NO-GHOST THEOREM FOR AN OPEN PARABOSONIC STRING*

N. MEBARKI, M. HAOUCHINE

Département de Physique Théorique, Université de Constantine
Constantine, Algeria**
and
Department of Physics Joseph Henri Labs, Princeton University
Jadwin Hall, P.O. BOX 708, Princeton N.J. 08544, USA

AND N. BELALOUI

Département de Physique Théorique, Constantine Université
Constantine, Algeria

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We show that for the critical dimension $D_c = 2 + \frac{24}{Q}$ of an open parabosonic string, the Fock space is free of negative norms (ghosts).

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1. Introduction

One of the main goals of quantum mechanics (QM) is to provide a consistent and unified description of the so-called wave particle duality which is direct consequence of the Heisenberg equations of motion. Now, to transit from classical to quantum theory, dynamical canonical variables have to verify the ordinary commutation relations which are consistent with the Heisenberg equations.

It turns out that these canonical commutation relations are necessary but not sufficient to guarantee the Heisenberg equations [1, 8]. The generalization of these commutation relations leads to a general quantum theory called "paraquantum theory" (PQM) characterized by an order parameter.

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** Permanent address.

The latter differs from the ordinary QM in the fact that the dynamical variables satisfy not bilinear but trilinear relations [1, 8].

Although it is, in principle, possible to study various features of the paraquantum observables within the Hilbert space, it is often convenient to put this space into a correspondence with a larger Hilbert space in which the operators satisfy bilinear relations [1, 9–12]. Traditionally, for Fock-type irreducible representation of paraquantum theories with a unique vacuum state, this is done by means of the Green decomposition [1, 9–12]

$$a_k^\mu = \sum_{\alpha=1}^{Q} a_k^{\mu(\alpha)}, \qquad \mu = \overline{1, D}, \tag{1.1}$$

where D is the space-time dimension, Q the order of paraquantization, α the Green index and $a_k^{\mu(\alpha)}$ are the annihilation operators Green components satisfying bilinear but anomalous commutation relations

$$\begin{aligned} \left[a_k^{\mu(\alpha)}, a_l^{+\nu(\alpha)}\right]_{\mp} &= \delta_{kl} g^{\mu\nu}, \\ \left[a_k^{\mu(\alpha)}, a_l^{+\nu(\beta)}\right]_{\pm} &= 0, \qquad \alpha \neq \beta \end{aligned} \tag{1.2}$$

(upper sign for parabosons and lower sign for parafermions).

The purpose of this paper is to show that the paraquantum Fock space of an open parabosonic string does not contain negative norm states for the space-time critical dimension $D = 2 + \frac{24}{Q}$ [13, 16]. In Section 2, we describe the formalism and in Section 3, we prove a no-ghost theorem and finally in Section 4 we draw our conclusions.

2. Formalism

The Nambu-Goto classical action of an open parabosonic string is given by [18]

$$S = -\frac{1}{2\pi\alpha'} \int d\tau\, d\sigma\, \left[(\dot{x} - x')^2 - \dot{x}^2\, x'^{\,2}\right]^{\frac{1}{2}}, \tag{2.1}$$

where τ, σ are dimensionless world-sheet parameters. Here α' is the string scale and $\dot{x}^\mu$(resp. $x'^{\,2}$) means $\frac{\partial x^\mu}{\partial \sigma}$ (resp. $\frac{\partial x^\mu}{\partial \tau}$). The general solution of equation of motion (in the orthonormal gauge) subject to the edge conditions

$$x'^{\,\mu}\,(\sigma = 0, \pi\ ;\ \tau) = 0 \tag{2.2}$$

is

$\ddot{x} - x'' = 0$

$$x^\mu = q^\mu + 2\alpha' p^\mu + i\sqrt{2\alpha'} \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} \left[a_n^\mu \exp(-in\tau) - a_n^{+\mu} \exp(in\tau)\right] \cos n\sigma \,, \tag{2.3}$$

where q^μ and p^μ are the string center of mass coordinates and the momentum respectively.

After quantization the physical states $|\Psi\rangle_{\text{phys}}$ are subject to the Virasoro conditions

$$L_n \, |\Psi\rangle_{\text{phys}} = 0 \,, \qquad n \geq 1 \,, \tag{2.4}$$

and

$$(L_0 - \alpha(0)) \, |\Psi\rangle_{\text{phys}} = 0 \,, \tag{2.5}$$

where the Virasoro generators L_n are given by

$$L_n = \frac{1}{2\alpha'} \sum_{n=1}^{+\infty} : \alpha_{n-m}^\mu \alpha_{m\mu} : \,, \tag{2.6}$$

where

$$\begin{aligned} \alpha_0^\mu &= 2\alpha' p^\mu \,, \\ \alpha_{-n}^\mu &= \sqrt{2\alpha' n} a_n^{+\mu} \,, \\ \alpha_n^\mu &= \sqrt{2\alpha' n} a_n^\mu \,, \end{aligned}$$

and the dynamical variables $q^\mu, p^\mu, a_n^\mu, a_n^{+\mu}$ verify the non vanishing canonical commutation relations

$$\begin{aligned} [q^\mu, p^\nu] &= i g^{\mu\nu} \,, \\ [a_n^\mu, a_m^{+\nu}] &= \delta_{n,m} g^{\mu\nu} \,. \end{aligned} \tag{2.7}$$

Now, the transition from the ordinary canonical commutation relations to paraquantum ones can be easily done by the use of the Green decomposition [1, 9–12]

$$q^\mu = \sum_{\beta=1}^{+\infty} q^{\mu(\beta)} \,, \quad p^\mu = \sum_{\beta=1}^{+\infty} p^{\mu(\beta)} \,, \quad a_n{}^\mu = \sum_{\beta=1}^{+\infty} a_n{}^{\mu(\beta)} \,, \quad a_n{}^{+\mu} = \sum_{\beta=1}^{+\infty} a_n{}^{+\mu(\beta)} \,. \tag{2.8}$$

It is to be noted that the string space-time properties should not be affected by the parquantization. In particular, and since q^μ and p^μ are observables characterizing the center of mass coordinates of the string; they have to satisfy ordinary commutation relations [13–16]. To respect this condition, one has to specify a direction in the Green para-space so that

$$q^{\mu(\beta)} = q^\mu \delta_{\beta 1} \,, \quad p^{\mu(\beta)} = p^\mu \delta_{\beta 1} \,. \tag{2.9}$$

Thus , the corresponding paraquantum commutation relations of (2.7) are

$$\begin{aligned}
[q^{\mu}, p^{\nu}] &= ig^{\mu\nu} , \\
\left[a_n^{\mu(\beta)}, a_m^{+\nu(\alpha)}\right]_+ &= 0 \qquad \alpha \neq \beta , \\
\left[a_n^{\mu(\alpha)}, a_m^{+\nu(\alpha)}\right]_- &= g^{\mu\nu}\delta_{mn} .
\end{aligned} \tag{2.10}$$

3. No-ghost theorem

Let us make a preliminary investigation of the conditions which ensure that there are no negative-norm physical states for an open parabosonic string. To delineate the regions of the parameter $\alpha(0)$ and the space-time dimension D in which there are no negative-norm states in the physical Fock space, it is very useful to look for physical states of zero norm. If one varies $\alpha(0)$ and D to cross from a region where the physical Fock space is positive semi-definite to the one which has negative-norm states, extra-physical states of zero norm are always present at the boundaries between the two regions. It is turn out, for some reasons which we will discuss, that these extra-states of zero norm are related to important physical principles and that the most interesting case is the critical space-time dimension $D = 2 + \frac{24}{D}$.

The discussion will be presented for an open parabosonic string, but the closed parabosonic string case is almost identical, with a doubling of the α oscillators and the Virasoro conditions.

An arbitrary state $|\Psi\rangle_{\mathrm{phys}}$ is called a physical state if it satisfies the constraints

$$\begin{aligned}
L_n|\Psi\rangle_{\mathrm{phys}} &= 0 \quad \text{for} \quad n > 0 , \\
[L_0 - \alpha(0)]|\Psi\rangle_{\mathrm{phys}} &= 0 ,
\end{aligned} \tag{3.1a}$$

where L_0 and L_n have the following Green decomposition [14, 16]

$$\begin{aligned}
L_0 &= \sum_{\alpha=1}^{Q} L_0^{(\alpha)} , \\
L_n &= \sum_{\alpha=1}^{Q} L_n^{(\alpha)} ,
\end{aligned} \tag{3.1b}$$

with

$$\begin{aligned}
L_n^{(\alpha)} &= \frac{1}{2\alpha'} \sum_{m=1}^{+\infty} : \ \alpha_{n-m}^{\mu(\alpha)} \alpha_{m\mu}^{(\alpha)} : , \\
L_0^{(\alpha)} &= \frac{1}{2\alpha'} \sum_{m=1}^{+\infty} \alpha_{-m}^{\mu(\alpha)} \alpha_{m\mu}^{(\alpha)} ,
\end{aligned}$$

($\alpha(0)$ is a c-number coming from the normal ordering " : :"). It is to be noted that the paraquantum Fock space is defined that [13–16]

$$a_n^{\mu(\alpha)}\left|0\right\rangle = 0\,,\qquad P^{\mu}\left|0\right\rangle = p^{\mu}\left|0\right\rangle\,, \tag{3.1c}$$

and

$$\langle 0|\,0\rangle = 1\,.$$

A state $|S\rangle$ which obeys

$$[L_n - \alpha(0)]\,|S\rangle = 0 \tag{3.2}$$

is called spurious state if it is orthogonal to all physical states, *i.e.*

$$\langle S|\,\Psi\rangle_{\rm phys} = 0\,, \tag{3.3}$$

and it can always be written in the form

$$|S\rangle = \sum_{n>0} L_{-n}\,|\Phi_n\rangle\,, \tag{3.4}$$

where $|\Phi_n\rangle$ is a state which obeys

$$[L_0 - \alpha(0) + n]\,|\Phi_n\rangle = 0\,. \tag{3.5}$$

Actually, the infinite series in equation (3.4) can be truncated, since the L_n *'s* for $n \geq 3$ can be represented as iterated commutators of L_{-1} and L_{-2}, *e.g.* $L_{-3} \propto [L_{-1}, L_{-2}]$. So we can simply write a spurious state as

$$|S\rangle = L_{-1}\,|\Phi_1\rangle + L_{-2}\,|\Phi_2\rangle\,, \tag{3.6}$$

where $|\Phi_1\rangle$ and $|\Phi_2\rangle$ obey (3.5). Notice that states of the form (3.6) are orthogonal to physical states $|\Psi\rangle_{\rm phys}$.

Now, if a state $|\,X>$ is both spurious and physical, then it has zero norm. Thus these states are orthogonal to all physical states, including themselves (and are sometimes said to be "null" physical states). We can construct states of this type by considering spurious states of the form

$$|X\rangle = \left(L_{-2} + \Omega L_{-1}^2\right)|\Theta\rangle\,, \tag{3.7}$$

where Ω is a c-number and $|\Theta\rangle$ an arbitrary state satisfying

$$L_m\,|\Theta\rangle = 0 \quad \text{for} \quad m > 0\,, \tag{3.8a}$$

and

$$[L_0 - \alpha(0) + 1]\,|\Theta\rangle = 0\,. \tag{3.8b}$$

Now, for $|X\rangle$ to have zero norm it must be physical and, in particular, it should be annihilated by L_m for $m > 0$. Since it is trivially annihilated by L_m with $m \geq 3$,we need only to impose the conditions

$$L_1 |\Theta\rangle = 0\,, \tag{3.9}$$

$$L_2 |\Theta\rangle = 0\,. \tag{3.10}$$

After straightforward calculations and making use of the modified Virasoro algebra [14]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{QD}{12}(n-n^3)\delta_{n+m,0} \tag{3.11}$$

together with equation (3.8), the conditions (3.9), (3.10) lead to

$$D = \frac{(42-16\alpha(0))\,(-\alpha(0)+2)}{Q\,(-2\alpha(0)+3)}\,. \tag{3.12a}$$

Now, as it will be clear in the next part,

$$0 \leq \alpha(0) \leq 1\,. \tag{3.12b}$$

Thus, one gets the following double inequality

$$\frac{26}{Q} \leq D \leq \frac{84}{Q}\,. \tag{3.13}$$

Notice that the critical dimension $D_c = 2 + \frac{24}{Q}$ [13, 15–16] verifies the inequalities (3.13). To be more explicit and as an alternative way to show that for critical dimension $D_c = 2 + \frac{24}{Q}$ the paraquantum Fock space is free of ghosts, let us consider the first three excited states of parabosonic string [13, 15–16]

$$|0\rangle\ ,\ |\varepsilon\rangle = \varepsilon_\mu a_i^{+\mu}|0\rangle\ ,\ |\lambda,\theta\rangle = \frac{1}{\sqrt{2}}\left[\lambda_\mu a_2^{+\mu} + \theta_{\mu\nu} a_1^{+\mu} a_1^{+\nu}\right]|0\rangle\,, \tag{3.14}$$

with the Green decomposition

$$a_{1,2}^{+\mu} = \sum_{\alpha=1}^{Q} a_{1,2}^{+\mu(\alpha)}\,, \tag{3.15}$$

and the normalization:

$$\langle 0|0\rangle = 1\,, \qquad |0\rangle \text{ is the vacuum state.}$$

Now, applying the mass operator M^2

$$M^2 = \frac{1}{\alpha'}\left[L_0 - \alpha(0)\right] \tag{3.16}$$

the vacuum state $|0\rangle$ one gets

$$M^2\,|0\rangle = -\frac{1}{\alpha'}\alpha(0)\,|0\rangle\ . \tag{3.17}$$

Since in the ordinary case (Q=1) this equation characterizes a tachyonic state, we should have $\alpha(0) \geq 0$. For the physical vectorial states $|\varepsilon\rangle$ and with the help of the Green components commutation relations (2.10) and equation (3.1c) we obtain

$$\langle\varepsilon|\varepsilon\rangle = \varepsilon_\mu \varepsilon^\mu Q\ . \tag{3.18}$$

Now the Virasoro condition

$$L_n\,|\varepsilon\rangle = 0 \qquad \text{for} \qquad n \geq 1 \tag{3.19}$$

is trivially verified for all values $n > 1$. However, for $n = 1$ one has to have

$$\varepsilon_\mu P^\mu = 0\ . \tag{3.20}$$

Moreover, the mass operator M^2 applied to the physical state $|\varepsilon\rangle$ implies that

$$-P^2 = \frac{1}{\alpha'}\left[1 - \alpha(0)\right] \tag{3.21}$$

with

$$P^2 = -P^{0^2} + P^{i^2}\ , \qquad i = \overline{1, D-1}\,,$$
$$\alpha(0) \leq 1\ . \tag{3.22}$$

In the rest frame one can write

$$P^\mu = \left(\left\{\frac{1}{\alpha'}\left[1 - \alpha(0)\right]\right\}^{\frac{1}{2}}, 0, 0, \ldots\right)\ . \tag{3.23}$$

Combining equations (3.20) and (3.23) one gets $\varepsilon^0 = 0$. Therefore, the norm (3.18) becomes

$$\langle\varepsilon|\varepsilon\rangle = \varepsilon^{i^2} Q\ . \tag{3.24}$$

This shows clearly that $\langle\varepsilon|\varepsilon\rangle \geq 0$, for each space-time dimension D.

Now, regarding the states $|\lambda, \theta\rangle$ and with the use of the commutation relations (2.10) one gets

$$\langle\lambda, \theta|\lambda, \theta\rangle = Q\left[\frac{\lambda^2}{2} + \theta_{(\mu\nu)}\theta^{(\mu\nu)}\right]\ , \tag{3.25}$$

and

$$M^2 \left|\lambda, \theta\right\rangle = -P^2 \left|\lambda, \theta\right\rangle = \frac{1}{\alpha'} \left[2 - \alpha(0)\right] \left|\lambda, \theta\right\rangle , \tag{3.26}$$

where $\theta_{(\mu\nu)}$ means the symmetric part of $\theta_{\mu\nu}$. For the Virasoro constraint

$$L_n \left|\lambda, \theta\right\rangle = 0 , \qquad n = 0 . \tag{3.27}$$

One can show that it is trivial except for $n = 1$ or 2. Straightforward calculations (for $n = 1$ and $n = 2$) lead to

$$\sqrt{2} \left(\lambda^v + \sqrt{4\alpha'}\theta_{(\mu\nu)} P^\mu\right) \sum_{\alpha=1}^{Q} a_1^{+\nu(\alpha)} \left|0\right\rangle = 0 , \tag{3.28}$$

and

$$-\frac{1}{\sqrt{2}} \left[\sqrt{4\alpha'}\lambda_\mu P^\mu + Q\theta_{(\mu\nu)}\eta^{\mu\nu}\right] \left|0\right\rangle = 0 . \tag{3.29}$$

As a result one has to have

$$\lambda^v + \sqrt{4\alpha'}\theta_{(\mu\nu)} P^\mu = 0 , \tag{3.30}$$

and

$$\sqrt{4\alpha'}\lambda_\mu P^\mu + Q\theta_{(\mu\nu)}\eta^{\mu\nu} = 0 . \tag{3.31}$$

In the rest frame where P^μ is given by equations (3.23); equations (3.30) and (3.31) become

$$\lambda_0 + \sqrt{4\alpha'}\theta_{(00)} P^0 = 0 , \tag{3.32}$$

$$\lambda_i + \sqrt{4\alpha'}\theta_{(0i)} P^0 = 0 , \tag{3.33}$$

and

$$\sqrt{4\alpha'}\varepsilon_0 \left\{\frac{1}{\alpha'} \left[2 - \alpha(0)\right]\right\}^{\frac{1}{2}} - Q\theta_{(00)} + Q \sum_{i=1}^{D-1} \theta_{(ii)} = 0 . \tag{3.34}$$

From equations (3.24), (3.32), (3.33) and (3.34) we deduce that

$$\begin{aligned} \left\langle\lambda, \theta | \lambda, \theta\right\rangle = Q \Bigg[& \frac{Q^2[1 - 2(2 - \alpha(0)]}{[4(2 - \alpha(0)) + Q]^2} \left(\sum_{i=1}^{D-1} \theta_{(ii)}\right)^2 \\ & + (2(2 - \alpha(0) - 2) \sum_{i=1}^{D-1} \theta_{(0i)}^2 + \sum_{i=1}^{D-1} \theta_{(ii)}^2 + \sum_{\substack{i,j \\ i \neq j}}^{D-1} \theta_{(ij)}^2 \Bigg] , \end{aligned} \tag{3.35}$$

where in the last term we have $i \neq j$.

Now, in order that the norm (3.35) is positive definite independently of the parameters $\theta_{\mu\nu}$, one should have

(a)

$$(2(2-\alpha(0)-2) \geq 0 \qquad i.e \qquad \alpha(0) \leq 1\,;$$

(b)

$$\frac{Q^2[1-2(2-\alpha(0)]}{[4(2-\alpha(0))+Q]^2}\left(\sum_{i=1}^{D-1}\theta_{(ii)}\right)^2+\sum_{i=1}^{D-1}\theta_{(ii)}^2 Q \geq 0 \quad \forall\theta_{(ii)}\,. \tag{3.36}$$

Now, using the fact that $x^2-x+1 \geq 0$ for all x (here in our case $x = \sum_{i=1}^{D-1}\theta_{(ii)}$), we obtain

$$-\frac{Q^2[1-2(2-\alpha(0)]}{[4(2-\alpha(0))+Q]^2}\left(1-\sum_{i=1}^{D-1}\theta_{(ii)}\right)+\sum_{i=1}^{D-1}\theta_{(ii)}^2 \geq 0\,.$$

Using (3.12b) direct simplifications lead to

$$D \leq 1+4\frac{[8+Q]^2}{Q^2}\,. \tag{3.37}$$

Notice that the critical dimension D_c verifies this inequality.

4. Conclusions

We conclude that the open parabosonic string critical dimension $D_c = 2+\frac{24}{Q}$ guaranties that all the paraquantum states subject to the paraquantum Virasoro conditions (3.10) are physical and consequently, the theory is free from negative norm states (ghosts).

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