

## ALGEBRAIC RENORMALIZATION OF THE ELECTROWEAK STANDARD MODEL\*

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*(Received July 10, 1998)*

The algebraic method of renormalization is applied to the standard model of electroweak interactions. We present the most important modifications compared to theories with simple groups.

PACS numbers: 11.10.Gh, 12.15.-y

### 1. Introduction

The Standard Model (SM) of electroweak interactions has been tested to high accuracy with the precision experiments at the Z-resonance at LEP [1]. With these experiments the SM in its perturbative formulation has been tested also beyond the tree approximation. For this reason an extensive calculation of 1-loop processes and also 2-loop processes has been carried out in the past years (see [2] for a review and references therein) and compared to the experimental results. A careful analysis shows that the theoretical predictions and the experiments are in excellent agreement with each other [3]. A necessary prerequisite for carrying out precision tests of the SM is the consistent mathematical and physical formulation in the framework of its perturbative construction. Due to the fact that parity is broken by weak interactions, higher orders in quantum field theory cannot be treated by referring to an invariant regularization scheme. In Ref. [4] we have carried out the renormalization of the electroweak SM to all orders by applying the algebraic method. It allows to prove renormalizability in a scheme-independent way just by using general properties of renormalized perturbation theory (see [5] for a review to the algebraic method). The algebraic method has been first applied to gauge theories with simple or semisimple groups [6],

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\* Presented at the DESY Zeuthen Workshop on Elementary Particle Theory “Loops and Legs in Gauge Theories”, Rheinsberg, Germany, April 19-24, 1998.

later on it has been extended to gauge theories with non-semisimple groups with several  $U(1)$ -factors [7]. The results obtained therein are only partially applicable to the SM due to the fact, that all particles are assumed to be massive, but it gives a complete discussion of anomalies in gauge theories with non-semisimple groups.

In the Section 2 of the paper we outline the procedure of algebraic renormalization and present the basic ingredients of the method, scheme dependence of counterterms and the quantum action principle. In Section 3 we discuss the characterization of invariant counterterms to the SM Green functions. There we direct our attention to three important results: The construction of the abelian Ward identity, the definition of symmetry operators in the on-shell schemes and consequences of rigid symmetry in the gauge-fixing and ghost sector of the action.

## 2. The algebraic method of renormalization

Starting point for the construction is the BRS-symmetric classical action of the electroweak SM. (We do not include strong interactions in the present analysis, but assume the quarks to be color vectors of global  $SU(3)$ .) It consists of the  $SU(2) \times U(1)$ -gauge invariant action  $\Gamma_{\text{GSW}}$  and the BRS-invariant gauge-fixing  $\Gamma_{\text{g.f.}}$  and ghost action  $\Gamma_{\text{ghost}}$ .

$$\Gamma_{\text{cl}} = \Gamma_{\text{GSW}} + \Gamma_{\text{g.f.}} + \Gamma_{\text{ghost}} \quad \text{with} \quad s\Gamma_{\text{cl}} = 0. \quad (1)$$

Here  $s$  denotes the nilpotent BRS-transformations ( $s^2 = 0$ ). The Glashow–Salam–Weinberg action  $\Gamma_{\text{GSW}}$  includes the massive gauge bosons of weak interactions,  $W_{\pm}, Z$ , and the massless photon  $A_{\mu}$ , the leptons  $e, \nu_e^L$ , the quarks  $u, d$ , the physical Higgs  $H$  and the unphysical scalar bosons  $\phi_{\pm}$  and  $\chi$ . Masses of gauge bosons, fermions and the Higgs are generated by spontaneous breaking of gauge symmetry to the electromagnetic subgroup. We do not consider mixing of different fermion families and assume CP-invariance throughout. Free parameters of the model are the masses and one coupling constant, which in the QED-like parametrization is chosen to be the electromagnetic coupling. For ensuring renormalizability and off-shell infrared existence by power counting we choose the restricted linear  $R_{\xi}$  gauge in the tree approximation. It contains two gauge parameters  $\xi$  and  $\zeta$  and turns out to be compatible with rigid symmetry:

$$\begin{aligned} \Gamma_{\text{g.f.}} &= \int d^4x \left( \frac{\xi}{2} B_a \tilde{I}_{ab} B_b + B_a \tilde{I}_{ab} F_b \right), \\ F_{\pm} &\equiv \partial_{\mu} W_{\pm}^{\mu} \mp i M_W \zeta \phi_{\pm}, \quad F_Z \equiv \partial_{\mu} Z^{\mu} - M_Z \zeta \chi, \quad F_A \equiv \partial_{\mu} A^{\mu}. \end{aligned} \quad (2)$$

The gauge fixing action breaks gauge invariance. Introducing the Faddeev–Popov ghosts  $c_a$  and the corresponding antighosts  $\bar{c}_a, a = +, -, Z, A$ , the

gauge-fixing action is complemented by the ghost action in such a way that the complete action is BRS-invariant.

In perturbation theory the Green functions are formally defined from the classical action by the Gell-Mann–Low formula and by Wick's theorem or equivalently by the Feynman diagrams and Feynman rules. (Feynman rules of the SM are given in several publications, see *e.g.* [8].) The loop corrections to Green functions are plagued with divergencies, which have to be consistently removed in the procedure of renormalization. Then one has to prove that the symmetries of the tree approximation can be established in the course of renormalization and that these symmetries uniquely fix the Green functions, if one imposes a finite number of normalization conditions.

For renormalization only the 1PI Green functions are relevant, which are summarized in the functional of 1PI Green functions  $\Gamma$ . The lowest order of  $\Gamma$  is the classical action,  $\Gamma = \Gamma_{\text{cl}} + O(\hbar)$ . For proceeding to higher orders the symmetry transformations have to be rewritten into functional form. The functional form of BRS-symmetry is the Slavnov–Taylor (ST) identity. By introducing an external scalar doublet  $\hat{\Phi}$  we are able to maintain rigid SU(2)-symmetry and spontaneously broken U(1)-gauge symmetry for the special choice of gauge parameters (2) and to establish the respective Ward identities:

$$\begin{aligned} \mathcal{S}(\Gamma_{\text{cl}}) &= 0, & \mathcal{W}_\alpha \Gamma_{\text{cl}} &= 0, \quad \alpha = +, -, 3 \\ \left( \frac{e}{\cos \theta_W} \mathbf{w}_4^Q - \sin \theta_W \frac{\delta}{\delta Z^\mu} - \cos \theta_W \frac{\delta}{\delta A^\mu} \right) \Gamma_{\text{cl}} &= \square (\sin \theta_W B_Z + \cos \theta_W B_A). \end{aligned} \quad (3)$$

The local Ward identity is crucial for the unique construction of higher orders. The symmetry operators depend explicitly on the free parameters of the theory. With the gauge choice (2) they depend on the mass ratio  $\frac{M_W}{M_Z}$  in the tree approximation [4]. We now assume that we have already calculated the 1-loop order in a specific scheme of renormalization and denote the finite Green functions by  $\Gamma^{\text{R}}$ . Regardless of special properties of the scheme we can apply the action principle in its quantized version to the Green functions [9], in order to get information of the possible breakings at 1-loop order. Applied to the symmetries of the tree approximation it tells that in the 1-loop order the symmetries are at most broken by *local* field polynomials with a definite UV and IR degree of power counting. Taking the most important example, the ST identity, it reads

$$\mathcal{S}(\Gamma^{\text{R}})^{(\leq 1)} = \Delta_{\text{brs}}^{(1)}, \quad \dim^{\text{UV}} \Delta_{\text{brs}}^{(1)} \leq 4, \quad \dim^{\text{IR}} \Delta_{\text{brs}}^{(1)} \geq 3. \quad (4)$$

The breakings include in a first step all local field polynomials compatible with the UV and IR dimension. They are restricted if one takes into account that the renormalization schemes do not break global symmetries

such as charge conservation, and discrete symmetries such as CP-invariance. For example  $\Delta_{\text{brs}}^{(1)}$  has Faddeev–Popov charge 1, is neutral with respect to electromagnetic charge and is even with respect to CP-transformations.

In the next step we have to prove that the breakings of the ST identity can be absorbed into counterterms to the classical action: For doing this one has to note that Green functions, when they are computed in a specific scheme, are only defined up to local counterterms. These counterterms are restricted by the global symmetries and the discrete symmetries of the SM. In order to maintain the properties of power counting renormalizability their UV and IR degree has to agree with the one of the classical action

$$\Gamma^{(\leq 1)} = \Gamma^{\text{R}(\leq 1)} + \Gamma_{\text{ct}}^{(1)} \quad \text{with} \quad \dim^{\text{UV}} \Gamma_{\text{ct}}^{(1)} \leq 4, \quad \dim^{\text{IR}} \Gamma_{\text{ct}}^{(1)} \geq 4. \quad (5)$$

Combining both the quantum action principle (4) and the scheme dependence of counterterms (5) we get

$$\mathcal{S}(\Gamma) = \mathcal{S}(\Gamma^{\text{R}} + \Gamma_{\text{ct}}^{(1)}) + O(\hbar^2) = \Delta_{\text{brs}}^{(1)} + s_{\Gamma_{\text{cl}}} \Gamma_{\text{ct}}^{(1)} + O(\hbar^2), \quad (6)$$

and a similar expression for rigid Ward identities. Eventually we have to prove that breakings of the ST identity can be written as  $s_{\Gamma_{\text{cl}}}$ -variations of counterterms to the classical action, *i.e.*

$$\Delta_{\text{brs}}^{(1)} \stackrel{!}{=} -s_{\Gamma_{\text{cl}}} \Gamma_{\text{ct}}^{(1)}. \quad (7)$$

Up to this point we did only use properties of renormalized perturbation theory. Finally one has to characterize both the counterterms and the breakings in terms of the symmetries: First, counterterms have to be decomposed into invariant and non-invariant counterterms

$$\Gamma_{\text{ct}} = \Gamma_{\text{inv}} + \Gamma_{\text{break}} \quad \text{with} \quad s_{\Gamma_{\text{cl}}} \Gamma_{\text{inv}} = 0. \quad (8)$$

The coefficients of invariants are not determined by the symmetries but have to be fixed by normalization conditions. Second, one restricts the breakings by using algebraic properties of the symmetry operators, as *e.g.* nilpotency of the ST operator:

$$s_{\Gamma} \mathcal{S}(\Gamma) = 0 \quad \text{and} \quad s_{\Gamma} s_{\Gamma} = 0 \quad \text{if} \quad \mathcal{S}(\Gamma) = 0. \quad (9)$$

Applying the  $s_{\Gamma_{\text{cl}}}$ -operator to Eq. (6) one obtains from (9) that  $\Delta_{\text{brs}}^{(1)}$  is  $s_{\Gamma_{\text{cl}}}$ -invariant

$$s_{\Gamma_{\text{cl}}} \Delta_{\text{brs}}^{(1)} = 0. \quad (10)$$

$\Delta_{\text{brs}}^{(1)}$  is invariant under the  $s_{\Gamma_{\text{cl}}}$ -transformation whenever it is a variation of the counterterms, *i.e.* if Eq. (7) is fulfilled. If there is only one field

polynomial which is  $s_{\Gamma_{\text{cl}}}$ -invariant, but not a variation of counterterms, one has an anomaly, and symmetries cannot be established by adjusting counterterms. In this way one has achieved an algebraic characterization of scheme dependent breakings and anomalies. The proof to all orders proceeds by induction, passing through the same steps as above from order  $n$  to  $n + 1$ .

In the SM the algebraic characterization of breakings can be proven to be the same as in the symmetric  $\text{SU}(2) \times \text{U}(1)$  theory and does not depend on the specific form of spontaneous symmetry breaking. In [7] it has been shown that there are only the well-known Adler–Bardeen anomalies. Their coefficients vanish in the SM, if we include lepton and colored quark doublets. The difficult and indeed specific part is the classification of invariant counterterms and of appropriate normalization conditions. Here mixing effects between neutral massless/massive fields have to be carefully analysed from the point of view of off-shell infrared existence. In the next section we present three important results of this analysis.

### 3. Invariant counterterms and normalization conditions

Invariant counterterms are determined if one solves the ST identity and the Ward identities for the most general action compatible with power counting renormalizability:

$$\mathcal{S}(\Gamma_{\text{cl}}^{\text{gen}}) = 0 \quad \mathcal{W}_\alpha \Gamma_{\text{cl}}^{\text{gen}} = 0 \quad \dim^{\text{UV}} \Gamma_{\text{cl}}^{\text{gen}} \leq 4. \quad (11)$$

In the SM one finds as solution an action, which contains in addition to the free parameters of the tree approximation two further undetermined couplings in each fermion family. They are couplings of abelian currents to the abelian component of vector fields and are not determined by the ST identity, but have to be fixed by a local gauge Ward identity. (In [10] these couplings are fixed by an antighost equation. From there the local Ward identity is defined by using the consistency with the ST identity.) In the SM classically we have three types of abelian currents, the currents of lepton and quark family number conservation,  $j_l^\mu$  and  $j_q^\mu$ , and the sum of the electromagnetic and neutral current of weak interactions. Being more specific we find as a special solution of Eq. (11)

$$\Gamma'_{\text{cl}} = \Gamma_{\text{cl}} + \int d^4x (g_l j_l^\mu + g_q j_q^\mu) (\sin \theta_W Z_\mu + \cos \theta_W A_\mu), \quad (12)$$

where  $\Gamma_{\text{cl}}$  is the classical action (1). In order to identify the action (1) as solution of symmetry identities, one has to impose the local  $\text{U}(1)$ -Ward identity as given in (3). The abelian local Ward operator is to all orders

fixed to be the sum of the non-integrated neutral SU(2) Ward operator and of the electromagnetic charge operator

$$\boldsymbol{w}_4^Q \equiv \boldsymbol{w}_{\text{em}} - \boldsymbol{w}_3 \quad \text{with} \quad [\boldsymbol{w}_4^Q, \mathcal{W}_\alpha] = 0 \quad (13)$$

In fact the local abelian Ward identity (3) with the abelian operator (13) is the functional generalization of the classical Gell-Mann–Nishijima relation. We want to point out that the abelian Ward identity has to be characterized to be abelian by its commutation relations with Ward operators of rigid SU(2)-symmetry. For this reason it is crucial to establish rigid symmetry in addition to the ST identity.

We did already mention that the symmetry operators depend explicitly on the mass ratio  $M_W/M_Z$  in the tree approximation. Solving Eq. (11) by inserting the tree operators we are not able to fix all mass parameters by normalization conditions and especially we are not able to diagonalize the mass matrix of neutral vector bosons on-shell at the same time:

$$\begin{aligned} \text{Re } \Gamma_{W+W-} \Big|_{p^2=M_W^2} &= 0, & \text{Re } \Gamma_{ZZ} \Big|_{p^2=M_Z^2} &= 0, & \Gamma_{AA} \Big|_{p^2=0} &= 0, \\ \text{Re } \Gamma_{ZA} \Big|_{p^2=M_Z^2} &= 0, & \Gamma_{ZA} \Big|_{p^2=0} &= 0. \end{aligned} \quad (14)$$

Mass diagonalization at  $p^2 = 0$  is crucial for obtaining infrared finite expression for off-shell Green functions and are implemented in the BPHZL scheme by the IR power counting. In order to fulfill the normalization conditions (14) one has to introduce a non-diagonal wave-function renormalization for the neutral vector bosons. If we carry out such field redefinitions also in the symmetry operators, the symmetry operators are renormalized and get higher order corrections,

$$\mathcal{S}(\Gamma) \rightarrow (\mathcal{S} + \delta\mathcal{S})(\Gamma), \quad \mathcal{W}_\alpha \Gamma \rightarrow (\mathcal{W}_\alpha + \delta\mathcal{W}_\alpha)\Gamma. \quad (15)$$

These corrections are in agreement with the algebra. For this reason one cannot fix the symmetry operators to their tree form but one has to take the most general ones compatible with the algebraic properties of the tree approximation (see (9)), when one solves eqs. (11). Then one has indeed the same number of free parameters as normalization conditions even in the complete on-shell scheme.

Requiring rigid symmetry has important restrictions on the gauge fixing. As we have already mentioned the choice (2) is covariant under rigid SU(2)×U(1)-transformations and can be constructed as being invariant under rigid transformation by introducing the external scalar field  $\hat{\Phi}$ . In this special gauge the ghost mass ratio is equivalent to the vector mass ratio,

$$M_W^{\text{ghost}}/M_Z^{\text{ghost}} = M_W/M_Z + \mathcal{O}(\hbar), \quad (16)$$

but it turns out to be differently renormalized from the vector mass ratio in higher orders. For this reason we have to introduce the ghost mass ratio as an independent parameter of the model. However, if one takes arbitrary gauge parameters  $\zeta_W$  and  $\zeta_Z$  in the in the gauge-fixing functions  $F_\pm$  and  $F_Z$  without changing  $F_A$  (2), one breaks rigid symmetry by the gauge fixing and cannot derive Ward identities of rigid and local U(1) symmetry. Consequently we loose the control about the gauged abelian currents in higher orders, namely we are not able to identify the electromagnetic current by means of a local Ward identity. Taking for the gauge fixing

$$\begin{aligned} F_\pm &\equiv \partial_\mu W_\pm^\mu \mp iM_W \zeta_W \phi_\pm, & F_Z &\equiv \partial_\mu Z^\mu - M_Z \zeta_Z \chi, \\ F_A &\equiv \partial_\mu A^\mu - \zeta_Z M_Z \frac{\cos \theta_W}{\sin \theta_W} \left(1 - \frac{\zeta_W}{\zeta_Z}\right) \chi, \end{aligned} \quad (17)$$

the gauge fixing is compatible with rigid symmetry and allows at the same time to treat the ghost mass ratio as an independent parameter of the theory. In order to avoid non-diagonal ghost mass terms BRS-transformations have to be generalized to  $s\bar{c}_a = \hat{g}_{ab} B_b$  and  $\hat{g}$  differs from the unit matrix by

$$\begin{aligned} \hat{g}_{ZZ} &= \cos(\theta_G - \theta_W) & \cos \theta_W &= \frac{M_W}{M_Z} \\ \hat{g}_{AZ} &= \sin(\theta_G - \theta_W) & \frac{\cos \theta_G}{\cos(\theta_W - \theta_G)} &= \frac{\zeta_W M_W}{\zeta_Z M_Z}. \end{aligned} \quad (18)$$

With this modification the ghost mass matrix is diagonal, and can be achieved to be diagonal on-shell in higher orders by adjusting the free parameters in the matrix  $\hat{g}_{ab}$ . Then also the Faddeev–Popov part is free from off-shell infrared divergencies. Contrary to simple gauge groups we have introduced not only independent wave function renormalizations for ghosts and vectors, but have also an independent wave function renormalization for antighosts.

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