NEW CRITICAL DIMENSIONS FOR PARA-BOSONIC STRINGS AND PARA-SUPERSTRINGS *

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Applying the mass shell condition for para-bosonic strings and superstrings, new critical space-time dimensions are derived.

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1. Introduction

The generalization of the ordinary canonical commutation relations between dynamical variables to trilinear relations [1–9] leads to a generalized quantum theory called "paraquantum theory" (PQM), characterized by an order parameter Q. In this new theory, the Heisenberg equations which guarantee the wave-particle duality are unchanged. Moreover, it turns out that a larger Hilbert space for the paraquantum observables is needed. In fact, it is convenient to put the Hilbert space in a correspondence with a larger one in which the operators satisfy bilinear and not trilinear relations [1–9].

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For Fock-type irreducible representation, this is done by means of the Green decomposition [1], [10–12]:

$$a_k^\mu = \sum_{\alpha=1}^{Q} a_k^{\mu(\alpha)} , \qquad b_k^\mu = \sum_{\alpha=1}^{Q} b_k^{\mu(\alpha)} , \tag{1.1}$$

($\mu = \overline{1, D}$; D is the space-time dimension), where Q is the order of the paraquantization, α is the Green index and $a_k^{\mu(\alpha)}$ and $b_k^{\mu(\alpha)}$ are the annihilation bosonic and fermionic operators Green components satisfying bilinear but anomalous commutation relations [1], [10–12]:

$$\left[a_k^{\mu(\alpha)}, a_l^{+\nu(\alpha)}\right]_- = \delta_{kl} g^{\mu\nu} , \quad \left[b_k^{\mu(\alpha)}, b_l^{+\nu(\alpha)}\right]_- = \delta_{kl} g^{\mu\nu} , \quad \left[a_k^{\mu(\alpha)}, a_l^{+\nu(\beta)}\right]_+ = 0 , \quad \left[b_k^{\mu(\alpha)}, b_l^{+\nu(\beta)}\right]_+ = 0 , \tag{1.2}$$

such that

$$a_k^{\mu(\alpha)}|0\rangle = 0 , \qquad b_k^{\mu(\alpha)}|0\rangle = 0$$

($|0\rangle$ is the vacuum state).

The purpose of this paper is to show that one can have a consistent parabosonic and super-parabosonic strings for the critical space-time dimensions $D_c = 2 + (24/Q)$ and $2 + (8/Q)$ respectively.

In Section 2, we describe the formalism and in Section 3, we derive the parabosonic string and parasuperstring critical dimensions. Finally, in Section 4, we draw our conclusions.

2. Formalism

The classical actions I_1 and I_2 of free relativistic bosonic string and superstring propagating in a flat space-time, are given by [13–14]:

$$I_1 = -\frac{1}{2\pi\alpha'} \int d\sigma \; d\tau \; \partial_a x^i \; \partial^a x^i , \tag{2.1a}$$

and

$$I_2 = -\frac{1}{2\pi\alpha'} \int d\sigma \; d\tau \; \left[\frac{1}{2}\partial_a x^i \; \partial^a x^i + \frac{1}{2} i \, \bar{\Psi} \gamma^a \; \partial^a \Psi\right] , \tag{2.1b}$$

where τ and σ are dimensionless world-sheet parameters, α' the string tension and $\partial_0 \equiv \partial_\tau$ and $\partial_1 \equiv \partial_\sigma$.

Here, Ψ is a Majorana–Weyl spinor with:

$$\Psi = \begin{bmatrix} \Psi^1 \\ \Psi^2 \end{bmatrix} , \; \bar{\Psi} = \Psi^+ \gamma^0 . \tag{2.2}$$

The general solutions of the equations of motion in the light-cone gauge are [13–14]:

$$x^i = q^i + 2\alpha' p^i \tau + i \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{\alpha_n^i}{n} \exp(-in\tau)\cos(n\sigma)\,, \tag{2.3a}$$

$$\begin{aligned} x^i &= q^i + 2\alpha' p^i \tau \\ &+ \frac{i}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n} \left[\alpha_n^i \exp(-2in(\tau-\sigma)) + \tilde{\alpha}_n^i \exp(-2in(\tau+\sigma))\right], \end{aligned} \tag{2.3b}$$

$$\begin{aligned} \Psi^1 &= \sum_{-\infty}^{+\infty} \Psi_n^1 \exp[-in(\tau-\sigma],\\ \Psi^2 &= \sum_{-\infty}^{+\infty} \Psi_n^2 \exp[-in(\tau+\sigma], \end{aligned} \tag{2.3c}$$

where q^i and p^i are the string center of mass coordinates and momenta respectively.

It is to be noted that the solutions (2.3a), (2.3b), (2.3c) are for the open and closed bosonic strings and open type I superstring respectively.

After quantization, the Mass operators are given by:

a) Open bosonic string:

$$\alpha' M^2 = \frac{1}{4\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} :\alpha_{-n}^i \alpha_n^i: \tag{2.4a}$$

b) Closed bosonic string:

$$\alpha' M^2 = \frac{1}{4\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \left[:\alpha_{-n}^i \alpha_n^i: + :\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i:\right] \tag{2.4b}$$

c) Open superstring:

$$\alpha' M^2 = \frac{1}{\alpha'} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} :\alpha_{-n}^i \alpha_n^i: + \frac{1}{4} \sum_{-\infty}^{+\infty} n :\bar{\Psi}_n\, \gamma^- \Psi_n: \tag{2.4c}$$

where

$$\begin{aligned} \alpha_0^i &= 2\alpha' p^i\,, & \tilde{\alpha}_{-n}^i &= \sqrt{2\alpha' n}\,\tilde{a}_n^{+i}\,,\\ \alpha_{-n}^i &= \sqrt{2\alpha' n}\, a_n^{+i}\,, & \tilde{\alpha}_n^i &= \sqrt{2\alpha' n}\,\tilde{a}_n^i\,,\\ \alpha_{-n}^i &= \sqrt{2\alpha' n}\, a_n^i\,, & \Psi_{-n} &= \Psi_n^+\,, \end{aligned} \tag{2.4d}$$

and

$$\gamma^- = \tfrac{1}{2}\left(\gamma^0 - \gamma^{D-1}\right).$$

It is to be noted that, in this case, the string and superstring dynamical variables $q^i, p^i, q^-, p^+, a_n^i$, $a_n^{+i}, \widetilde{a}_n^i, \widetilde{a}_n^{+i}, \Psi_n^+$ and Ψ_n verify the following non vanishing canonical commutation relations:

$$\begin{aligned} \left[q^i, p^j\right]_- &= i\delta^{ij}, & \left[q^-, p^+\right]_- &= -1, \\ \left[a_n^i, a_m^{+j}\right]_- &= \delta ij \delta_{nm}, & \left[\widetilde{a}_n^i, \widetilde{a}_m^{+j}\right]_- &= \delta ij \delta_{nm}, \\ \left[\Psi_n^a, \Psi_m^b\right]_+ &= \left(\gamma^+ h \gamma^0\right)^{ab} \delta_{n+m,0}, \end{aligned} \tag{2.5a}$$

where

$$\gamma^+ = \tfrac{1}{2}\left(\gamma^0 + \gamma^{D-1}\right) \tag{2.5b}$$

and the helicity projection operator h has as expression

$$h = \tfrac{1}{2}\left(1 + \gamma^{D+1}\right), \tag{2.5c}$$

where

$$\gamma^{D+1} = \gamma^0 \gamma^1 \ldots \gamma^D .$$

Now, for the paraquantization, the commutation relations (2.5a) become:

$$\begin{aligned} &\left[q^i, p^j\right] = i\delta^{ij}, & &\left[q^-, p^+\right] = -1, \\ &\left[a_n^{i(\alpha)}, a_m^{+j(\alpha)}\right]_- = \delta ij \delta_{nm}, & &\left[\widetilde{a}_n^{i(\alpha)}, \widetilde{a}_m^{+j(\alpha)}\right]_- = \delta ij \delta_{nm}, \\ &\left[a_n^{i(\alpha)}, a_m^{+j(\beta)}\right]_+ = 0, & &\left[\widetilde{a}_n^{i(\alpha)}, \widetilde{a}_m^{+j(\beta)}\right]_+ = 0, \ \alpha \neq \beta, \\ &\left[\Psi_n^{a(\alpha)}, \Psi_m^{b(\alpha)}\right]_+ = \left(\gamma^+ h \gamma^0\right)^{ab} \delta_{n+m,0}, & &\left[\Psi_n^{a(\alpha)}, \Psi_m^{b(\beta)}\right]_- = 0, \ \alpha \neq \beta, \end{aligned} \tag{2.6}$$

where we have used the Green ansatz [1], [10–12]:

$$\begin{aligned} q^i &= \sum_{\beta=1}^{Q} q^{i(\beta)}, \quad p^i = \sum_{\beta=1}^{Q} p^{i(\beta)}, \quad a_n = \sum_{\beta=1}^{Q} a_n^{(\beta)}, \quad a_n^+ = \sum_{\beta=1}^{Q} a_n^{+(\beta)}, \\ q^- &= \sum_{\beta=1}^{Q} q^{-(\beta)}, \quad p^+ = \sum_{\beta=1}^{Q} p^{+(\beta)}, \quad \widetilde{a}_n = \sum_{\beta=1}^{Q} \widetilde{a}_n^{(\beta)}, \quad \widetilde{a}_n^+ = \sum_{\beta=1}^{Q} \widetilde{a}_n^{+(\beta)}, \\ \Psi_n^a &= \sum_{\beta=1}^{Q} \Psi_n^{a(\beta)} \end{aligned} \tag{2.7}$$

and the fact that the observables like q^i, p^i, q^-, p^+ which describe the center of mass coordinates and momentum of the string should not be affected by the paraquantization. In fact, this is achieved by choosing a specific direction in the Green para-space, let us say [15–18]:

$$q^{i(\beta)} = q^i \delta_{\beta 1}, \quad q^{-(\beta)} = q^- \delta_{\beta 1}, \quad p^{+(\beta)} = p^+ \delta_{\beta 1}, \quad p^{i(\beta)} = p^i \delta_{\beta 1}.$$

3. The mass shell condition

For the open parabosonic string, the mass operator can be easily constructed. In fact, using the commutation relations (2.6) one can show that:

$$\sum_{\alpha \neq \beta} \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} : \alpha_{-n}^{i(\alpha)} \alpha_n^{i(\beta)} := 0 \tag{3.1}$$

and consequently

$$\alpha' M^2 = \frac{1}{4\alpha'} \sum_{\alpha=1}^{Q} \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} : \alpha_{-n}^{i(\alpha)} \alpha_n^{i(\alpha)} : . \tag{3.2}$$

Regarding the normal ordering ambiguity, one can simplify equation (3.2) by means of the Riemann zeta function regularization and get:

$$\alpha' M^2 = \frac{-Q\,(D-2)}{24} + \frac{1}{2\alpha'} \sum_{\alpha=1}^{Q} \sum_{n=1}^{+\infty} \alpha_{-n}^{i(\alpha)} \alpha_n^{i(\alpha)} . \tag{3.3}$$

Now, let us consider vectorial states of the form $\varepsilon_j \alpha_{-1}^j \left|0\right\rangle$ (ε_j is a polarization vector). To preserve the covariance after the paraquantization, these states should define an irreducible representation of their little group which is the vectorial representation of the SO$(D-2)$ group. This implies that these states have a zero mass *i.e.*:

$$M^2 \varepsilon_j \alpha_{-1}^j \left|0\right\rangle = 0 . \tag{3.4}$$

Using the Green components decomposition as well as the commutation relations (2.6) and equation (3.4) we obtain:

$$\frac{-Q\,(D-2)}{24} \varepsilon_j \alpha_{-1}^j \left|0\right\rangle + \frac{1}{2\alpha'} 2\alpha' \varepsilon_j \alpha_{-1}^j \left|0\right\rangle = 0 , \tag{3.5}$$

where the paraquantum Fock space is defined so that

$$\alpha_n^{i(\alpha)} \left|0\right\rangle = 0 , \qquad n \geq 1 ,$$

and consequently

$$D = 2 + \frac{24}{Q}\,. \tag{3.6}$$

Regarding the closed parabosonic string, one can show easily that the mass operator M^2 is given by:

$$\alpha' M^2 = \frac{-Q\,(D-2)}{12} + \frac{1}{2\alpha'} \sum_{\alpha=1}^{Q} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \left[: \alpha_{-n}^{i(\alpha)} \alpha_{n}^{i(\alpha)} : + : \tilde{\alpha}_{-n}^{i(\alpha)} \tilde{\alpha}_{n}^{i(\alpha)} : \right]. \tag{3.7}$$

Similar simplifications as in the previous case lead to the expression:

$$\alpha' M^2 = \frac{-Q\,(D-2)}{12} + \frac{1}{2\alpha'} \sum_{\alpha=1}^{Q} \sum_{n=1}^{+\infty} \left[: \alpha_{-n}^{i(\alpha)} \alpha_{n}^{i(\alpha)} : + : \tilde{\alpha}_{-n}^{i(\alpha)} \tilde{\alpha}_{n}^{i(\alpha)} : \right]. \tag{3.8}$$

Now, let us take symmetric tensorial states $|i,j\rangle$ of the form $\varepsilon_{ij}\,(\alpha_{-1}^{j}\alpha_{-1}^{i} + \alpha_{-1}^{i}\alpha_{-1}^{j})\,|0\rangle$. In order that these states have a zero mass, one has to have

$$M^2\,|i,j\rangle = 0\,. \tag{3.9}$$

After straightforward manipulations (as in the previous case) equation (3.9) becomes

$$\left[\frac{-Q\,(D-2)}{12} + \frac{1}{2\alpha'}4\alpha'\right] |i,j\rangle = 0 \tag{3.10}$$

and again one gets the result (3.6). Notice that the spectrum of the closed parabosonic string contains a massless spin two particle (the symmetric tensor state $|i,j\rangle$) which can be interpreted as the graviton.

Regarding the type I open para-super-string and using the paraquantum Fierz like rearrangement formula one has

$$\bar{\Psi}_{n}^{(\alpha)}\, \gamma^{-} \Psi_{n}^{(\beta)} = -\, \bar{\Psi}_{n}^{(\beta)}\, \gamma^{-} \Psi_{n}^{(\alpha)} \qquad \text{for} \qquad \alpha \neq \beta\,. \tag{3.11}$$

One can show that

$$\alpha' M^2 = \frac{1}{2\alpha'} \sum_{\beta=1}^{Q} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} : \alpha_{-n}^{i(\beta)} \alpha_{n}^{i(\beta)} : + \frac{1}{4} \sum_{\beta=1}^{Q} \sum_{n=-\infty}^{+\infty} n : \bar{\Psi}_{n}^{(\beta)}\, \gamma^{-} \Psi_{n}^{(\beta)} : \tag{3.12}$$

and

$$\sum_{\alpha\neq\beta}^{Q} \sum_{n=-\infty}^{+\infty} : \alpha_{-n}^{i(\alpha)} \alpha_{n}^{i(\beta)} := \sum_{\alpha\neq\beta}^{Q} \sum_{n=-\infty}^{+\infty} n : \bar{\Psi}_{n}^{(\beta)}\, \gamma^{-} \Psi_{n}^{(\beta)} := 0$$

and because of the algebra of $(\Psi_0^a)^{(\beta)}$, the spectrum in the massless ground state has to contain a vector state $|i\rangle = p^i|0\rangle$ and the spinor partner $|a\rangle$ given by

$$|a\rangle = \sum_{\beta=1}^{Q} |a\rangle^{(\beta)}\,, \tag{3.13}$$

where

$$|a\rangle^{(\beta)} = \frac{i}{8}\,(\gamma_i\Psi_0^a)^{(\beta)}\,|i\rangle$$

which are annihilated by $(\alpha_n^i)^{(\beta)}$ and $(\Psi_0^a)^{(\beta)}$ $(n \geq 1)$.

If $|i\rangle$ represents the $(D-2)$ physical transverse polarizations of the massless vector field, the spinor partner $|a\rangle$, which is represented by equation (3.13), has to have $2^{Q(D-2)/2}$ components (degrees of freedom). Therefore, the conditions

$$\begin{aligned} M^2\,|i\rangle &= 0\,,\\ M^2\,|a\rangle &= 0\,. \end{aligned} \tag{3.14}$$

as well as the definition of the unique paraquantum Fock space vacuum

$$\begin{aligned} \alpha^{i(\beta)}\,|0\rangle &= 0\,,\\ \Psi_n^{a(\beta)}\,|0\rangle &= 0\,, \quad n \geq 1\,, \end{aligned} \tag{3.15}$$

lead to

$$\left[Q\frac{(D-2)}{2} - \frac{1}{4}Tr\left(\gamma^0\gamma^-\gamma^+h\gamma^-\right)\right]\sum_{n=1}^{+\infty} n\,|i\rangle = 0\,. \tag{3.16}$$

After direct simplifications and using the fact that

$$[\gamma^+,\gamma^-]_+ = 2$$

and

$$\gamma^{0^2} = 1\,, \quad Tr\gamma^+\gamma^-\gamma^{D+1} = 0\,, \quad Tr1 = 2^{Q(D-2)/2} \tag{3.17}$$

we obtain

$$Q\,(D-2) = \tfrac{1}{2}2^{Q(D-2)/2}\,. \tag{3.18}$$

Equation (3.18) has a solution if and only if

$$D = 2 + \frac{8}{Q}\,. \tag{3.19}$$

4. Conclusion

Throughout this work, we have derived a new critical space-time dimension for bosonic string as well as superstring in a general framework of the quantization procedure.

Thus one can have a paraquantum consistent bosonic strings and superstrings at $D = 26, 14, 10, 4$ and $D =, 10, 6, 4, 2$ respectively (which is not the case in ordinary strings). This means that one may have $D = 4$ for the bosonic strings and superstrings which corresponds to $Q = 12$ and 4 respectively without having recourse to the compactification procedure. Moreover, with these new critical space-time dimensions, one may have interesting phenomenological implications (more details are under investigations) [19].

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