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SINGULARITIES OF THE S-MATRIX FOR A COMPLEX SQUARE WELL POTENTIAL* **

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Trajectories of S-matrix poles in complex k plane are presented for a complex square well potential. The conformal character of the connection between the potential and the location of the poles is used to deduce the properties of the trajectories.

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1. Introduction

Whereas complex optical potentials $\mathcal{V} = V + iW$ have been widely used in nuclear physics, the analytical properties of the S-matrix for these potentials were investigated in only few papers (see [1–5]), mainly restricted to the s wave or states with positive energy.

One problem which should be clarified is the question which poles of the S-matrix in the complex k plane move with increasing absorptive potential W into the part of the second quadrant of this plane, below its bisector. The problem appears relevant in the theory of Σ hypernuclear states. Stępień-Rudzka and Wycech [6] noticed that in that part of the k plane the Σ single particle Hamiltonian with a complex potential has eigenvalues which may explain the observed Σ states with positive energy. These unstable bound states (UBS's) and the corresponding poles of the S-matrix have been discussed in detail by Gal, Toker, and Alexander [7]. In the discussion of the Σ hypernuclear states Gal *et al.* [7] and also Oset *et al.* [8] made the statement that with increasing strength of the absorptive potential W the virtual poles (located on the negative imaginary axis for $W = 0$) move

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clockwise into the third quadrant of the complex k plane. This statement, which in general is not correct, has been repeated recently by Bonetti *et al.* [9] in their review of the multistep direct reaction theory. A similar incorrectness may be found in the discussion of Badalyan *et al.* [10] of the S-matrix for coupled channels.

Second problem which deserves attention is the motion of the poles of the S-matrix in the case when the strength of W tends to infinity. The statement made by Faxedas and Sesma [5] that in this case all poles move to infinity applies only to the s wave. Consequently, in a complete discussion of the trajectories of the S-matrix poles in the case of very strong W, one has to consider partial waves higher then the s wave.

In the present paper, trajectories of S-matrix poles in complex k plane are investigated for the nonrelativistic Schrödinger equation with a complex potential $\mathcal{V}$ of finite range. Special attention is paid to the two problems mentioned above. For $\mathcal{V}$ we assume the square well shape for which one has an explicit expression for the S-matrix. The present paper is an extension of [11] to the case of an arbitrarily strong absorption W.

2. Properties of the pole trajectories

Let us consider a particle of mass m, whose motion obeys the nonrelativistic Schrödinger equation with the complex square well potential

$$\mathcal{V}(r) = -(V_0 + iW_0)\theta(R - r)\,. \tag{1}$$

The strength of $\mathcal{V}$ will be measured by dimensionless parameters

$$v = (2m/\hbar^2)R^2V_0, \;\; w = (2m/\hbar^2)R^2W_0\,. \tag{2}$$

For the S-matrix in the state with the angular momentum l and wave vector k, one has

$$S_l(k) = -\frac{\beta h_l^{(2)}(\beta)' j_l(\alpha) - \alpha j_l(\alpha)' h_l^{(2)}(\beta)}{\beta h_l^{(1)}(\beta)' j_l(\alpha) - \alpha j_l(\alpha)' h_l^{(1)}(\beta)}\,, \tag{3}$$

where $j_l, h_l^{(1)}$ and $h_l^{(2)}$ are the spherical Bessel, and Hankel functions of the first and second kind, and primes denote derivatives with respect to the argument of the functions. The dimensionless wave numbers outside and inside of the potential are denoted by β and α:

$$\beta = kR, \quad \alpha = \sqrt{\beta^2 + \gamma}\,, \tag{4}$$

where $\gamma = v + iw$.

Poles of the S-matrix in the complex k plane are determined by the condition that the denominator of expression (3) vanishes, which may be written in the form:

$$\alpha j_l(\alpha)'/j_l(\alpha) = \beta h_l^{(1)}(\beta)'/h_l^{(1)}(\beta)\,. \tag{5}$$

Condition (5) determines the location of the poles in the complex k plane as a function of γ,

$$\beta = F_l(\gamma)\,. \tag{6}$$

By varying $\gamma = v + iw$ in definite ways, we get various trajectories of the S-matrix poles in the complex $\beta = kR$ plane.

2.1. Conformal mapping of γ onto β

In the region of γ where the function $F_l(\gamma)$ is analytic, it represents a conformal transformation of γ into β. The angle preserving property of this transformation leads us immediately to the simple rule:

If we move on a $w = const$ pole trajectory in the direction of increasing v, then to swtch to a $v = cost$ trajectory in the direction of increasing w we have to make a $90°$ turn left.

In discussing the pole trajectories, we find it necessary to know the "critical points" at which $F_l(z)$ is singular and transformation (6) is not conformal. At these points $dF_l/d\gamma \to \infty$, and for the inverse transformation $\gamma = f_l(\beta)$ one has $df_l/d\beta = 0$. To find these critical points, at which $dF_l/d\gamma = d\beta/d\gamma \to \infty$, we use condition (5) and the known properties of the spherical Bessel and Hankel functions. For the crtical values of $\alpha = \alpha_c$ and $\beta = \beta_c$ (the corresponding value of γ is $\gamma_c = \alpha_c^2 - \beta_c^2$), we get the following result:

Let us denote by $x_{l,\nu}$ the ν-th zero (not counting $x = 0$) of j_l (all the $x_{l,\nu}$'s are real), and by y_l zeros of $h_l^{(1)}$ (there are l complex y_l's [12]). The critical points of transformation (6) are:

$$\text{for } l \ = \ 0: \quad \beta_c = y_1, \quad \alpha_c = x_{1,\nu}\,. \tag{7}$$

$$\text{for } l \ = \ 1: \quad \beta_c = \begin{cases} y_2\,, \\ 0\,, \end{cases} \quad \alpha_c = \begin{cases} x_{2,\nu}\,, \\ x_{0,\nu}\,, \end{cases} \tag{8}$$

$$\text{for } l > 1: \quad \beta_c \ = \ \begin{cases} y_{l-1}\,, \\ y_{l+1}\,, \\ 0\,, \end{cases} \quad \alpha_c = \begin{cases} x_{l-1,\nu}\,, \\ x_{l+1,\nu}\,, \\ x_{l-1,\nu}\,. \end{cases} \tag{9}$$

2.2. The limit of $w \to \infty$

We consider trajectories along which the poles move when v is kept constant, and w is varied from 0 to ∞. The location of the starting points $w = 0$ of these trajectories, *i.e.*, the location of the poles for real potential with the depth v, is well known from the early work of Nussenzweig [13]. For the end points of the trajectories, *i.e.*, in the limit $w \to \infty$, there are two possibilities: 1. $\beta \to \infty$, or 2. $\beta \to \beta_F$, where β_F is finite. We shall see, that the separation of the two types of trajectories is connected with some of the critical points discussed above. To get approximate expressions for β when $w \to \infty$, one starts from condition (5) and uses the asymptotic form of the Hankel functions. In this way one gets the following results.

2.2.1. The case of $\lim_{w\to\infty} \beta = \infty$

Here, one gets the following approximate expressions for β and α valid for large values of w:

$$\beta \simeq \pm(-1+i)\sqrt{w/2}\left[1 + i\frac{(x_{l,\nu})^2 - v}{2w}\right], \quad \alpha \simeq x_{l,\nu}\left[1 \pm \frac{(-1+i)}{\sqrt{2w}}\right]. \tag{10}$$

2.2.2. The case of $\lim_{w\to\infty} \beta = \beta_F$

Here, one gets

$$\beta_F = y_l \, . \tag{11}$$

The Hankel function $h_l^{(1)}(y)$ has l complex zeros y_l, thus $h_0^{(1)}$ has no zeros at all. Consequently, in the s state ($l = 0$) for $w \to \infty$ all the trajectories move to infinity. On the other hand, for $l > 0$ some of the trajectories move to infinity, and some of them converge to the points $\beta_F = y_l$.

3. Results and conclusions

The results for the $1s$, $2s$, $1p$, and $2p$ states are shown in figures 1–4.

In the case of the $1s$ state there are no critical points, and in the case of the $2s$ state there is one critical point, $\beta = -i$. In the case of the p states there are three critical points at $\beta_c = \sqrt{3}/2 - i3/2$, $-\sqrt{3}/2 - i3/2$, and 0. The second critical points lies in the third quadrant in which there are no trajectories for absorptive potentials ($w > 0$) considered here. The first critical point is indicated in figures 3 and 4 by an open square. The critical trajectories (indicated by broken lines), which go through the critical point, saparete the trajectories which tend to $y_1 = -i$ when $w \to \infty$ from those which tend to ∞ when $w \to \infty$.

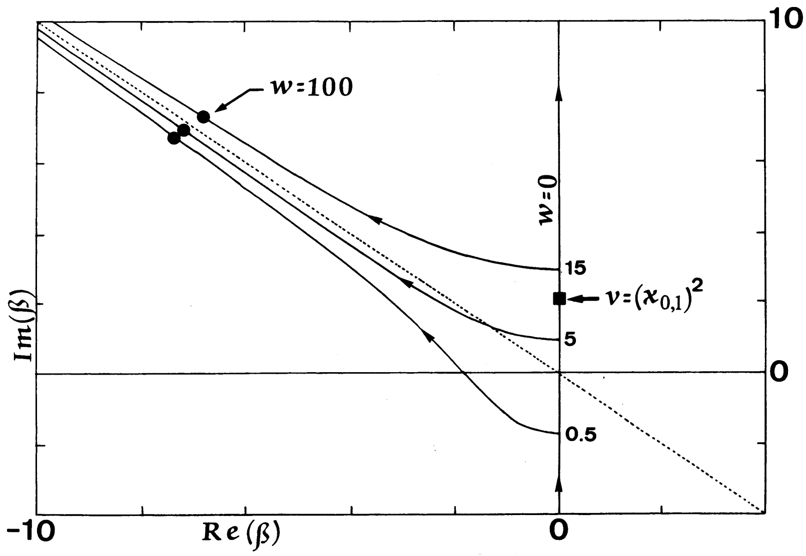

Fig. 1. Trajectories of the poles of the S-matrix for the $1s$ state. Arrows on the $w = 0$ trajectory indicate the direction of increasing v, and on the $v =$ *constant* trajectories the direction of increasing w. Numbers at the starting points of the $v =$ *constant* trajectories are the corresponding values of v. The straight dotted line is the bisector.

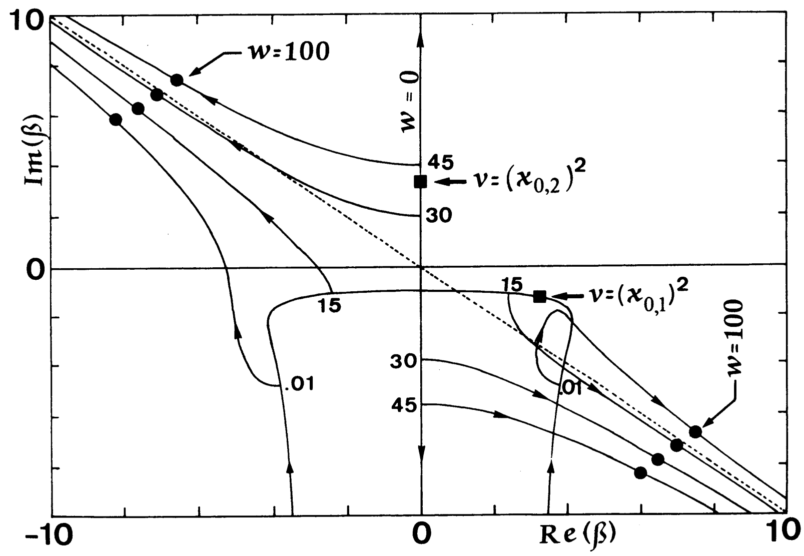

Fig. 2. As figure 1 but for the $2s$ state.

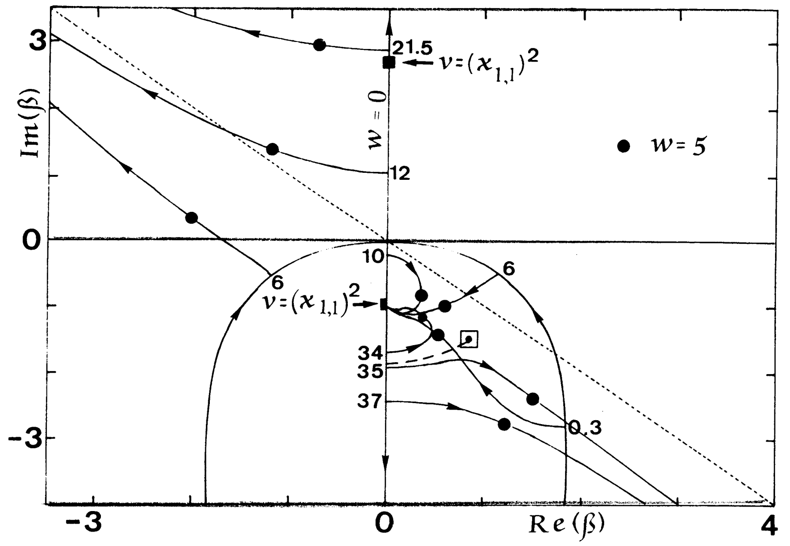

Fig. 3. As figure 1 but for the $1p$ state. The open square is the critical point, and the broken line is the critical trajectory.

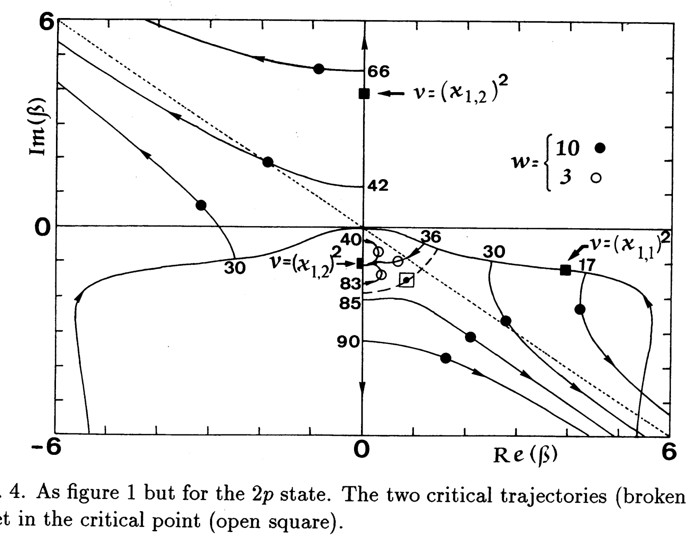

Fig. 4. As figure 1 but for the $2p$ state. The two critical trajectories (broken lines) meet in the critical point (open square).

Fig. 4. As figure 1 but for the $2p$ state. The two critical trajectories (broken lines) meet in the critical point (open square).

When $w \to \infty$ the $v = constant$ trajectories, which at $w = 0$ start from the bound or twin resonance states, approach the bisector of the second quadrant. The trajectories for the nl states with $v < (x_{l,n})^2$ approach the bisector from below. This means that the nl unstable bound states (UBS's), *i.e.*, states in the part of the second quadrant of the complex k plane below its bisector, may be reached (by increasing w) starting from and only from bound states with $v < (x_{l,n})^2$ or from the twin resonance states. An exception is the $1s$ state, for which all virtual states , *i.e.*, states on the negative imaginary axis, are good starting points (there are no resonances in the $1s$ state). To be quite precise, one should add that for the ns states with $n \geq 1$ the small part of the negative imaginary axis above -1 also presents good starting points.

All $v = constant$ trajectories of the S_l poles, which start from the resonance and virtual states, remain in the fourth quadrant (except of course for the $1s$ state trajectories, and also those ns trajectories which start from the negative imaginary axis above -1). When $w \to \infty$, they either escape to infinity, or they tend to a finite value of $\beta = \beta_F = y_l$. The last thing happens to the $v = constant$ trajectories for v within intervals determined by critical points of the conformal connection $\beta = F_l(\gamma)$. Again the ns state trajectories which appear in the fourth quadrant, *i.e.*, those with $n \geq 1$, are an exeption: they all escape to infinity, when $w \to \infty$.

Note added in proof. A more detailed presentation of the results of this paper is given in [14].

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