Vol. 29 (1998) *ACTA PHYSICA POLONICA B* No 1–2

OPTIMIZED EXPANSION FOR THE NAMBU AND JONA-LASINIO MODEL* **

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(Received November 25, 1997)

The optimized expansion is applied to calculate the effective action for the Nambu-Jona-Lasinio model. The method is non-perturbative, the results derived from the effective action calculated to the first order of the optimized expansion corespond to an infinite summation of perturbative Feynman diagrams both in the Schwinger–Dyson equation for propagator and in the two-body Bethe–Salpeter equation. We show that this is equivalent to the mean field (relativistic Hartree plus random phase) approximation. The optimized expansion offers thus a systematic method to improve the relativistic mean field approximation in a consistent way.

PACS numbers: 03.70. +k, 21.60. Jz

1. Introduction

Quantum field theory of a self-interacting massless Dirac field has been proposed by Nambu and Jona-Lasinio (NJL) as a model of nucleon interactions [1]. Formulating a nonperturbative self-consistent approximation, inspired by the microscopic theory of superconductivity, Nambu and Jona-Lasinio demonstrated that a chiral symmetry is spontaneously broken and a nucleon mass is generated dynamically in the model. In terms of Schwinger-Dyson equations their approximation corresponds to keeping only the lowest

* Presented at the XXV Mazurian Lakes School of Physics, Piaski, Poland, August 27–September 6, 1997.

** This work has been supported in part by Grant 2-P03B-048-12 of the Committee for Scientific Research.

order diagrams both in the one-body equation for the propagator (Hartree approximation) and in the two-body Bethe-Salpeter equation (random phase approximation). Nowaday the NJL model, reinterpreted as a theory with quark degrees of freedom, is widely used as an effective theory which displays essential features of QCD in the self-consistent approximation of Nambu and Jona-Lasinio [2]. A nonperturbative treatment allowing to go beyond this approximation is thus of great interest, and such a possibility is offered by the effective action (EA). The EA is a generating functional for one-particle-irreducible (1-PI) Green's functions (proper vertices), an approximation to this functional generates thus a consistent set of approximations for all Green's functions of the theory. A systematic approximation scheme, based on the $\frac{1}{N}$ expansion of the EA, in the theory of N color degrees of freedom, has been applied recently to the NJL model [3, 4]. The leading order gives a self-consistent approximation of Nambu and Jona-Lasinio (both for the propagator as well as for the Bethe-Salpeter equation), higher orders provide a method for systematic improvement of this result. In the chiral limit ($m_0 \to 0$) the method is symmetry conserving - although the chiral symmetry is spontaneously broken, the definite symmetry relations (Ward-Takahashi identities) are preserved order by order in $\frac{1}{N}$ expansion. The method relies however on auxiliary field formulation and can be applied only to very limited class of field theory models. Here we propose an alternative scheme, based on the optimized expansion (OE) which is formulated in terms of fundamental fermion fields without introducing auxiliary fields. The OE can be thus useful to discuss models where the auxiliary field method does not work, *e.g.* to the tree-flavor NJL model with t'Hooft interaction. Another advantage of the OE is the fact that the expansion parameter is not related to the number of fields.

The OE has been developed [5] in scalar QFT with $\lambda\Phi^4$ interaction. In this case the first order effective action coincides with the Gaussian approximation obtained by applying the time dependent variational principle to the functional Schrodinger equation [7, 8]. The OE has been also applied in the fermion theory with $(\bar{\Psi}\Psi)^2$ interaction (Gross-Neveu model) and it has been shown [6] that the first order result for the EA gives account of fermion mass generation and provides the exact result in the large N limit.

Here we consider the simplest version of the NJL model with a classical action given by

$$\begin{aligned} S[\bar{\Psi},\Psi] \;=\; \int d^n x \, \big(\bar{\Psi}^i_A(x)\,(i\;\partial\!\!\!/ - m_0)\,\Psi^i_A(x) \\ + \frac{g}{2N}\left((\bar{\Psi}^i_A(x)\Psi^i_A(x))^2 + (\bar{\Psi}^i_A(x)(i\gamma_5)_{AB}\Psi^i_B(x))^2\right)\Big)\,, \qquad (1) \end{aligned}$$

where Ψ represents the quark field with N colors (flavours are ignored for simplicity) and a current mass m_0 has been included for generality ($m_0 \to 0$ gives the chirally-symmetric NJL theory). We shall suppress the space arguments and integration, as well as the color (i) and Dirac (A) indices and summation over them, writing the NJL action (1) as

$$S[\bar{\Psi}, \Psi] = \bar{\Psi} D^{-1} \Psi + \frac{g}{2N} \left((\bar{\Psi}\Psi)^2 + (\bar{\Psi} i\gamma_5 \Psi)^2 \right), \tag{2}$$

where

$$D_{ij}^{-1}(x,y) = (\not{\partial} - m_0)\delta(x-y)\delta_{ij}. \tag{3}$$

Quantization is done by representing the generating functional for Green's functions as a path integral over the fields

$$Z[\eta, \bar{\eta}] = e^{iW[\eta,\bar{\eta}]} = \int D\Psi D\bar{\Psi} e^{i\left(S[\Psi,\bar{\Psi}] + \bar{\eta}\Psi + \bar{\Psi}\eta\right)} \tag{4}$$

with Grassmann sources η and $\bar{\eta}$ introduced. The EA is defined as

$$\Gamma[\psi, \bar{\psi}] = W[\eta, \bar{\eta}] - \bar{\psi}\eta - \bar{\eta}\psi, \tag{5}$$

where the background fields, defined by means of left variational derivatives

$$\psi = \frac{\delta W}{\delta \bar{\eta}} \qquad \text{and} \qquad \bar{\psi} = -\frac{\delta W}{\delta \eta}, \tag{6}$$

are the vacuum expectation values of Grassmann fields in the presence of external sources. When the sources are turned off the background Grassmann fields should vanish, since Lorentz invariance is not expected to be broken. The vacuum energy density is thus given by

$$\mathcal{E}_{\text{vac}} = - \left. \frac{\Gamma[\psi, \bar{\psi}]}{\int d^4x} \right|_{0,0} \tag{7}$$

and n-particle proper vertices can be generated as

$$\Gamma^n(x_1, ..., x_n, y_1, ..., y_n) = \left. \frac{\delta^{2n} \Gamma[\psi, \bar{\psi}]}{\delta\psi(x_1)...\delta\psi(x_n)\delta\bar{\psi}(y_1)...\delta\bar{\psi}(y_n)} \right|_{0,0}. \tag{8}$$

2. Optimized expansion

The path integral for the generating functional $Z[\eta, \bar{\eta}]$ cannot be evaluated analytically and approximation methods are necessary. The steepest-descent method, representing the path integral (4) as a series of calculable

Gaussian integrals proves to be very useful to this end. Upon translating the integration variables to the stationary point of the exponent and rescaling them by $\hbar$, the functional $Z[\eta,\bar{\eta}]$ is obtained as a series in $\hbar$, providing the usual loop expansion for the effective action [9]. The optimized expansion (OE) is obtained in an analogous way, but with a classical action (2) written in a modified form

$$S_{\mathrm{mod}}[\bar{\Psi},\Psi] = \bar{\Psi}G^{-1}\Psi + \epsilon\left(\bar{\Psi}(D^{-1}-G^{-1})\Psi + \frac{g}{2N}(\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5\Psi)^2\right), \quad (9)$$

where a fermion propagator, $G(x,y)$, is arbitrary. A formal parameter ε has been introduced to identify orders of the perturbation and its value has to be set equal to one at the end. Upon translating the integration variables by ψ^0 and $\bar{\psi}^0$ which make the exponent in Eq. 4 stationary:

$$\left.\frac{\delta S_{\mathrm{mod}}}{\delta\Psi}\right|_{\Psi=\psi^0,\bar{\Psi}=\bar{\psi}^0} = \bar{\eta} \qquad \text{and} \qquad \left.\frac{\delta S_{\mathrm{mod}}}{\delta\bar{\Psi}}\right|_{\Psi=\psi^0,\bar{\Psi}=\bar{\psi}^0} = -\eta, \quad (10)$$

and expanding the exponential into a Taylor series, we obtain $Z[\eta,\bar{\eta}]$ as a series in a formal parameter ε:

$$\begin{aligned} Z[\eta,\bar{\eta}] &= e^{iS_{\mathrm{mod}}[\bar{\psi}^0,\psi^0]+i\bar{\eta}\psi^0+i\bar{\psi}^0\eta}\int D\Psi D\bar{\Psi} e^{i\bar{\Psi}G^{-1}\Psi} \\ &\times\Bigg\{1+i\varepsilon\Bigg[\bar{\Psi}(D^{-1}-G^{-1})\Psi + \frac{g}{N}(\bar{\psi}^0\psi^0)(\bar{\Psi}\Psi) \\ &+\frac{g}{N}(\bar{\psi}^0\Psi)(\bar{\Psi}\psi^0) + \frac{g}{N}(\bar{\psi}^0 i\gamma_5\psi^0)(\bar{\Psi}i\gamma_5\Psi) \\ &+\frac{g}{N}(\bar{\psi}^0 i\gamma_5\Psi)(\bar{\Psi}i\gamma_5\psi^0) + \frac{g}{2N}(\bar{\Psi}\Psi)^2 + \frac{g}{2N}(\bar{\Psi}i\gamma_5\Psi)^2\Bigg] + O(\varepsilon^2)\Bigg\}. \quad (11)\end{aligned}$$

Taking the trial propagator G diagonal in the color indices, $G_{ii} = \mathcal{G}$, and performing the Gaussian integration term by term we have

$$\begin{aligned} Z[\eta,\bar{\eta}] &= e^{iS_{\mathrm{mod}}[\bar{\psi}^0,\psi^0]+i\bar{\eta}\psi^0+i\bar{\psi}^0\eta} Det^N\mathcal{G}^{-1}\Bigg\{1+\varepsilon\Bigg[\Bigg(N(\mathcal{D}^{-1}-\mathcal{G}^{-1})_{AB} \\ &+\frac{g}{N}\psi^0_A\bar{\psi}^0_B + g(\bar{\psi}^0 i\gamma_5\psi^0)(i\gamma_5)_{AB} + \frac{g}{N}(i\gamma_5)_{AC}\psi^0_C\bar{\psi}^0_D(i\gamma_5)_{DB}\Bigg)\mathcal{G}_{BA} \\ &+g(\psi^0_B\bar{\psi}^0_B)\mathcal{G}_{AA} - \frac{Nig}{2}\mathcal{G}_{AA}\mathcal{G}_{BB} + \frac{ig}{2}\mathcal{G}_{AB}\mathcal{G}_{BA} \\ &-\frac{Nig}{2}\mathcal{G}_{BA}(i\gamma_5)_{AB}\mathcal{G}_{DC}(i\gamma_5)_{CD} \\ &+\frac{ig}{2}\mathcal{G}_{DA}(i\gamma_5)_{AB}\mathcal{G}_{BC}(i\gamma_5)_{CD}\Bigg] + O(\varepsilon^2)\Bigg\}, \quad (12)\end{aligned}$$

where the determinant is taken with respect to space arguments and to Dirac indices. Performing the Legendre transform the EA can be obtained to the k-th order in ε. We can take the advantage of the freedom of choosing the trial propagator, $\mathcal{G}(x,y)$, to optimize the expansion. The exact EA does not depend on the trial propagator. We require thus the k-th order approximant, $\Gamma_{(k)}$, to be as insensitive as possible to the small variation of the trial propagator, $\mathcal{G}(x,y)$, by choosing that propagator to satisfy the gap equation:

$$\frac{\delta\Gamma_{(k)}}{\delta\mathcal{G}(x,y)} = 0. \tag{13}$$

To the first order of the OE the EA is obtained equal to

$$\begin{aligned}
&\Gamma_1\bigg[\psi,\bar{\psi},\mathcal{G}] = \bar{\Psi}D^{-1}\Psi + \frac{g}{2N}\bigg((\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5\Psi)^2\bigg) \\
&-iN\mathrm{TrLn}\mathcal{G}^{-1} + i\bigg(N(\mathcal{G}^{-1} - \mathcal{D}^{-1})_{AB} - g\bar{\psi}_C\psi_C\delta_{AB} - \frac{g}{N}\psi_A\bar{\psi}_B \\
&-g(i\gamma_5)_{AB}\bar{\psi}_C(i\gamma_5)_{CD}\psi_D - \frac{g}{N}(i\gamma_5)_{AC}\psi_C\bar{\psi}_D(i\gamma_5)_{DB}\bigg)\mathcal{G}_{BA} \\
&-\frac{g}{2}\bigg(N\mathcal{G}_{AA}\mathcal{G}_{BB} - \mathcal{G}_{AB}\mathcal{G}_{BA} + N\mathcal{G}_{AB}(i\gamma_5)_{BA}\mathcal{G}_{DC}(i\gamma_5)_{CD} \\
&-\mathcal{G}_{DA}(i\gamma_5)_{AB}\mathcal{G}_{BC}(i\gamma_5)_{CD}\bigg)\bigg].
\end{aligned} \tag{14}$$

The gap equation reads

$$\begin{aligned}
\frac{\delta\Gamma_1}{\delta\mathcal{G}_{\mathcal{BA}}(x,y)} &= iN\bigg(\mathcal{G}^{-1}_{AB} - \mathcal{D}^{-1}_{AB} - g\bar{\psi}_C\psi_C\delta_{AB} \\
&-\frac{g}{N}\psi_A\bar{\psi}_B - g(i\gamma_5)_{AB}\bar{\psi}_C(i\gamma_5)_{CD}\psi_D \\
&-\frac{g}{N}(i\gamma_5)_{AC}\bar{\psi}_C\psi_D(i\gamma_5)_{DB}\bigg) - g\bigg(N\mathcal{G}_{CC}\delta_{AB} - \mathcal{G}_{BA} \\
&+N(i\gamma_5)_{AB}\mathcal{G}_{CD}(i\gamma_5)_{DC} - (i\gamma_5)_{AC}\mathcal{G}_{CD}(i\gamma_5)_{DB}\bigg) = 0
\end{aligned} \tag{15}$$

and can be fulfilled by the trial propagator of the form

$$\mathcal{G}^{-1}(x,y) = (\partial\!\!\!/ - \Omega)\delta(x-y). \tag{16}$$

The energy density (7) is given by

$$\mathcal{E}_{\mathrm{vac}} = -4NI_1(\Omega) + 4N(\Omega - m_0)\Omega I_0(\Omega) + (4N-2)2g(\Omega I_0(\Omega))^2\,, \tag{17}$$

where Ω fulfills the algebraic equation

$$m_0 - \Omega = \frac{(4N-2)g}{N} \Omega I_0(\Omega) \tag{18}$$

with

$$I_1(\Omega) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln(k^2 + \Omega^2) \quad \text{and} \quad I_0(\Omega) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \Omega^2}. \tag{19}$$

Our Eq. (18) coincides with the gap equation obtained by Nambu and Jona-Lasinio. By using the gap equation, the energy density can be represented in the form

$$\begin{aligned} \mathcal{E}_{\text{vac}} &= -4NI_1(\Omega) + 2N(\Omega - m_0)\Omega I_0(\Omega) \\ &= 2N \int \frac{d^3k}{(2\pi)^3} \left[-2\sqrt{\mathbf{k}^2 + \Omega^2} + \frac{\Omega(\Omega - m_0)}{\sqrt{\mathbf{k}^2 + \Omega^2}} \right], \end{aligned} \tag{20}$$

which coincides with the result obtained by a variational method by Suzuki [10]. He pointed out that the result of Nambu and Jona-Lasinio (where the second term is missing) is in error which was later attributed to double counting of the interaction in the self-consistent mean-field approach by Hatsuda and Kunihiro [11]. We want to point out that in the EA formalism double counting is avoided automatically.

The implicit expression for the EA allows one to calculate all proper vertices in the OE as derivatives with respect to the background fields ψ and $\bar{\psi}$. The inverse of the full propagator is obtained in the form

$$\Gamma^2_{ij}(x,y) = \left. \frac{\delta^2 \Gamma[\psi, \bar{\psi}]}{\partial \psi_i(x) \delta \bar{\psi}_j(y)} \right|_{0,0} = (\not{\partial} - \Omega)\, \delta(x-y) \delta_{ij} = \mathcal{G}^{-1}(x,y) \delta_{ij}, \tag{21}$$

where Ω, as a solution of the gap equation (18), coincides with the self-consistent mass obtained by Nambu and Jona-Lasinio. Our approach can be regarded as an extension of the self-consistent perturbation method developed in the original work of Nambu and Jona-Lasinio [1]. They calculate the self-energy using a free propagator with an arbitrary mass, which is later fixed by requiring the given order radiative corrections to the self-energy to vanish. In the OE the EA is calculated using an arbitrary trial propagator, which is fixed later by requiring the given order approximation to the EA do not depend on that propagator (13). Since the EA is a generating functional for 1-PI Green's functions, this pushes further the idea of self-consistency - all the proper vertices are generated from the approximate expression for the EA. In the NJL approach only the mass is determined self-consistently, other vertices are calculated perturbatively with the use of the improved propagator. Even in the lowest order higher 1-PI vertices differ, although the propagator happens to be the same in both approaches. In the NJL approach the four-fermion vertex is equal to that obtained in perturbation theory, in the OE the fourth derivative of the implicit expression for the EA (14) and (15) generates an approximation to the two-particle Bethe-Salpeter equation. In the NJL approach the Bethe-Salpeter equation is treated independently of the gap equation for the self-consistent mass, it is thus unclear how to relate the approximations to these two equations beyond the first order. This is in contrast to the case of the OE where differentiation of the aproximate expression for the EA generates a definite approximations both to the gap equation and to the two-particle Bethe-Salpeter equation. The systematical improvement of the self-consistent approximation of Nambu and Jona-Lasinio offered by the OE requires further study.

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