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RELATIONS AMONG POLARIZED AND UNPOLARIZED SPLITTING FUNCTIONS BEYOND LEADING ORDER*

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The role of various symmetries in the evaluation of splitting functions and coefficient functions is discussed. The scale invariance in hard processes is known to be a guiding tool to understand the dynamics. We discuss the constraints on splitting functions coming from various symmetries such as scale, conformal and supersymmetry. We also discuss the Drell–Levy–Yan relation among splitting and coefficient functions in various schemes. The relations coming from conformal symmetry are also presented.

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1. Scale transformation

Symmetries are known to be very useful guiding tool to understand the dynamics of various physical phenomena. Particularly, continuous symmetries played an important role in particle physics to unravel the structure of dynamics at low as well as high energies. In hadronic physics, such symmetries at low energies were found to be useful to classify various hadrons. At high energy, where the masses of the particles can be neglected, one finds in addition to the above mentioned symmetries new symmetries such as conformal and scale invariance. This for instance happens in deep inelastic lepton-hadron scattering (DIS) where the energy scale is much larger than the hadronic mass scale. At these energies one can in principle ignore the mass scale and the resulting dynamics is purely scale independent.

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Limiting ourselves to scale transformations the latter is defined by $x_\mu \to e^t\ x_\mu$. An arbitrary quantum field $\hat{\phi}(x)$ is then transformed as follows

$$\hat{\phi}(x) \to U^\dagger\ \hat{\phi}(x)\ U = e^{d_0 t}\ \hat{\phi}(e^{-t}x)\,, \tag{1.1}$$

where U is the unitary operator and d_0 is its canonical dimension. Under this transformation the n-point Green's function $E_n(p_i, g)$ behaves like

$$E_n(e^t p, g) = E_n(p, g) e^{t(D-n\ d_0)}\,, \tag{1.2}$$

where p_i are the momenta and g denotes the coupling constant. However in perturbation theory, like QCD, scale invariance is broken due to the introduction of a regulator scale which is rigid under conformal and scale transformation. Even if the regulator is removed in the renormalized Green's function a renormalization scale μ is left which is rigid too. In this case the Green's function does not satisfy a simple scaling equation anymore and the latter is replaced by the Callan Symmanzik (CS) equation [1] which reads

$$\left[\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} - D + n\ (d_0 + \gamma(g))\right] E_n(e^t p_i, g, \mu) = 0\,, \tag{1.3}$$

where $\beta(g)$ and $\gamma(g)$ denote the beta-function and the anomalous dimension respectively with the property $\beta \to 0$ and $\gamma \to 0$ as $g \to 0$. In the case $\beta(g_c) = 0$ at some fixed point $g = g_c \neq 0$ scale invariance is restored and the solution to this equation becomes

$$E_n(e^t p, g_c, \mu) = E_n(p, g_c, \mu) e^{t(D - n(d_0 + \gamma(g_c))}\,. \tag{1.4}$$

Let us discuss the beta-function and the anomalous dimensions of composite operators for QCD. The latter are derived from the Green's function

$$\begin{aligned} &E_{ij}^{(n)}(p, g, \mu) \\ &= \int d^4x_1\, d^4x_2\, e^{ip.(x_1 - x_2)} \langle 0 \mid T\Big(\hat{\phi}_j(x_1) O_i^{(n)}(0) \hat{\phi}_j(x_2)\Big) \mid 0\rangle\,, \end{aligned} \tag{1.5}$$

Here $O_i^{(n)}$ denotes the composite operator of spin n which is build out of quark and gluon fields $\hat{\phi}_j$ with $i, j = q, g$. If one chooses D-dimensional regularization the renormalized Green's functions and the bare Green's functions (indicated by the subscript u) are related by

$$E_{ij}^{(n)}(p, g, \mu) = \Big(Z^{(n)}\Big)^{-1}_{il}(\varepsilon, g, \mu) E_{lj,u}^{(n)}(p, g, \varepsilon)\,, \tag{1.6}$$

where $Z_{ij}(\varepsilon, g, \mu)$ is the operator renormalization constant and ε indicates the ultraviolet pole terms in D-dimensional regularization ($D = 4 - 2\varepsilon$).

Notice that there is more than one operator involved in the renormalization so that we have to deal with mixing. If we amputate the external legs of the Green function in (1.5) the anomalous dimension of the composite operator $O^{(n)}$ is given by

$$\gamma_{ij}^{(n)}(g) = \beta(g,\varepsilon)\Big(Z^{(n)}\Big)^{-1}_{il}(g,\varepsilon)\frac{d\ Z_{lj}^{(n)}(g,\varepsilon)}{dg}\,. \tag{1.7}$$

The renormalization constant $Z(\varepsilon, g)$ has an expansion in $1/\varepsilon$ as

$$Z_{ij}^{(n)}(\varepsilon,g) = 1 + \frac{1}{\varepsilon}Z_{ij}^{(n),(1)}(g) + \frac{1}{\varepsilon^2}Z_{ij}^{(n),(2)}(g) + \cdots\,. \tag{1.8}$$

Since the beta-function has the following form

$$\beta(g,\varepsilon) = \varepsilon\frac{g}{2} + \beta_0\frac{g^3}{16\pi^2} + \cdots\,, \tag{1.9}$$

the $\gamma_{ij}^{(n)}$ are finite in the limit $\varepsilon \to 0$ and one gets

$$\gamma(n)_{ij}(g) = \frac{g}{2}\Big(Z^{(n),(1)}\Big)^{-1}_{il}\frac{d\ Z_{lj}^{(n),(1)}}{dg}\,. \tag{1.10}$$

For the amputated Green's function the CS equation (1.3) reads

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_{il}^{(n)}(g)\right] E_{lj}^{(n)}(p,g,\mu) = 0\,. \tag{1.11}$$

In the case of scale invariance i.e. $\beta(g_c) = 0$ and no mixing the above CS equation has the simple solution

$$E^{(n)}(\mu^2) = E^n(\mu_0^2)\left(\frac{\mu^2}{\mu_0^2}\right)^{\gamma^{(n)}}\,, \tag{1.12}$$

with

$$\gamma^{(n)}(\alpha_s) = \left(\frac{\alpha_s}{2\pi}\right)\gamma^{(n),(0)} + \left(\frac{\alpha_s}{2\pi}\right)^2\gamma^{(n),(1)} + \cdots \qquad \alpha_s \equiv \frac{g^2}{4\pi}\,. \tag{1.13}$$

The splitting functions $P(x,\alpha_s)$ are related to these anomalous dimensions via a Mellin transformation given by

$$\gamma^{(n),(i)}(\alpha_s) = \int_0^1 dx\, x^{n-1}P^{(i)}(x,\alpha_s) \qquad i = 0,1,2\cdots\,. \tag{1.14}$$

The above analysis based on scale transformation suggests that only in a scale invariant theory, the Green's function has the form given in the Eq. (1.12). This will be no longer true in a scale breaking theory like QCD. The same will hold for the anomalous dimension which in the case of no mixing and scale invariance is independent of the subtraction scheme. This will change when this symmetry is broken as we will show below.

2. Supersymmetric relations

In this section we discuss some relations among splitting functions which govern the evolution of quark and gluon parton densities. These relations are valid when QCD becomes a supersymmetric $\mathcal{N} = 1$ gauge field theory where both quarks and gluons are put in the adjoint representation with respect to the local gauge symmetry SU(N). In this case one gets a simple relation between the colour factors which become $C_F = C_A = 2T_f = N$. In the case of spacelike splitting functions, which govern the evolution of the parton densities in deep inelastic lepton-hadron scattering, one has made the claim (see [2]) that the combination defined by

$$\mathcal{R}^{(i)} = P_{qq}^{(i)} - P_{gg}^{(i)} + P_{gq}^{(i)} - P_{qg}^{(i)}\,, \tag{2.1}$$

is equal to zero, *i.e.*, $\mathcal{R}^{(i)} = 0$. This relation should follow from an $\mathcal{N} = 1$ supersymmetry although no explicit proof has been given yet. An explicit calculation at leading order(LO) confirms this claim so that we have $\mathcal{R}^{(0)} = 0$ However at Next to Leading Order (NLO), when these splitting functions are computed in the $\overline{\rm MS}$-scheme, it turns out that $\mathcal{R}^{(1)}_{\overline{\rm MS}} \neq 0$. Actually one finds in the unpolarized case (see [8])

$$\begin{aligned}\mathcal{R}^{\rm unpol,(1)}_{\overline{\rm MS}} \;=\; & -\frac{5}{6} - \frac{2}{3x} + \frac{23x}{3} - 7x^2 - 4\delta(1-x) \\ & -(1 - 2x - 4x^2)\ln(x)\,,\end{aligned} \tag{2.2}$$

whereas in the polarized case [4] one obtains

$$\mathcal{R}^{\rm pol,(1)}_{\overline{\rm MS}} = \frac{28}{3} - \frac{44x}{3} - 8(1-x)\ln(x) - 4\delta(1-x)\,. \tag{2.3}$$

The reason that this relation is violated can be attributed to the regularization method and the renormalization scheme in which these splitting functions are computed. In this case it is D-dimensional regularization and the $\overline{\rm MS}$-scheme which breaks the supersymmetry. In fact, the breaking occurs already at the ε dependent part of the leading order splitting functions. Although this does not affect the leading order splitting functions in the limit $\varepsilon \to 0$ it leads to a finite contribution at the NLO level via the $1/\varepsilon^2$ terms which are characteristic of a two-loop calculation (see Eq. (1.8)). If one carefully removes such breaking terms at the LO level consistently, one can avoid these terms at NLO level. They can be avoided if one uses D-dimensional reduction which preserves the supersymmetry. An other possibility is that one can convert the splitting functions from one scheme to another by the

following transformation

$$Z^{(n)} \to Z^{(n)\prime} = \hat{Z}_F Z^{(n)}\,, \tag{2.4}$$

where $\hat{Z}_F$ is a finite renormalization. Under this transformation the anomalous dimensions in the new scheme become

$$\gamma^{(n)} \to \gamma^{(n)\prime} = \hat{Z}_F^{-1}\gamma^{(n)}\hat{Z}_F - \beta(g)\frac{d\hat{Z}_F^{-1}}{dg}\hat{Z}_F\,. \tag{2.5}$$

After Mellin inversion, see (1.14), one gets in the unpolarized case

$$\hat{Z}_F^{\rm unpol} = \begin{pmatrix} -2+2x+\delta(1-x) & 0 \\ -2x & -4x+4x^2+\frac{1}{3}\delta(1-x) \end{pmatrix}, \tag{2.6}$$

and for the polarized case we have

$$\hat{Z}_F^{\rm pol} = \begin{pmatrix} -2+2x+\delta(1-x) & 0 \\ 0 & \frac{1}{3}\delta(1-x) \end{pmatrix}. \tag{2.7}$$

In this new (primed) scheme it turns out that $\mathcal{R}^{(1)\prime} = 0$.

The above observations also apply to the timelike splitting functions, denoted by a tilde, which govern the evolution of fragmentation functions. Substitution of their expressions [3] into Eq. (2.1) yields the $\overline{\rm MS}$-scheme results

$$\tilde{\mathcal{R}}^{\rm unpol,(1)} = \frac{5x}{3} - \frac{2}{3x} + \frac{13}{6} - x^2 - \frac{1}{2}\delta(1-x) - (1-2x-4x^2)\ln(x)\,. \tag{2.8}$$

For the polarized case we need the splitting functions in [5] so that we get

$$\tilde{\mathcal{R}}^{\rm pol,(1)} = \frac{11}{6} - \frac{7x}{6} + (1-x)\ln(x) - \frac{1}{2}\delta(1-x)\,. \tag{2.9}$$

It appears that the scheme transformation, introduced for the spacelike case in Eqs. (2.6), (2.7), or the use of supersymmetric reduction also lead in the timelike case to the result $\tilde{\mathcal{R}}^{(1)\prime} = 0$ in Eq. (2.1).

3. Drell–Levy–Yan relation

The Drell–Levy–Yan relation (DLY) [6] relates the structure functions $F(x,Q^2)$ measured in deep inelastic scattering to the fragmentation functions $\tilde{F}(\tilde{x},Q^2)$ observed in $e^+\,e^-$-annihilation. Here x denotes the Bjorken scaling variable which in deep inelastic scattering and $e^+\,e^-$-annihilation

is defined by $x = Q^2/2p.q$ and $\tilde{x} = 2p.q/Q^2$, respectively. Notice that in deep inelastic scattering the virtual photon momentum q is spacelike i.e. $q^2 = -Q^2 < 0$ whereas in $e^+ e^-$-annihilation it becomes timelike $q^2 = Q^2 > 0$. Further p denotes the in or outgoing hadron momentum. The DLY relation looks as follows

$$\tilde{F}_i(\tilde{x}, Q^2) = x\mathcal{A}c\left[F_i(1/x, Q^2)\right] , \tag{3.1}$$

where $\mathcal{A}c$ denotes the analytic continuation from the region $0 < x \leq 1$ (DIS) to $1 < x < \infty$ (annihilation region). At the level of splitting functions we have

$$\tilde{P}_{ij}(\tilde{x}) = x\mathcal{A}c\left[P_{ji}(1/x)\right] . \tag{3.2}$$

At LO, one finds $\tilde{P}^{(0)}(\tilde{x}) = P^{(0)T}(x)$. Explicitly,

$$\tilde{P}^{(0)}_{qq}(\tilde{x}) = -xP^{(0)}_{qq}(1/x) \qquad \tilde{P}^{(0)}_{qg}(\tilde{x}) = \frac{2T_f}{C_F} xP^{(0)}_{gq}(1/x) ,$$

$$\tilde{P}^{(0)}_{gq}(\tilde{x}) = \frac{C_F}{2T_f} xP^{(0)}_{qg}(1/x) \qquad \tilde{P}^{(0)}_{gg}(\tilde{x}) = -xP^{(0)}_{gg}(1/x) . \tag{3.3}$$

At the leading order level, one finds that

$$\tilde{P}^{(0)}_{qq}(\tilde{x}) = P^{(0)}_{qq}(x) \tag{3.4}$$

which is nothing but Gribov–Lipatov relation [7]. This relation in terms of physical observables is known to be violated when one goes beyond leading order [8]. On the other hand the DLY (analytical continuation) relation defined above holds at the level of physical quantities provided the analytical continuation is performed in both x as well as the scale Q^2 ($Q^2 \to -Q^2$) (see below).

In analytical continuation, care is needed when one goes beyond LO when dimensional regularization is adopted. The correct $\mathcal{A}c$ relation in DR scheme reads as follows [8]:

$$\tilde{P}_{ij}(\tilde{x}) = x^{1-2\varepsilon}\mathcal{A}c\left[P_{ji}(1/x)\right] . \tag{3.5}$$

The extra term $x^{-2\varepsilon}$ arises due to the difference between the spacelike and timelike phase space integrations. Starting from the definitions of splitting functions,

$$P(x) = \beta(\alpha_s, \varepsilon)\frac{d\ln Z(x, \alpha_s, \varepsilon)}{d\ln \alpha_s} \qquad \tilde{P}(\tilde{x}) = \beta(\alpha_s, \varepsilon)\frac{d\ln \tilde{Z}(\tilde{x}, \alpha_s, \varepsilon)}{d\ln \alpha_s} , \tag{3.6}$$

and

$$x^{-2\varepsilon}\mathcal{A}c\left[Z(1/x,\varepsilon)\right] = Z_F(x)\mathcal{A}c\left[Z(1/x,\varepsilon)\right] \,, \tag{3.7}$$

one finds that the splitting functions are related by simple relation

$$\tilde{P}(\tilde{x}) = x\mathcal{A}c\left[P(1/x)\right] + \text{contributions coming from } Z_F \,. \tag{3.8}$$

The DLY relation between NLO coefficient functions appearing in DIS and e^+e^- can be worked out in the same way as we did for the splitting functions above. In the subsequent part of this section we will only study the gluonic coefficient functions corresponding to the deep inelastic structure functions and the fragmentation functions. The conclusions also apply to the quark coefficient functions as well.

The spacelike gluonic coefficient function for the polarized case in DIS originates from photon-gluon fusion process and is given by [9]

$$\mathcal{C}_{1,g}(x,Q^2)|_{\overline{\rm MS}} = e_q^2\frac{\alpha_s}{4\pi}\left((2x-1)\ln\left(\frac{Q^2(1-x)}{\mu^2 x}\right) + 3 - 4x\right) \,. \tag{3.9}$$

In the above, the collinear singularity is treated in D-dimensional regularization and the scale μ is the factorization scale. For e^+ e^--annihilation the timelike coefficient function becomes [10]

$$\tilde{\mathcal{C}}_{1,g}(\tilde{x},Q^2)|_{\overline{\rm MS}} \;=\; x\mathcal{C}_{1,g}(1/x,Q^2)|_{\overline{\rm MS}} + 2\,P_{gq}^{(0)}\ln(x) \,. \tag{3.10}$$

The violation of DLY relation is due to the regularization method and the scheme we have adopted to remove the collinear singularities from the partonic cross sections. This is the reason we get a mismatch between the phase space integrations in the spacelike and timelike case which is equal to $x^{-2\varepsilon}$. This factor is multiplied with the lowest order pole term which leads to the finite contribution on the right hand side of Eq. (3.10).

The violation is an artifact of dimensional regularization and the choice of the $\overline{\rm MS}$-scheme. For example if one chooses a regularization where the gluon gets a mass m_g and one removes the mass singularity $\ln(\mu^2/m_g^2)$ only, the space-and timelike coefficient functions become [11]

$$\mathcal{C}_{1,g}(x,Q^2)|_{m_g\neq 0} = e_q^2\frac{\alpha_s}{4\pi}(2x-1)\left(\ln\left(\frac{Q^2}{\mu^2x^2}\right) - 2\right) \,, \tag{3.11}$$

and

$$\tilde{\mathcal{C}}_{1,g}(\tilde{x},Q^2)|_{m_g\neq 0} = x\mathcal{C}_{1,g}(1/x,Q^2)|_{m_g\neq 0} \,, \tag{3.12}$$

respectively, so that the DLY relation is satisfied. The same happens when the quark gets a mass m_q. After removing the mass singularity $\ln(\mu^2/m_q^2)$ one gets [12]

$$\mathcal{C}_{1,g}(x,Q^2)|_{m_q\neq 0} = e_q^2\frac{\alpha_s}{4\pi}\left((2x-1)\ln\left(\frac{Q^2(1-x)}{\mu^2 x}\right)+3-4x\right), \quad (3.13)$$

and

$$\tilde{\mathcal{C}}_{1,g}(\tilde{x},Q^2)|_{m_q\neq 0} = x\mathcal{C}_{1,g}(1/x,Q^2)|_{m_q\neq 0}\,. \quad (3.14)$$

Hence the violation of the DLY relation for the splitting functions and the coefficient functions separately is just an artifact of the adopted regularization method and the subtraction scheme. When these coefficient functions are combined with the splitting functions in a scheme invariant way as for instance happens for the structure functions and fragmentation functions the above relation holds. The reason for the cancellation of the DLY violating terms among the splitting functions and coefficient functions is that the former are generated by simple scheme transformations.

4. Supersymmetric and conformal relations

In this section we study the constraints coming from the conformal symmetry on the splitting functions in an $\mathcal{N}=1$ supersymmetry. The following set of relations have been derived [13] between the unpolarized (P_{ij}) and polarized (ΔP_{ij}) splitting functions:

$$(P_{qq}-P_{qg})+(\Delta P_{qq}-\Delta P_{qg}) \;=\; x\,(P_{qq}+P_{gq}+\Delta P_{qq}+\Delta P_{gq})\,, \quad (4.1)$$

$$(P_{qq}-P_{qg})-(\Delta P_{qq}-\Delta P_{qg}) \;=\; -x\,(P_{qq}+P_{gq}-\Delta P_{qq}-\Delta P_{gq})\,. \quad (4.2)$$

The LO splitting functions satisfy the above relations but at NLO level they are violated in the $\overline{\rm MS}$-scheme. In the latter scheme the difference between the left- and right-hand side of Eqs. (4.1) and (4.2) is given by

$$\frac{1}{3}-8x+\frac{29x^2}{6}+\frac{4x^3}{3}-2x\ln(x)-5x^2\ln(x)\,, \quad (4.3)$$

and

$$-2+\frac{8x}{3}+\frac{13x^2}{6}-\frac{4x^3}{3}-2\ln(x)-2x\ln(x)-x^2\ln(x)\,, \quad (4.4)$$

respectively. Following the discussion below Eq. (2.1) these relations can be preserved by making finite scheme transformations. Another interesting relation in [13] is the one between the non-diagonal entries of splitting function matrix:

$$x\frac{d}{dx}\left((\Delta)P_{qg} - (\Delta)P_{gq}\right) = 2(\Delta)P_{qg} + (\Delta)P_{gq}\,. \tag{4.5}$$

The known LO splitting functions satisfy this relation but it is violated by NLO splitting functions in $\overline{\rm MS}$ scheme. Interestingly, the violation comes from the terms such as $\ln(x)\ln(1-x)$. These terms can not be removed by finite scheme transformation so that the above equation does not hold anymore in NLO irrespective of the chosen scheme.

5. Conclusions

We have discussed the relations between the splitting functions coming from various symmetries such as scale symmetry, conformal symmetry and supersymmetry on NLO splitting functions and coefficient functions. The Drell–Levy–Yan relation among them is also discussed at NLO level. Most of the relations coming from these symmetries are violated in dimensional regularization with $\overline{\rm MS}$ prescription. The breaking terms can be identified at the leading order level. By simple finite renormalization one can preserve the relations coming from scale and supersymmetric constraints. The breaking due to conformal non-invariant terms (see Eq. (4.5)) cannot be cured by a simple finite renormalization.

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