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# REMARKS ON TREE-LEVEL TOPOLOGICAL STRING THEORIES

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A few observations concerning topological string theories at the string-tree level are presented: (1) The tree-level, large phase space solution of an arbitrary model is expressed in terms of a variational problem, with an "action" equal, at the solution, to the one-point function of the puncture operator, and found by solving equations of Gauss-Manin type; (2) For $A_k$ Landau-Ginzburg models, an extension to large phase space of the usual residue formula for three-point functions is given.

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## 1. Introduction

Topological string theory [1] has been successful in providing a rather simple and convenient language to describe the minimal models of noncritical string theory [2, 3]. There has also been some progress in attempts to extend this language to noncritical strings in two-dimensional target space, though this has proved to be not quite straightforward [4–9]. The topological description is especially efficient in reproducing the string tree level (genus zero) approximation to noncritical string amplitudes.

A number of open questions remain, however, concerning the relevance of the topological formulation to more physically interesting string theory models. On one hand, it seems rather straightforward to couple any topological matter theory, as originally defined by Dijkgraaf, Verlinde and Verlinde (see [10] for an early review), to topological gravity — at least in the genus zero approximation, but what remains unclear is whether consistency of such a construction can in general be preserved at higher genera, or whether requiring this imposes additional constraints on the topological matter theory, beyond those laid down in [10]. In other words, we lack a well defined and universal prescription to compute loop corrections to topological string theory amplitudes from the data defining the theory at the level of the spherical approximation, and thus we cannot test the consistency and uniqueness of string loop corrections for a generic model.

A second open question concerns the generalization of the topological string theory formalism to models with an infinite number of topological primary fields (of which $c = 1$ strings appear to provide an example), and to theories with (physical) fermionic degrees of freedom.

The former question was raised in [12], and the interesting example of a topological string theory based on the $CP^1$ topological matter theory was worked out in [13] (see also [18, 19] for further developments). As for the latter question, existing studies of the $c = 1$ theories do not yet seem to point to any general approach to the issue of models with infinite primary fields, while the topological formulation of models with fermionic degrees of freedom has not been much studied; perhaps further work using the results of [14–16] will provide clues in this direction.

In the present note, I first derive a few simple properties of tree-level topological string theory that follow rather directly from its basic axioms, and will therefore hold for the coupling of any topological matter model (with a finite number of topological primary fields) to topological gravity. This is a reformulation and extension of the discussion in Section 2 of [12]: namely, the tree-level, large phase space solution of an arbitrary topological string theory may be expressed in terms of a variational problem, with an "action" (Eq. (17)) equal, at the solution, to the one-point function of the puncture operator, and found by solving differential equations of the Gauss-Manin type (Eq. (24)). It should be stressed that the stated property holds independently of whether or not a topological string theory admits a Landau-Ginzburg realization; this fact has not been clearly stated in the literature, and hopefully, it may be of some use in answering the open questions discussed above. In the last section, I derive a simple generalization (Eq. (46)) of the usual residue formula for three-point functions of topological primaries in $A_k$ Landau-Ginzburg matter theories, that is valid for general three-point correlators on large phase space, and that follows straightforwardly from the integrable (Gelfand-Dikii) structure of these theories, or — more precisely — from its genus-zero (nondispersive) limit.

The common theme of the observations presented herein is that working in terms of the "order parameter" variables $u_\alpha$, defined as two-point functions of a primary field and the puncture operator, makes it explicit that the large phase space solution (incorporating couplings to all gravitational descendants) of a topological string theory in the genus-zero approximation is fully determined from the structure of its underlying topological matter system — the small phase space.

## 2. The action for the string equations

For the purposes of the present discussion, a tree-level topological string theory is defined by the free energy of the underlying topological matter system, *i.e.* a function $\mathcal{F}$ of the couplings $t_{0,\alpha}$ to the topological primary fields $\Phi_\alpha$, $\alpha = 0, \ldots, k$, whose derivatives with respect to the couplings provide the correlators of primary fields on the "small phase space" of the theory. $\mathcal{F}$ is required to satisfy the constraints that

$$\langle P\Phi_\alpha\Phi_\beta\rangle \equiv \frac{\partial^3\mathcal{F}}{\partial t_{0,0}\partial t_{0,\alpha}\partial t_{0,\beta}} = \eta_{\alpha\beta}, \tag{1}$$

with $\eta_{\alpha\beta}$ a nondegenerate symmetric tensor independent of the couplings (the topological metric), and that

$$c^\gamma_{\alpha\beta} \equiv \langle\Phi_\alpha\Phi_\beta\Phi_\mu\rangle\eta^{\mu\gamma}, \tag{2}$$

(where $\eta^{\alpha\beta}$ is the inverse of $\eta_{\alpha\beta}$) provide, for all values of the primary field couplings, a set of structure constants for an associative, commutative algebra. That is, one may define a law of multiplication of primary fields, by

$$\Phi_\alpha \cdot \Phi_\beta = c^\gamma_{\alpha\beta}\Phi_\gamma, \tag{3}$$

such that

$$\Phi_\alpha \cdot \Phi_\beta = \Phi_\beta \cdot \Phi_\alpha \tag{4}$$

and

$$(\Phi_\alpha \cdot \Phi_\beta) \cdot \Phi_\gamma = \Phi_\alpha \cdot (\Phi_\beta \cdot \Phi_\gamma), \tag{5}$$

and a nondegenerate symmetric scalar product,

$$(\Phi_\alpha, \Phi_\beta) = \eta_{\alpha\beta}, \tag{6}$$

obeying

$$(\Phi_\alpha, \Phi_\beta \cdot \Phi_\gamma) = (\Phi_\alpha \cdot \Phi_\beta, \Phi_\gamma). \tag{7}$$

The *puncture operator* $P \equiv \Phi_0$ is seen to be the unit element of this primary field algebra.

Given a topological matter system, *i.e.* a solution $\mathcal{F}$ to the above constraints, the full topological string theory (at string-tree level) is understood to be given by the extension of $\mathcal{F}$ to a function on the "large phase space" of all (primary and descendant) couplings $t_{n,\alpha}$ $(n = 0, 1, \ldots\infty)$, that reduces to the small phase space free energy for $t_{n,\alpha} = 0$ $(n > 0)$, and is determined by Witten's tree-level factorization relation:

$$\langle\sigma_{n+1}(\Phi_\alpha)XY\rangle = \langle\sigma_n(\Phi_\alpha)\Phi_\beta\rangle\langle\Phi^\beta XY\rangle, \tag{8}$$

for arbitrary (primary or descendant) fields $X$, $Y$ (and $\sigma_0(\Phi_\alpha) \equiv \Phi_\alpha$), together with the puncture equation

$$\langle P\rangle = \frac{1}{2}\sum_{\alpha,\beta} t_{0,\alpha}t_{0,\beta}\eta_{\alpha\beta} + \sum_{n=0}^{\infty} t_{n+1,\alpha}\langle\sigma_n(\Phi_\alpha)\rangle. \tag{9}$$

It has been pointed out in [2] that, as a consequence of the above properties, all two-point functions, as functions on large phase space, depend on the couplings $t_{n,\alpha}$ only through the $k+1$ variables

$$u_\alpha = \langle P\Phi_\alpha\rangle. \tag{10}$$

In other words, the expressions of arbitrary two-point functions in terms of $u^\alpha$ are a universal characteristic of a given model, *i.e.* do not contain the couplings explicitly, and are known as *constitutive relations.* Of special interest are the constitutive relations

$$\langle\Phi_\alpha\sigma_n(\Phi_\beta)\rangle = R_{\alpha;n,\beta}(u). \tag{11}$$

The form of these constitutive relations can be rather straightforwardly determined from the topological matter system, by using (derivatives of) the puncture equation (9), and the factorization equation (8), restricted to small phase space. Inserting the relations (11) into the equations obtained by taking the first derivatives of (9) with respect to the primary couplings, one obtains

$$u_\alpha = \sum_\beta \eta_{\alpha\beta}t_{0,\beta} + \sum_\beta\sum_{n=0}^{\infty} t_{n+1,\beta}R_{\alpha;n,\beta}(u), \tag{12}$$

a set of $k+1$ equations for the $k+1$ unknowns $u_\alpha$ in terms of the couplings $t_{n,\alpha}$, called the "generalized Landau-Ginzburg equations" in [2]. These determine the full large phase space solution of the topological string theory at genus zero.

Let us now consider the expression

$$-\frac{1}{2}\eta^{\alpha\beta}u_\alpha u_\beta + \sum_\alpha\sum_{n=0}^{\infty} t_{n,\alpha}\langle P\sigma_n(\Phi_\alpha)\rangle, \tag{13}$$

where the two-point functions are understood as functions of $u_\alpha$ given by the constitutive relations.

On small phase space,

$$\frac{\partial}{\partial t_{0,\beta}}\langle P\sigma_n(\Phi_\alpha)\rangle = \langle P\Phi_\beta\sigma_n(\Phi_\alpha)\rangle = \langle\Phi_\beta\sigma_{n-1}(\Phi_\alpha)\rangle \tag{14}$$

for $n \neq 0$, where I used the small phase space form of the puncture equation. But, being a relation between two-point functions, the above extends to large phase space via the constitutive relations, as

$$\frac{\partial}{\partial u_\beta}\langle P\sigma_n(\Phi_\alpha)\rangle = \eta^{\beta\gamma}\langle\Phi_\gamma\sigma_{n-1}(\Phi_\alpha)\rangle, \tag{15}$$

while, for $n = 0$,

$$\frac{\partial}{\partial u_\beta}\langle P\Phi_\alpha\rangle = \delta_\alpha^\beta. \tag{16}$$

Therefore, the $u$-derivatives (at constant $t$'s) of

$$\langle P\rangle = -\frac{1}{2}\eta^{\alpha\beta}u_\alpha u_\beta + \sum_\alpha\sum_{n=0}^{\infty} t_{n,\alpha}\langle P\sigma_n(\Phi_\alpha)\rangle(u)\,. \tag{17}$$

yield the generalized Landau-Ginzburg equations (12). Furthermore, we can indeed show that the expression (17) is equal (up to an integration constant independent of all the couplings) to $\langle P\rangle$: this follows by observing that its total derivative with respect to any coupling $t_{n,\alpha}$ equals $\langle P\sigma_n(\Phi_\alpha)\rangle$ at the solutions of the string equations (12).

At this point, it is useful to observe that

$$\frac{\partial}{\partial u_\alpha}\langle P\sigma_1(P)\rangle = \eta^{\alpha\beta}u_\beta, \tag{18}$$

hence the first, quadratic term in (17) may be absorbed into a shift of the coupling $t_{1,0} \mapsto t_{1,0} - 1$ (so that $t_{1,0} = -1$ on small phase space).

In fact, a more general statement may be proven: namely, up to an additive constant,

$$\langle X\rangle = \sum_\alpha\sum_{n=0}^{\infty} t_{n,\alpha}\langle X\sigma_n(\Phi_\alpha)\rangle, \tag{19}$$

for any operator $X$, primary or descendant. To see this, take the derivative of the RHS of (19) with respect to any of the couplings $t_{n,\alpha}$, and use (8) to show that the contribution of terms where the derivative acts on the two-point functions is proportional to the Landau-Ginzburg equations (12). Thus the derivatives of both sides of Eq. (19) with respect to any of the couplings are equal, up to terms that vanish by the string equations, proving our claim.

It should also be noted that Eq. (19) is nothing else than a derivative of the (tree-level) *dilaton equation*,

$$\sum_\alpha\sum_{n=0}^{\infty} t_{n,\alpha}\langle\sigma_n(\Phi_\alpha)\rangle = 2\mathcal{F}, \tag{20}$$

which was not itself assumed in the above.

We have thus seen that the tree level solution of an arbitrary topological string theory satisfying our axioms is determined from a variational problem, by the extremum of the "action" given by Eq. (17) with respect to the "order parameters" $u^\alpha$. The form of Eq. (17) for a given model is determined by its constitutive relations, and those can be worked out from the small phase space solution, *i.e.* from the underlying topological matter theory. More explicitly, the small phase space form of the puncture equation,

$$\langle PP\sigma_{n+1}(\Phi_\alpha)\rangle = \langle P\sigma_n(\Phi_\alpha)\rangle \tag{21}$$

leads to the following recursion for the constitutive relations:

$$\frac{\partial}{\partial u^0}\langle P\sigma_{n+1}(\Phi_\alpha)\rangle = \langle P\sigma_n(\Phi_\alpha)\rangle, \tag{22}$$

while the small phase space equality

$$\begin{aligned} \langle P\sigma_{n+2}(\Phi_\alpha)\Phi_\beta\Phi_\gamma\rangle &= \langle \sigma_{n+1}(\Phi_\alpha)\Phi_\beta\Phi_\gamma\rangle \\ = \langle \sigma_n(\Phi_\alpha)\Phi_\delta\rangle\langle\Phi^\delta\Phi_\beta\Phi_\gamma\rangle &= \langle PP\sigma_{n+2}(\Phi_\alpha)\Phi_\delta\rangle\langle\Phi^\delta\Phi_\beta\Phi_\gamma\rangle \end{aligned}$$

may be rewritten as

$$\frac{\partial}{\partial t_{0,\beta}}\frac{\partial}{\partial t_{0,\gamma}}\langle P\sigma_{n+2}(\Phi_\alpha)\rangle = c^\delta_{\beta\gamma}(t)\frac{\partial}{\partial t_{0,\delta}}\frac{\partial}{\partial t_{0,0}}\langle P\sigma_{n+2}(\Phi_\alpha)\rangle, \tag{23}$$

which leads to a constraint on constitutive relations, valid on large phase space, when one substitutes $t_{0,\alpha} \mapsto u^\alpha$:

$$\frac{\partial}{\partial u^\beta}\frac{\partial}{\partial u^\gamma}\langle P\sigma_n(\Phi_\alpha)\rangle = c^\delta_{\beta\gamma}(u)\frac{\partial}{\partial u^\delta}\frac{\partial}{\partial u^0}\langle P\sigma_n(\Phi_\alpha)\rangle, \tag{24}$$

(it is easily verified by hand that this equation is valid also for $n = 0,\ 1$).

Equations (22) and (24), together with the initial conditions

$$\langle P\sigma_0(\Phi_\alpha)\rangle \equiv \langle P\Phi_\alpha\rangle = u_\alpha \tag{25}$$

determine all the constitutive relations in question, *i.e.* determine the dependence of the "action" of Eq. (17) on $u_\alpha$, up to additional arguments that must be invoked to fix integration constants in (22). The equations (24) generalize the Gauss-Manin equations, known from the restricted context of Landau-Ginzburg topological matter theories [11]; here, they are seen to hold for arbitrary topological matter theories.

## 3. Large phase space residue formula for $A_k$ models

To fix notations, I begin by recalling the formulas that state the defining properties of $A_k$ topological strings [10]. The algebra of primary fields is given in terms of the superpotential $W(X)$,

$$W(X) = \frac{1}{k+2}X^{k+2} + \sum_{i=0}^{k} g_i(t)X^i, \tag{26}$$

where $t = \{t_{0,\alpha}\}$, $(\alpha = 0, \ldots, k)$ are the couplings to the primary fields, as the algebra of polynomials in $X$ modulo the relation

$$\frac{dW}{dX} = 0. \tag{27}$$

The topological metric is provided by the formula

$$\eta_{\alpha\beta} = \mathrm{Res}_X \left[\frac{\Phi_\alpha(X)\Phi_\beta(X)}{W'(X)}\right], \tag{28}$$

where $\mathrm{Res}_X$ denotes the residue at infinity in $X$, *i.e.* the coefficient of the $X^{-1}$ term in a Laurent series expansion for large $X$. A preferred basis of primary fields is determined by requiring $\eta_{\alpha\beta}$ to be constant, and

$$\Phi_\alpha(X) = X^\alpha + \mathcal{O}(X^{\alpha-2}). \tag{29}$$

The $\Phi_\alpha$ are then given by the ($t$-dependent) polynomials

$$\Phi_\alpha(X) = \frac{1}{\alpha+1}\frac{d}{dX}[L^{\alpha+1}(X)]_+, \tag{30}$$

where $L(X) = [(k+2)W(X)]^{\frac{1}{k+2}}$, $L(X) = X + \mathcal{O}(X^{-1})$, and $[\cdots]_+$ denotes the polynomial part of the large-$X$ Laurent series expansion of the expression inside the brackets. The relation between the coefficients $g_i$ in the superpotential and the primary couplings $t_{0,\alpha}$ is determined by the equations

$$\frac{\partial W}{\partial t_\alpha} = \Phi_\alpha. \tag{31}$$

Further on, the structure constants of the primary field algebra — and equivalently, the small phase space three-point functions of the primary fields, are given by the well-known residue formula

$$\langle\Phi_\alpha\Phi_\beta\Phi_\gamma\rangle = \eta_{\gamma\mu}c^\mu_{\alpha\beta} = \mathrm{Res}_X\left(\frac{\Phi_\alpha\Phi_\beta\Phi_\gamma}{W'}\right). \tag{32}$$

Finally, the solution to Eq. (31) may be written as

$$\eta_{\alpha\beta}t_\beta = t_{k-\alpha} = \frac{1}{\alpha+1}\,\mathrm{Res}_X(L^{\alpha+1}), \tag{33}$$

completing the brief review of the well-known solution of the $A_k$ topological matter theories at genus zero.

To extend the description of these theories to correlators of descendant fields and large phase space, observe first that, since on small phase space

$$t_{k-\alpha} = \eta_{\alpha\beta} = \langle P\Phi_\alpha\rangle, \tag{34}$$

we may write a large phase space version of (33) as

$$u_\alpha = \langle P\Phi_\alpha\rangle = \frac{1}{\alpha+1}\,\mathrm{Res}_X(L^{\alpha+1}). \tag{35}$$

Effectively, we are extending the notion of superpotential to large phase space, by defining it to be the same function as on small phase space, but with the replacement $t_{0,\alpha} \mapsto u^\alpha$. The well-known small phase space formulas that express various correlators in terms of $W$ (or $L$) are then easily converted into constitutive relations, provided we take care to use formulas that involve two-point functions only. And thus, two-point functions of primary fields are given by [10]

$$\langle\Phi_\alpha\Phi_\beta\rangle = \frac{1}{\alpha+1}\,\mathrm{Res}_X(L^{\alpha+1}\Phi_\beta), \tag{36}$$

with $\Phi_\beta$ given in terms of $L$ by Eq. (30), and the basic constitutive relations involving descendants read [11]

$$\langle P\sigma_n(\Phi_\alpha)\rangle = \frac{\mathrm{Res}_X(L^{\alpha+1+n(k+2)})}{\prod_{i=0}^{n}(\alpha+1+i(k+2))}. \tag{37}$$

It is now easy to write a large phase space residue formula for three-point functions involving a single descendant field insertion. Since two-point functions depend on all couplings only through the $u^\alpha$, we have

$$\langle\Phi_\alpha\Phi_\beta\sigma_n(\Phi_\gamma)\rangle = \frac{\partial}{\partial t_{n,\gamma}}\langle\Phi_\alpha\Phi_\beta\rangle = \frac{\partial u^\mu}{\partial t_{n,\gamma}}\frac{\partial}{\partial u^\mu}\langle\Phi_\alpha\Phi_\beta\rangle, \tag{38}$$

and therefore

$$\langle\Phi_\alpha\Phi_\beta\sigma_n(\Phi_\gamma)\rangle = \frac{\partial u^\mu}{\partial t_{n,\gamma}}\,\mathrm{Res}_X\left(\frac{\Phi_\alpha\Phi_\beta\Phi_\mu}{W'}\right). \tag{39}$$

On the other hand, as the superpotential $W$ is also a function only of the $u^\alpha$, we have

$$\frac{\partial W}{\partial t_{n,\gamma}} = \frac{\partial u^\mu}{\partial t_{n,\gamma}}\frac{\partial W}{\partial u^\mu} = \frac{\partial u^\mu}{\partial t_{n,\gamma}}\Phi_\mu(X). \tag{40}$$

By combining the two above equations we obtain

$$\langle \Phi_\alpha \Phi_\beta \sigma_n(\Phi_\gamma)\rangle = \mathrm{Res}_X\left(\frac{\Phi_\alpha\Phi_\beta \frac{\partial W}{\partial t_{n,\gamma}}}{W'}\right), \tag{41}$$

the simplest instance of the claimed large phase space residue formula.

To further generalize this result, we recall here that the dependence of $W$ on the (primary and descendant) couplings may be described by an infinite family of commuting Hamiltonian flows:

$$\frac{\partial W}{\partial t_{n,\alpha}} = \{H_{n,\alpha}, W\}, \tag{42}$$

with a Poisson bracket given by

$$\{F,G\} = \frac{\partial F}{\partial X}\frac{\partial G}{\partial t_{0,0}} - \frac{\partial F}{\partial t_{0,0}}\frac{\partial G}{\partial X}, \tag{43}$$

and the Hamiltonians

$$H_{n,\alpha} = \frac{[L^{\alpha+1+n(k+2)}]_+}{\prod_{i=0}^n(\alpha+1+i(k+2))}; \tag{44}$$

in our notation, $\Phi_\alpha(X) = \partial_X H_{0,\alpha}$. The above is simply the genus-zero limit of the Gelfand-Dikii integrable hierarchy structure of these models that holds to all genera [17]. Now, since the second term of the Poisson bracket Eq. (42) is a polynomial in $X$ times $\frac{\partial W}{\partial X}$, the residue formula (41) may be written as

$$\langle \Phi_\alpha \Phi_\beta \sigma_n(\Phi_\gamma)\rangle = \mathrm{Res}_X\left(\frac{H'_{0,\alpha}H'_{0,\beta}H'_{n,\gamma}}{W'}\frac{\partial W}{\partial t_{0,0}}\right). \tag{45}$$

The claim is that for general three-point functions on large phase space, this formula generalizes to

$$\langle \sigma_l(\Phi_\alpha)\sigma_m(\Phi_\beta)\sigma_n(\Phi_\gamma)\rangle = \mathrm{Res}_X\left(\frac{H'_{l,\alpha}H'_{m,\beta}H'_{n,\gamma}}{W'}\frac{\partial W}{\partial t_{0,0}}\right). \tag{46}$$

This formula may be verified by using twice Witten's factorization equation (8) to reduce the general three-point function to one with a single descendant field, and exploiting the identity

$$\frac{\partial W}{\partial t_{n,\alpha}} = \langle \sigma_{n-1}(\Phi_\alpha)\Phi^\mu \rangle \frac{\partial W}{\partial t_{0,\mu}}. \tag{47}$$

Equation (47) is itself a consequence of the validity of constitutive relations, and of the puncture equation: namely,

$$\frac{\partial W}{\partial t_{n,\alpha}} = \langle P\Phi_\mu \sigma_n(\Phi_\alpha)\rangle \Phi^\mu(X); \tag{48}$$

reexpressing the three-point function on the RHS as a derivative of a two-point function, and using the small phase space form of the puncture equation, the above equals

$$\frac{\partial u^\nu}{\partial t_{0,\mu}} \left( \frac{\partial}{\partial u^\nu} \langle P\sigma_n(\Phi_\alpha)\rangle \right) \Phi^\mu(X) = \langle P\Phi^\nu \Phi_\mu \rangle \langle \sigma_{n-1}(\Phi_\alpha)\Phi_\nu \rangle \Phi^\mu(X) \, . \tag{49}$$

Finally, the first and last factor on the LHS above combine to $\frac{\partial W}{\partial t_{0,\nu}}$, completing the proof of Eq. (47). It is now straightforward to demonstrate the validity of Eq. (46), as outlined above.

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