ \tilde{x}^3 &= \xi\,. \end{aligned} \tag{6}$$

In the rescaled variables the evolution equation and the static solution take the form:

$$\begin{aligned} \tilde{\partial}^\mu \tilde{\partial}_\mu \phi - 2(\phi^2 - 1)\phi &= 0\,, \\ \phi_0 &= \tanh(\xi)\,, \end{aligned} \tag{7}$$

where $\tilde{\partial}^\mu = \frac{\partial}{\partial \tilde{x}_\mu}$. The first step of our considerations is to find excitations of the planar domain wall. As it was stated in the Introduction the excitations are considered as small corrections to the basic static domain wall solution. Thus the corresponding equation for the excitation field ϕ_1 we shall obtain as the linear approximation to the initial evolution equation. Inserting the expansion (1) into Eq. (7) and keeping the terms of the first order in the expansion parameter A we obtain:

$$\tilde{\partial}^\mu \tilde{\partial}_\mu \phi_1 - 2(3{\phi_0}^2 - 1)\phi_1 = 0. \tag{8}$$

The planar domain wall distinguishes the direction perpendicular to the wall plane, given in our case by the coordinate lines of ξ. Small correction ϕ_1 do not change this asymmetry. Therefore we assume a special form of the perturbation in which dependence on the coordinate ξ is separated,

$$\phi_1 = \psi(\xi)\chi(\tau, \tilde{x}^1, \tilde{x}^2). \tag{9}$$

Inserting the above Ansatz into Eq. (8) we obtain the system of two equations for the functions ψ and χ with a separation constant c,

$$\begin{aligned} \frac{d^2\psi}{d\xi^2} - 6{\phi_0}^2\psi + 2\psi &= -c\psi\,, \\ \tilde{\partial}^a \tilde{\partial}_a \chi &= c\chi\,, \end{aligned} \tag{10}$$

where $a = 0, 1, 2$. From that follows a restriction for constant c. If c were less than zero the solution of the second equation would increase with the time infinitely and so would not meet the main requirement of the perturbative calculation.

Let us solve the first equation of the system (10). It can be transformed into the following form:

$$-\frac{1}{2}\frac{d^2\psi}{d\xi^2} - \frac{3}{\cosh^2\xi}\psi = (\frac{c}{2} - 2)\psi. \tag{11}$$

It is the special case of Schrödinger equation with generalized Pöschel-Teller potential. Solutions of this equation can be found in [15]. There exist two bound states enumerated by $n = 0, 1$,

$$\begin{aligned} n = 0 \quad & \psi_0(\xi) = \tfrac{1}{\cosh^2\xi}, \quad c_0 = 0, \\ n = 1 \quad & \psi_1(\xi) = \tfrac{\sinh\xi}{\cosh^2\xi}, \quad c_1 = 3. \end{aligned} \tag{12}$$

Both are localized on the wall in the sense that they exponentially decrease for large $|\ \xi\ |$. The first, even solution $\psi_0(\xi)$ can be interpreted as a small displacement of the domain wall,

$$\phi = \phi_0 + \psi_0\chi = \tanh\xi + \frac{1}{\cosh^2\xi}\chi \simeq \tanh(\xi + \chi). \tag{13}$$

This is the zero mode related to the translational symmetry of the model. In what follows we will not consider this kind of excitation. The excitation given by the second, odd function ψ_1 does not posses such interpretation. This solution we shall accept as the proper bound state of the domain wall.

The second equation of the system (10) is of the wave type. In this Section we shall consider the special case of excitation, homogeneous on the whole wall and given by the solution independent of the coordinates $\tilde{x}^1, \tilde{x}^2$,

$$\chi(\tau) = A\cos(\sqrt{3}\tau + \delta), \tag{14}$$

where δ is a constant phase which we shall put equal to zero, and A is a constant amplitude. As mentioned above the described procedure assumes that A is sufficiently small. Finally, the field of the homogeneously excited planar domain wall is given by the formula:

$$\phi = \tanh\xi + A\frac{\sinh\xi}{\cosh^2\xi}\cos(\sqrt{3}\tau). \tag{15}$$

From this formula one can see that the excitation introduces periodic changes of the thickness of the domain wall.

The second step in our procedure is to find the backreaction. Inserting the expansion (1) into the Eq. (7) and keeping all terms of order A^2 we obtain the equation for the backreaction ϕ_2,

$$\tilde{\partial}^\mu \tilde{\partial}_\mu \phi_2 - 2(3{\phi_0}^2 - 1)\phi_2 = 6\phi_0 {\phi_1}^2. \tag{16}$$

It is an inhomogeneous equation with the r.h.s. including the square of the excitation component. We will denote the inhomogeneous term by $N = 6\phi_0{\phi_1}^2$. For a homogeneous excitation we are dealing with, N is a function of the two variables (ξ, τ). Inserting the formulae for the functions ϕ_0 and ϕ_1 we obtain:

$$N = N_1 + N_2 = 3A^2 \frac{\sinh^3\xi}{\cosh^5\xi} + 3A^2 \frac{\sinh^3\xi}{\cosh^5\xi} \cos(2\sqrt{3}\tau). \tag{17}$$

We shall solve Eq. (16) in two steps, considering each of the two parts of the inhomogeneous term N separately. At the first step we consider the equation:

$$\frac{1}{2}\frac{d^2\phi_2}{d\xi^2} - (3\tanh^2\xi - 1)\phi_2 = \frac{3}{2}A^2 \frac{\sinh^3\xi}{\cosh^5\xi}. \tag{18}$$

The general solution of this equation can be found by the standard Green's function technique, see *e.g.* [16]. As the two linearly independent solutions of the homogeneous part of Eq. (16) we take:

$$\begin{aligned} f_1(\xi) &= \frac{1}{\cosh^2\xi}, \\ f_2(\xi) &= \frac{1}{8}\sinh(2\xi) + \frac{3}{8}\tanh\xi + \frac{3}{8}\frac{\xi}{\cosh^2\xi}. \end{aligned} \tag{19}$$

As the Green's function we take

$$G(\xi, x) = 2f_1(x)f_2(\xi)\theta(\xi - x) - 2f_1(\xi)f_2(x)[\theta(\xi - x) - \theta(-x)]. \tag{20}$$

The Green's function was chosen in such a manner as to obey the condition

$$G(\xi = 0, x) = 0 \qquad \bigwedge x \in R. \tag{21}$$

Such a choice ensures that an inhomogeneity will not produce any displacement of the domain wall as a whole. The general solution of Eq. (18) has the form

$$\phi_2(\xi) = af_1(\xi) + bf_2(\xi) + A^2 \int_{-\infty}^{+\infty} G(\xi, x)h(x)dx, \tag{22}$$

where

$$h(x) = \frac{3}{2}\frac{\sinh^3 x}{\cosh^5 x},$$

and a, b are constants. Formula (22) gives:

$$\phi_2(\xi) = A^2[c_1(\xi)f_1(\xi) + c_2(\xi)f_2(\xi)], \tag{23}$$

$$c_1(\xi) = \frac{a}{A^2} - 2\int_0^{\xi} f_2(x)h(x)dx,$$

$$c_2(\xi) = \frac{b}{A^2} + 2\int_{-\infty}^{\xi} f_1(x)h(x)dx. \tag{24}$$

The function f_1 is even while f_2 is odd. Because $f_2(\xi)$ exponentially grows for $\xi \to \pm\infty$ the coefficient function c_2 has to vanish in this limit. Therefore we have to put $b = 0$ while a is still arbitrary. But we put a to be zero too because keeping it nonzero would amount to including uninteresting solution of the homogeneous equation. In the second step we have to solve the Eq. (16) with the second inhomogeneous term N_2 containing a periodic time-dependence.

The perturbatively obtained Eq. (16) for the beackreaction has radiation type solutions which do not vanish for large ξ as will be shown further. In this case we adopt the Helmholtz condition which states that for $|\xi| \to \infty$ only outgoing radiation waves are present.

Hence we consider the following form of the solution:

$$\phi_2 = \tfrac{1}{2}A^2\left[\varphi_+(\xi)\exp(-i2\sqrt{3}\tau) + \varphi_-(\xi)\exp(i2\sqrt{3}\tau)\right],$$

where the positive and negative frequency components are related by complex conjugation,

$$\varphi_- = [\varphi_+]^*$$

This Ansatz leads to the following equation for the functions $\varphi_\pm$,

$$\left[\frac{1}{2}\frac{d^2}{d\xi^2} + (4 + \frac{3}{\cosh^2\xi})\right]\varphi_\pm(\xi) = \frac{3}{2}\frac{\sinh^3\xi}{\cosh^5\xi}, \tag{25}$$

which we solve analogously as Eq. (18). As two linearly independent solutions we take

$$g_1(\xi) = \cos(2\sqrt{2}\xi) - \sqrt{2}\tanh\xi\sin(2\sqrt{2}\xi) + \frac{1}{2}\frac{\cos(2\sqrt{2}\xi)}{\cosh^2\xi},$$

$$g_2(\xi) = \sin(2\sqrt{2}\xi) + \sqrt{2}\tanh\xi\cos(2\sqrt{2}\xi) + \frac{1}{2}\frac{\sin(2\sqrt{2}\xi)}{\cosh^2\xi}. \tag{26}$$

The Green's function is given by the formula

$$G(\xi,x)=\frac{1}{6\sqrt{2}}[g_2(\xi)g_1(x)\theta(\xi-x)-g_1(\xi)g_2(x)(\theta(\xi-x)-\theta(-x))]\,. \quad (27)$$

The general solution has the form

$$\varphi_\pm(\xi)=[d_1(\xi)+a_\pm]g_1(\xi)+[d_2(\xi)+b_\pm]g_2(\xi)\,, \quad (28)$$

where

$$d_1(\xi)=-\frac{1}{6\sqrt{2}}\int_0^\xi g_2(x)h(x)dx\,, \quad (29)$$

$$d_2(\xi)=\frac{1}{6\sqrt{2}}\int_{-\infty}^\xi g_1(x)h(x)dx\,, \quad (30)$$

and $a_\pm, b_\pm$ are constants.

In order to satisfy the Helmholtz condition imposed on the solution, we have to find solutions with the appropriate asymptotics given by the formula below

$$\varphi_\pm(\xi\to\pm\infty)\sim\exp\left[\pm i2\sqrt{2}\mid\xi\mid\right]. \quad (31)$$

We consider separately the regions $\xi>0$ and $\xi<0$ and choose the solutions which will have the proper asymptotic behaviour given by the formula (31). Next, we impose the matching conditions at the point $\xi=0$, *i.e.* the continuity conditions of the solution and its first derivative. The solutions found in this way are given by the formula

$$\varphi_\pm=d_1(\xi)g_1(\xi)+[d_2(\xi)\pm id_1(\infty)]g_2(\xi), \quad (32)$$

where

$$d_1(\infty)=\lim_{\xi\to+\infty}d_1(\xi)$$

and is finite. Finally, the asymptotic form of the solution is the following

$$\phi_2(\xi\to\pm\infty)\sim\pm\sqrt{3}d_1(\infty)A^2\cos(2\sqrt{2}\mid\xi\mid-2\sqrt{3}\tau\pm\beta), \quad (33)$$

where $\beta=\arctan\sqrt{2}, \beta\in(0,\frac{\pi}{2})$.

Let us summarize results of our calculations. The full solution consisting of the static domain wall, the excitation and backreaction is the following,

$$\begin{aligned}\phi=\tanh\xi\;+\;&A\frac{\sinh\xi}{\cosh^2\xi}\cos(\sqrt{3}\tau)+A^2\left[c_1(\xi)f_1(\xi)+c_2(\xi)f_2(\xi)\right]\\&+\;\tfrac{1}{2}A^2\left[\varphi_+(\xi)\exp(-i2\sqrt{3}\tau)+\varphi_-(\xi)exp(i2\sqrt{3}\tau)\right]. \quad (34)\end{aligned}$$

Backreaction consists of two terms. The first one, independent of time, consists of the static reaction of the domain wall to the homogeneous excitation while the second one depends on time. Its asymptotics is given by the formula (33) and describes the radiation with the energy fluxes given by the Poynting vectors:

$$\vec{S}_{\pm} = 6\sqrt{3}d_1^2(\infty)A^4v^2\alpha^2\sin^2(k_{\pm}^{\mu}\tilde{x}_{\mu} \pm \beta)\vec{k}_{\pm}\,, \tag{35}$$

where

$$k_{\pm}^{\mu} = (2\sqrt{3}, 0, 0, \pm 2\sqrt{2})\,. \tag{36}$$

and the signs $\pm$ correspond to the limits $+\infty$ and $-\infty$ respectively. The fact that the energy of the radiation is infinite is consistent with the physical situation considered here where the excitation of the domain wall is constantly maintained for instance by the interaction with other fields.

3. The method for the general forms of excitations

Our goal in this Section is to present the method of calculating backreaction for a general form of excitation ϕ_1 defined by the formula:

$$\phi_1 = \psi(\xi)\chi(\tau, \tilde{x}^1, \tilde{x}^2)\,, \tag{37}$$

where

$$\psi(\xi) = \frac{\sinh\xi}{\cosh^2\xi}\,,$$

and the function χ is any bounded solution of the wave equation

$$\tilde{\partial}^a\tilde{\partial}_a\chi = 3\chi\,. \tag{38}$$

Let us recall the equation for the backreaction,

$$\tilde{\partial}^{\mu}\tilde{\partial}_{\mu}\phi_2 - 2(3\tanh^2\xi - 1)\phi_2 = 6\phi_0{\phi_1}^2\,. \tag{39}$$

The first step is to define the operator $\hat{L}_{\xi}$ by the formula:

$$\hat{L}_{\xi} = \frac{d^2}{d\xi^2} + \frac{6}{\cosh^2\xi}\,. \tag{40}$$

Then the Eq. (39) takes the form:

$$\tilde{\partial}^a\tilde{\partial}_a\phi_2 + \hat{L}_{\xi}\phi_2 - 4\phi_2 = 6\phi_0{\phi_1}^2\,. \tag{41}$$

The second step is to solve the eigenvalue problem for this operator. We shall find all solutions of the equation:

$$\hat{L}_\xi \psi_\lambda(\xi) = \lambda \psi_\lambda(\xi)\,, \tag{42}$$

where λ is the eigenvalue and $\psi_\lambda(\xi)$ - the normalized eigenfunction corresponding to this eigenvalue. The backreaction ϕ_2 we treat as the expansion in the eigenfunctions of $\hat{L}_\xi$ given by the formula:

$$\phi_2(\xi, \tilde{x}^a) = \sum\!\!\!\!\!\!\int_\lambda a_\lambda(\tilde{x}^a)\psi_\lambda(\xi)\,, \tag{43}$$

where the coefficients a_λ depend on the coordinates $\tilde{x}^a$. Inserting this expansion into the Eq. (41) we obtain

$$\sum\!\!\!\!\!\!\int_{\lambda'} \psi_{\lambda'}(\xi)[\tilde{\partial}^a\tilde{\partial}_a + \lambda' - 4]a_{\lambda'}(\tilde{x}^a) = N(\xi, \tilde{x}^a)\,, \tag{44}$$

where $N(\xi, \tilde{x}^a) = 6\phi_0{\phi_1}^2$. Multiplying the equation above by the eigenfunction $\psi_\lambda(\xi)$ (provided the set of them is complete) and integrating over the full range of variation of ξ and using the orthonormality condition for the system of the eigenfunctions $\psi_\lambda(\xi)$ we obtain the system of differential equations for the coefficient functions $a_\lambda(\tilde{x}^a)$,

$$(\tilde{\partial}^a\tilde{\partial}_a + \lambda - 4)a_\lambda(\tilde{x}^a) = h_\lambda(\tilde{x}^a)\,, \tag{45}$$

where

$$h_\lambda(\tilde{x}^a) = \int_{-\infty}^{+\infty} d\xi \psi_\lambda(\xi) N(\xi, \tilde{x}^a)\,.$$

The last step in this procedure is to solve Eq. (45). It may be done by the standard method of the Green's function. Let us denote by $G_\lambda(\tilde{x}^a)$ the retarded Green's function of the operator,

$$\hat{D}_\lambda = \tilde{\partial}^a\tilde{\partial}_a + \lambda - 4\,. \tag{46}$$

Then the solution of the inhomogeneous equations (45) is given by the formula

$$a_\lambda(\tilde{x}^a) = \int G_\lambda(\tilde{x}^a - \tilde{x'}^a)h_\lambda(\tilde{x'}^a)d\tilde{x'}^a\,. \tag{47}$$

In this solution we have dropped the homogeneous part because it does not contain any information about the excitation.

The results of the calculations are the following. The spectrum of the operator $\hat{L}_\xi$ consists of two parts. The discrete part, for which $\lambda > 0$, contains two eigenfunctions,

$$\psi_1 = \frac{\sqrt{3}}{2}\frac{1}{\cosh^2\xi}, \quad \lambda_1 = 4,$$
$$\psi_2 = \sqrt{\frac{3}{2}}\frac{\sinh\xi}{\cosh^2\xi}, \quad \lambda_2 = 1. \tag{48}$$

The continuous part, for which $\lambda \leq 0$, includes twice degenerate subspaces corresponding to the eigenvalues $\lambda = -k^2 (k \in R_+)$ spanned by the eigenfunctions:

$$\psi_k{}^{(1)}(\xi) = n(k)[(k^2-2)\cos(k\xi) - 3k\sin(k\xi)\tanh\xi + \frac{3\cos(k\xi)}{\cosh^2\xi}],$$
$$\psi_k{}^{(2)}(\xi) = n(k)[(k^2-2)\sin(k\xi) + 3k\cos(k\xi)\tanh\xi + \frac{3\sin(k\xi)}{\cosh^2\xi}], \tag{49}$$

where

$$n(k) = [\pi k(k^2+1)(k^2+4)]^{-\frac{1}{2}}.$$

One can prove that the set of eigenfunctions is complete [17]. Therefore we have to find the retarded Green's function for the following operators:

$$\hat{D}_1 = \tilde{\partial}^a\tilde{\partial}_a(\lambda = 4),$$
$$\hat{D}_2 = \tilde{\partial}^a\tilde{\partial}_a - 3(\lambda = 1),$$
$$\hat{D}_k = \tilde{\partial}^a\tilde{\partial}_a - k^2 - 4(\lambda = -k^2). \tag{50}$$

They are obtained by the Fourier transform method and given by the formulae:

$$G_1(\tilde{z}^a) = -\frac{1}{2\pi}\frac{\theta(\tilde{z}^0)}{\sqrt{(\tilde{z}^0)^2 - (\tilde{z})^2}},$$
$$G_2(\tilde{z}^a) = -\frac{\theta(\tilde{z}^0)}{2\pi}\int_0^\infty dK\,K\frac{\sin(\sqrt{K^2+3}\tilde{z}^0)}{\sqrt{K^2+3}}J_0(K\tilde{z}),$$
$$G_k(\tilde{z}^a) = -\frac{\theta(\tilde{z}^0)}{2\pi}\int_0^\infty dK\,K\frac{\sin(\sqrt{K^2+k^2+4}\tilde{z}^0)}{\sqrt{K^2+k^2+4}}J_0(K\tilde{z}), \tag{51}$$

where

$$\tilde{z}^a = \tilde{x}^a - \tilde{x}'^a,$$
$$\tilde{z} = \sqrt{(\tilde{z}^1)^2 + (\tilde{z}^2)^2}.$$

The procedure presented above enables us to find the backreaction for the general form of excitation of domain wall. We shall use it in the next Section.

4. The backreaction for the plane wave and wave packet excitations

In this section we analyse the backreaction of the excitation of the plane wave and wave packet type. In the first case χ is given by formula:

$$\chi = A\cos(\omega(k_0^1, k_0^2)\tau - k_0^1\tilde{x}^1 - k_0^2\tilde{x}^2)\,, \tag{52}$$

where $\omega(k_0^1, k_0^2) = \sqrt{(k_0^1)^2 + (k_0^2)^2 + 3}$.

In the second case we consider the approximate solution of Eq. (38):

$$\chi \cong A\exp[-\frac{(\tilde{x}^1)^2 + (\tilde{x}^2)^2}{\Lambda^2}]\sin(\sqrt{3}\tau) = Aw(\tilde{x})\sin(\sqrt{3}\tau)\,, \tag{53}$$

where A and Λ are constants. We assume that $\Lambda \gg 1$ and that the wave packet has momentum cutoff $k \sim \Lambda^{-1}$. Then we may neglect for the finite time interval $0 \leq \tau \leq\sim \Lambda$ the spreading out of the wave packet, which is of course present in exact wave packet solutions of Eq. (38).

4.1. The plane wave case

The inhomogeneous terms in the backreaction equation has the following form:

$$N^{(1)}(\tau, \tilde{x}^1, \tilde{x}^2, \xi) = 3A^2\frac{\sinh^3\xi}{\cosh^5\xi}[1 - \cos(2\omega(k_0^1, k_0^2)\tau - 2k_0^1\tilde{x}^1 - 2k_0^2\tilde{x}^2)]\,. \tag{54}$$

It is convenient to pass to the Fourier transform with respect to the coordinates $\tilde{x}^1, \tilde{x}^2, \tau$:

$$\phi_2^{(1)}(\xi, \tilde{x}^1, \tilde{x}^2, \tau) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int dk^1\int dk^2\int d\omega\exp[-i\omega\tau + ik^1\tilde{x}^1 + ik^2\tilde{x}^2]\hat{\phi}_2^{(1)}(\xi, k^1, k^2, \omega)\,, \tag{55}$$

$$N^{(1)}(\xi, \tilde{x}^1, \tilde{x}^2, \tau) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int dk^1\int dk^2\int d\omega\exp[-i\omega\tau + ik^1\tilde{x}^1 + ik^2\tilde{x}^2]\hat{N}^{(1)}(\xi, k^1, k^2, \omega)\,. \tag{56}$$

In this case,

$$\hat{N}^{(1)}(\xi,k^1,k^2,\omega)=3(2\pi)^{\frac{3}{2}}A^2\frac{\sinh^3\xi}{\cosh^5\xi}[\delta(\omega)\delta(k^1)\delta(k^2)$$
$$-\tfrac{1}{2}\delta(\omega+2\omega(k_0^1,k_0^2))\delta(k^1+2k_0^1)\delta(k^2+2k_0^2)$$
$$-\tfrac{1}{2}\delta(\omega-2\omega(k_0^1,k_0^2))\delta(k^1-2k_0^1)\delta(k^2-2k_0^2)]\,. \tag{57}$$

The equation for the Fourier transform $\hat{\phi}_2^{(1)}$ has the following form:

$$\left[\omega^2-k^2+\frac{d^2}{d\xi^2}-2(3\tanh^2\xi-1)\right]\hat{\phi}_2^{(1)}(\xi,k^1,k^2,\omega)=\hat{N}^{(1)}(\xi,k^1,k^2,\omega). \tag{58}$$

It is clear from the above equation that the solution has the form:

$$\hat{\phi}_2^{(1)}(\xi,k^1,k^2,\omega)=A^2(2\pi)^{\frac{3}{2}}\Big[\varphi_0^{(1)}(\xi)\delta(\omega)\delta(k^1)\delta(k^2)$$
$$-\tfrac{1}{2}\varphi_-^{(1)}(\xi)\delta(\omega+2\omega(k_0^1,k_0^2))\delta(k^1+2k_0^1)\delta(k^2+2k_0^2)$$
$$-\tfrac{1}{2}\varphi_+^{(1)}(\xi)\delta(\omega-2\omega(k_0^1,k_0^2))\delta(k^1-2k_0^2)\delta(k^2-2k_0^2)\Big]\,. \tag{59}$$

The negative and positive frequency components are related by complex conjugation,

$$\varphi_-^{(1)}=\left[\varphi_+^{(1)}\right]^*\,, \tag{60}$$

while $\varphi_0^{(1)}$ is real valued. The functions $\varphi_0^{(1)}$, $\varphi_\pm^{(1)}$ have to satisfy the following equations:

$$\left[\frac{1}{2}\frac{d^2}{d\xi^2}+\frac{3}{\cosh^2\xi}-2\right]\varphi_0^{(1)}=\frac{3}{2}\frac{\sinh^3\xi}{\cosh^5\xi}\,, \tag{61}$$

$$\left[\frac{1}{2}\frac{d^2}{d\xi^2}+\frac{3}{\cosh^2\xi}+4\right]\varphi_\pm^{(1)}=\frac{3}{2}\frac{\sinh^3\xi}{\cosh^5\xi}\,. \tag{62}$$

The function $\varphi_0^{(1)}$corresponding to the frequency $\omega=0$ contains the information about the static backreaction of the domain wall while two remaining functions describe the dynamic backreaction. We shall concentrate on these two functions only. They must satisfy identical conditions as the functions $\varphi_\pm$ in Section 2 and they also satisfy the same equation, see formulae (25)–(31). Therefore, the solutions are given by the formula (32). The asymptotic form of the radiation part of the backreaction is then the following:

$$\phi_2^{(1)}(\tilde{x}^1, \tilde{x}^2, \xi, \tau)$$
$$\sim \mp\sqrt{3}A^2 d_1(\infty)\cos[2k_0^1\tilde{x}^1 + 2k_0^2\tilde{x}^2 + 2\sqrt{2}\mid\xi\mid - 2\omega(k_0^1, k_0^2)\tau \pm \beta], \tag{63}$$

where $\beta = \arctan\sqrt{2}$, $\beta \in (0, \frac{\pi}{2})$. The corresponding wave vectors have the components:

$$(k_\pm^\mu) = (2\omega(k_0^1, k_0^2), 2k_0^1, 2k_0^2, \pm 2\sqrt{2}), \tag{64}$$

where the signs $\pm$ correspond to the limit in the $\pm\infty$ respectively. The energy fluxes due to these waves are given by the Poynting vectors:

$$\vec{S}_\pm = 6d_1^2(\infty)A^4 v^2\alpha^2\omega(k_0^1, k_0^2)\sin^2(k_\pm^\mu\tilde{x}_\mu \pm \beta)\vec{k}_\pm . \tag{65}$$

4.2. The wave packet case

Calculations of backreaction are carried out in the analogous steps as in plane wave case. Instead of formulae (54), (57) we have now:

$$N^{(2)}(\xi, \tilde{x}^1, \tilde{x}^2, \tau) = 3A^2\frac{\sinh^3\xi}{\cosh^5\xi}w^2(\tilde{x})[1 - \cos(2\sqrt{3}\tau)], \tag{66}$$

$$\hat{N}^{(2)}(\xi, k^1, k^2, \omega) = 3A^2\frac{\sinh^3\xi}{\cosh^5\xi}w(k)[\delta(\omega) - \tfrac{1}{2}\delta(\omega + 2\sqrt{3}) - \tfrac{1}{2}\delta(\omega - 2\sqrt{3})], \tag{67}$$

where

$$w(k) \equiv \frac{1}{2\pi}\int d\tilde{x}^1\int d\tilde{x}^2\exp[-ik^1\tilde{x}^1 - ik^2\tilde{x}^2]w^2(\tilde{x}),$$
$$k = \sqrt{(k^1)^2 + (k^2)^2}.$$

We next obtain the same equation for the Fourier transform $\hat{\phi}_2^{(2)}(\xi, k^1, k^2, \omega)$ of $\phi_2^{(2)}(\xi, \tilde{x}^1, \tilde{x}^2, \tau)$:

$$\left[\omega^2 - k^2 + \frac{d^2}{d\xi^2} - 2(3\tanh^2\xi - 1)\right]\hat{\phi}_2^{(2)} = \hat{N}^{(2)}. \tag{68}$$

Analogously to the previous case the solution can be written as

$$\hat{\phi}_2^{(2)} = A^2\frac{\sinh^3\xi}{\cosh^5\xi}w(k)[\varphi_0^{(2)}(\xi)\delta(\omega) - \tfrac{1}{2}\varphi_-^{(2)}(\xi)\delta(\omega + 2\sqrt{3}) - \tfrac{1}{2}\varphi_+^{(2)}(\xi)\delta(\omega - 2\sqrt{3})]. \tag{69}$$

In the exact solution the functions $\varphi_0^{(2)}, \varphi_\pm^{(2)}$ dependent on k, but in the case at hand the expression in square brackets on the left hand side of Eq. (68) can be simplified. Namely, we may neglect the term k^2 because of the cutoff $\Lambda \gg 1$. This simplification leads to the same set of equations (61), (62) for the functions $\varphi_0^{(2)}, \varphi_\pm^{(2)}$ as in the previous case and the solution given by the formula (69) is then the approximate one. The asymptotic conditions (31) remain unchanged. Thus the asymptotic form of the radiation part of the backreaction is the following:

$$\phi_2^{(2)}(\xi \to \pm\infty, \tilde{x}^1, \tilde{x}^2, \tau) \sim \mp\sqrt{3} d_1(\infty) A^2 \cos(2\sqrt{2} \mid \xi \mid -2\sqrt{3}\tau \pm \beta) w^2(\tilde{x}). \tag{70}$$

The corresponding wave vector has the components:

$$k_\pm^\mu = (2\sqrt{3}, 0, 0, \pm 2\sqrt{2}). \tag{71}$$

The energy flux is given by the formula:

$$\vec{S}_\pm = 6\sqrt{3} d_1^2(\infty) A^4 v^2 \alpha^2 w^4(\tilde{x}) \sin^2(k_\pm^\mu \tilde{x}_\mu \pm \beta) \vec{k}_\pm . \tag{72}$$

5. Remarks

1. Let us summarize the main points of our work. We have presented the calculations of the backreaction in the cases of the homogeneous, plane wave and wave packet type excitations of the domain wall. We also have described the method enabling us to analyse the more general cases of the excitations. The main result of our work is the existence of the long range component in the backrection which is interpreted as the radiation from the excited domain wall. The frequency of the radiation, given by ϕ_2,(*c.f.* formulae (63), (70)), is twice of that of the excitation function χ, (*c.f.* formulae (52), (53)).

2. The idea of the expansion in the amplitude of excitation, which all the calculations were based on, was applied to the simplest model of the real scalar field and the potential $V(\phi) = \frac{\lambda}{2}(\phi^2 - v^2)^2$. It seems that without much trouble this method could be applied also to the other field-theoretical models containing domain wall configuration. On the other hand, we should remember that this method is based on the linear approximation, what implies that it is reliable for small amplitudes of the excitations only. In order to consider stronger excitations we have to work out another method which will take into account the nonlinearity of the evolution equation in a better manner. One could for instance use the polynomial approximation in the vicinity of the domain wall and the proper asymptotics at the infinity and smoothly match them in the intermediate region.

I would like to thank Professor H. Arodź for his help, interest and stimulating discussions. I am also grateful to dr L. Hadasz for his help in the editing of this paper.

REFERENCES

[1] See, *e.g.*, M. Baker, J.S. Ball, F. Zachariasen, *Phys. Rep.* **209**, 73 (1991).
[2] T.W.B. Kibble, *J. Phys.* **A9**, 1387 (1976).
[3] A.L. Vilenkin, *Phys. Rep.* **121**, 263 (1985).
[4] See, *e.g.*, J. Slonczewski, in *Physics of Defects*, Les Houches Session XXXV, 1980, North-Holland Publ. Comp., Amsterdam 1981.
[5] R.P. Huebener, *Magnetic Flux Sructures in Superconductors,* Springer-Verlag, Berlin-Heidelberg-New York 1979.
[6] R.J. Donnelly, *Quantized Vortices in HeliumII*, Cambridge University Press, Cambridge 1991.
[7] S. Chandrasekhar, G.S. Ranganath, *Adv. Phy.* **35**, 507 (1986).
[8] W.H. Żurek, *Phys. Rep.* **276**, 177 (1996).
[9] H. Arodź, *Phys. Rev.* **D52**, 1082 (1995).
[10] H. Arodź, A.L. Larsen, *Phys. Rev.* **D49**, 4154 (1994).
[11] H. Arodź, *Nucl. Phys.* **B450**, 174 (1995).
[12] L.M. Widrow, *Phys. Rev.* **D40**, 1002 (1989).
[13] H. Arodź, L. Hadasz, *Phys. Rev.* **D54**, 4004 (1996).
[14] H. Arodź, L. Hadasz, *Phys. Rev.* **D55**, 942 (1997).
[15] S. Flügge, *Practical Quantum Mehanics I*, Springer-Verlag, 1971.
[16] G. Korn, T. Korn, *Sprawocznik po matematike*, Moskwa 1970, in Russian.
[17] R. Pełka, MSc Thesis, Jagellonian University, June 1996.